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<th>A Remark on Nowhere Dense Closed P-Sets (General Topology, Geometric Topology and Related Problems)</th>
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A Remark on Nowhere Dense Closed P-Sets

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Abstract. Using the methods from continua theory of $R^*$, we prove that NCF implies that $\omega^*$ can be covered by an increasing sequence of nowhere dense closed P-sets.

Key words: $\omega^*$, $R^*$, continuum, P-set and NCF.
Kunen, van Mill and Mills proved in [4] that no compact space of weight $2^\omega$ can be covered by nowhere dense closed P-sets under CH. It was proved in [1] that, in the model obtained by adding $\omega_1$ Cohen reals to a model of MA+$\neg$CH, $\omega^*$ can be covered by nowhere dense closed P-sets. It is not difficult to show that the axiom of near coherence of filters, abbreviated as NCF (See [2]), implies that $\omega^*$ can be covered by nowhere dense closed P-sets. Our purpose in this note is to strengthen the conclusion as follows:

**Theorem 1.** NCF implies that $\omega^*$ can be covered by an increasing sequence of nowhere dense closed P-sets.

Our way is to use the methods from continua theory of $R^*$ to guarantee an induction construction going smoothly through the limit steps. Actually, we shall prove, (See also Corollary 5.7 in [5]).

**Theorem 2.** NCF is equivalent to that $\mathbb{B}[0,\omega)-[0,\omega)$ can be covered by a strictly increasing sequence of subcontinua which are nowhere dense P-sets.
It is not difficult to show that if $\omega^*$ can be covered by nowhere dense closed $\mathbb{P}$-sets then so can $\mathbb{R}^*$ (See Corollary 4). But the author don't know whether or not the converse is true.


Let $\Omega$ be the collection of all families of infinite discrete non-degenerate closed interval of the half real line $[0,\omega)$. For $\mathcal{J} \in \Omega$, we let $i : \omega \rightarrow \mathcal{J}$ be the bijection such that $i(n) < i(n+1)$ for $n \in \omega$, where $i(n) < i(n+1)$ means that $r < s$ for all $(r, s) \in i(n) \times i(n+1)$. Let $i : \mathcal{U} \rightarrow \omega$ be such that $i(x) = n$ if and only if $x \in i(n)$. Let $\beta i$ be the Stone-Čech extension of $i$ from $\text{cl}_{\beta \mathbb{R}}(\mathcal{U})$ to $\beta \omega$. For $B \subseteq \omega^*$, we define

$$M(\mathcal{J}, B) = \beta i^{-1}(B)$$

and, if $B = \{u\}$, then $M(\mathcal{J}, \{u\})$ is denoted by $M(\mathcal{J}, u)$. It is well-known that $M(\mathcal{J}, u)$ is a continuum for any $u \in \omega^*$. Moreover, a subcontinuum $C$ of $\beta[0,\omega)-[0,\omega)$ is called a standard continuum if $C = M(\mathcal{J}, u)$ for some $\mathcal{J} \in \Omega$ and $u \in \omega^*$. Note that every proper subcontinuum of $\beta[0,\omega)-[0,\omega)$ is nowhere dense since $\beta[0,\omega)-[0,\omega)$ is an indecomposable continuum.

Recall that a subset $B$ of a space $X$ is called a $\mathbb{P}$-set
provided that the intersection of countably many neighbourhoods of \( B \) is again a neighbourhood of \( B \). A point \( x \) of \( X \) is called a P-point if the singleton \( \{x\} \) is a P-set.

For an open set \( U \) of a metric space \( X \), we let \( O(U) = \{x \in B\cdot X : \exists F \in X(F \subseteq U)\} \). Then \( \{O(U): U \text{ is open in } X\} \) is a base for \( B\cdot X \). \([\omega]^\omega\) is the set of all infinite subsets of \( \omega \). As usual, \( O(A) \cap \omega^* \) is denoted by \( A^* \) for \( A \in [\omega]^\omega \).

For \( \mathcal{J}, \mathcal{J}' \in \Omega \), we say that \( \mathcal{J}' \) is an expander of \( \mathcal{J} \) if \( i(n) \) is contained in the interior of \( i'(n) \) for all \( n \in \omega \).

**Lemma 3.** \( B \in \omega^* \) is a nowhere dense closed P-set if and only if \( M(\mathcal{J}, B) \) is a nowhere dense closed P-set of \( \beta([0, \omega) - [0, \omega) \) for \( \mathcal{J} \in \Omega \).

**Proof.** Assume that \( M(\mathcal{J}, B) \) is a nowhere dense closed P-set of \( \beta([0, \omega) - [0, \omega) \). It is easily seen that \( B \) is nowhere dense closed in \( \omega^* \). Let \( \mathcal{J}' \in \Omega \) be an expander of \( \mathcal{J} \). Then \( M(\mathcal{J}, B) \) is a P-set of \( \text{cl}_{\beta \cdot R}(\mathcal{U}\mathcal{J}') \). Suppose that \( \{A^*_n : n \in \omega\} \) is a family of countably many neighbourhoods of \( B \). Then \( \{i^{-1}(A^*_n) : n \in \omega\} \) is a family of neighbourhoods of \( M(\mathcal{J}, B) \). Therefore, there is a basic open set \( O(U) \) such that \( M(\mathcal{J}, B) \subseteq O(U) \cap R^* \subseteq \beta i^{-1}(A^*_n) \) for all \( n \in \omega \). Note that, for \( u \in \omega^* \), \( M(\mathcal{J}, u) = \wedge \{\text{cl}_{\beta \cdot R}(\mathcal{U}\mathcal{J}) : i^{-1}(\mathcal{J}) \in u\} \). Therefore, for each \( u \in B \), there is \( A_u \in \mathcal{U} \) such that \( \bigcup \{i(n) : n \in A_u\} \subseteq U \). Let \( \mathcal{A} = \{i^{-1}(I) : I \in \mathcal{J} \text{ and } I \subseteq U\} \). Then \( A_u \subseteq \mathcal{A} \) for \( u \in B \). So \( A^* \) is a neighbourhood of \( B \). Since \( O(U) \cap R^* \subseteq \beta i^{-1}(A_n) \),
we have that $A^* \subseteq A_n^*$ for all $n \in \omega$.

Assume that $B$ is a nowhere dense closed $P$-set of $\omega^*$. Let $O(U)$ be a basic open set of $B$ and $O(U) \cap M(\mathcal{F}, B) \neq \emptyset$. Let $A = \{ n \in \omega : \mathcal{I}(n) \cap U \neq \emptyset \}$. Then $A \in [\omega]^\omega$. Since $B$ is nowhere dense, there is $A \in [\omega]^\omega$ such that $A \subseteq A$ and $A \cap B = \emptyset$. Therefore, $M(\mathcal{F}, A^*) \cap M(\mathcal{F}, B) = \emptyset$. But $O(U) \cap M(\mathcal{F}, A^*) \neq \emptyset$. So $O(U) \cap R^* \subseteq M(\mathcal{F}, B) \neq \emptyset$. It follows that $M(\mathcal{F}, B)$ is nowhere dense. Suppose that $\{ O(U_n) : n \in \omega \}$ is a family of neighbourhoods of $M(\mathcal{F}, B)$. Let $A_n = \{ i(I) : I \in \mathcal{F}$ and $I \subseteq U_n \}$ for $n \in \omega$. As we showed in the last paragraph, $A_n^*$ is a neighbourhood of $B$ for all $n \in \omega$. Since $B$ is a $P$-set, there is $A \in [\omega]^\omega$ such that $B \subseteq A^*$ and $A^* \subseteq A_n^*$ for all $n \in \omega$. We choose a strictly increasing sequence $\{ m : n \in \omega \}$ of integers so that for each $n \in \omega$, $A \setminus m_n \subseteq A_n$ and $[m_n, m_{n+1}) \cap A \neq \emptyset$, where $[m_n, m_{n+1}) = \{ i \in \omega : m_n \leq i < m_{n+1} \}$. For each $i \in [m_n, m_{n+1}) \cap A$, let $J_i$ be an open interval of $R$ such that $i(1) \subseteq J_i \subseteq U_n$. Let $V = \cup \{ J_n : n \in \omega \}$. Then $M(\mathcal{F}, B) \subseteq M(\mathcal{F}, A^*) \subseteq O(V)$ and $O(V) \subseteq O(U_n)$ for $n \in \omega$. This completes the proof of Lemma 3.

Since we can easily choose $\mathcal{F}, \mathcal{F}' \subseteq \omega$ such that $U(\mathcal{F} \cup \mathcal{F}') = [0, \omega)$ and $R^*$ is the topological sum of $\beta(-\omega, 0) - (-\omega, 0]$ and $\beta(0, \omega) - [0, \omega)$, we have

**Corollary 4.** If $\omega^*$ can be covered by nowhere dense closed $P$-sets, then so can $R^*$. 
Blass proved in [2] that, under NCF, for any \( u \in \omega^* \) there is a finite-to-one non-decreasing function \( f: \omega \to \omega \) such that \( v = \beta f(u) \) is a P-point. It is easily seen that \( \beta f^{-1}(v) \) is a nowhere dense closed P-set of \( \omega^* \) and \( u \in \beta f^{-1}(v) \). Therefore, NCF implies that \( \omega^* \) can be covered by nowhere dense closed P-sets. Our purpose is to sharpen the conclusion so that \( \omega^* \) can be covered by an increasing sequence of nowhere dense closed P-sets under NCF.

We regard \( \omega^* \) as a subspace of \( \beta(0,\omega) \setminus (0,\omega) \). The following lemma is an easy observation.

**Lemma 5.** If \( u \in \omega^* \) is a P-point, then \( M(\mathcal{B},u) \cap \omega^* \) is a nowhere dense closed P-set of \( \omega^* \) for \( \mathcal{B} \in \Omega \).

**Proof.** Let \( X = \omega \cap (\cup \mathcal{B}) \) and \( Y = \{ i^{-1}(I) : I \cap \omega \neq \emptyset \} \). If \( Y \notin u \), then, \( M(\mathcal{B},u) \cap \omega^* = \emptyset \). So we assume that \( Y \in u \). We define a finite to one function \( f: X \to Y \) from \( X \) onto \( Y \) by \( f(n) = m \) if and only if \( n \in i(m) \). Then \( M(\mathcal{B},u) \cap \omega^* = \beta f^{-1}(u) \). Since \( \beta f^{-1}(u) \) is a nowhere dense closed P-set in \( \omega^* \), \( M(\mathcal{B},u) \cap \omega^* \) is a nowhere dense closed P-set in \( \omega^* \).

By Lemma 3 and 5, our Theorem 1 and 2 follows easily from the following theorem.
Theorem 2'. NCF is equivalent to that there is a family
\{(\mathcal{Y}_\alpha, u_\alpha) : \alpha < \lambda\} such that

1. \mathcal{Y}_\alpha \in \mathcal{O} and \, u_\alpha \in \omega^* is a P-point for all \, \alpha < \lambda;

2. M(\mathcal{Y}_\alpha, u_\alpha) \subseteq M(\mathcal{Y}_\beta, u_\beta) for all \, \alpha < \beta < \lambda;

3. \beta[0, \omega) \setminus \{0, \omega\} = \bigcup \{M(\mathcal{Y}_\alpha, u_\alpha) : \alpha < \lambda\}.

Theorem 2' will be proved along the line of the proof of Corollary 5.7 in [5]. We first recall some properties of NCF and standard continua. We refer to [2] and [5] for details.

A subset \( C \) of a continuum \( K \) is a composant if, for some point \( p \in C \), \( C \) is the set of all points \( x \) such that there is a proper subcontinuum of \( K \) containing both \( p \) and \( x \). It is well-known that NCF is equivalent to that \( \beta[0, \omega) \setminus \{0, \omega\} \) is a composant of itself (See [3]). Therefore, our conditions in Theorem 2' implies NCF.

Recall that there is a natural partial order \( <_{\mathcal{Y}, u} \) on \( M(\mathcal{Y}, u) \) for \( \mathcal{Y} \in \mathcal{O} \) and \( u \in \omega^* \), defined as follows: For any \( x, y \in M(\mathcal{Y}, u) \),

\[ x <_{\mathcal{Y}, u} y \text{ if there are } F \in x \text{ and } H \in y \text{ such that } \{\iota^{-1}(I) : I \in \mathcal{Y} \text{ and } F \cap I \in H \cap I\} \in u, \]

For \( x \in M(\mathcal{Y}, u) \), we let
\[ [x]_u^\vartheta = \{ y \in M(\vartheta, u) : y \text{ is } \leq u \text{-incomparable with } x \text{ or } y = x \}. \]

\[ [x]_u^\vartheta \] is called a layer of \( M(\vartheta, u) \). It is well-known that layers are indecomposable subcontinua of \( M(\vartheta, u) \) and every indecomposable subcontinuum of \( M(\vartheta, u) \) is contained in a layer.

**Lemma 6** (Corollary 2.11 in [5]). *Let \( C \) and \( D \) be subcontinua of \( R^* \). If one of them is indecomposable, then \( C \cap D = \emptyset \).*

A point \( u \in \omega^* \) is a Q-point if every finite-to-one function from \( \omega \) to \( \omega \) is one-to-one on a set in \( u \). By Proposition 5.1 in [5], it is equivalent to require the functions in the definition of Q-points to be non-decreasing. Blass proved in [2] that NCF implies that there is no Q-points.

**Lemma 7.** Under NCF, for every proper subcontinuum \( C \) of \( \beta(0, \omega) - [0, \omega) \), there is a standard continuum \( M(\vartheta, u) \) and a layer \( T \) of \( M(\vartheta, u) \) such that \( C \subseteq T \) and \( M(\vartheta, u) \) is a nowhere dense P-set of \( \beta(0, \omega) - [0, \omega) \).

**Proof.** Since every proper subcontinuum of \( \beta(0, \omega) - [0, \omega) \) is contained in a standard subcontinuum, we assume that \( C \subseteq M(\vartheta_1, u_1) \) for some \( \vartheta_1 \in \Omega \) and \( u \in \omega^* \). Since NCF implies that there is no
Q-points, there is a finite-to-one non-decreasing function \( f: \omega \to \omega \) which witnesses that \( u_1 \) is not a Q-point. We define \( \mathcal{G}_2 = \{ I_n : n \in \omega \} \) as follows: \( I_n \) is the convex hull of the set \( \bigcup \{ i_1(m) : m \in f^{-1}(n) \} \). Let \( u_2 = f(u_1) \). Then, \( M(\mathcal{G}_1, u_1) \subset M(\mathcal{G}_2, u_2) \). Moreover, for any \( x, y \in M(\mathcal{G}_1, u_1) \), \( x \) and \( y \) are \( \prec \mathcal{G}_2 \)-incomparable or \( x = y \). Therefore, \( M(\mathcal{G}_1, u_1) \) is contained in a layer \( T' \) of \( M(\mathcal{G}_2, u_2) \). By NCF, there is a finite-to-one non-decreasing function \( g: \omega \to \omega \) such that \( u = g(u_2) \) is a P-point. By the same method as above, we can find \( \mathcal{F} \in \Omega \) such that \( M(\mathcal{G}_2, u_2) \subset M(\mathcal{F}, u) \).

Since \( T' \) is an indecomposable subcontinuum of \( M(\mathcal{F}, u) \), there is a layer \( T \) of \( M(\mathcal{F}, u) \) such that \( C \subset T' \subset T \). By Lemma 3, \( M(\mathcal{F}, u) \) is a nowhere dense P-set of \( \beta[0, \omega) - [0, \omega) \).

Now we are in a position to complete the proof of Theorem 2'. We assume NCF. We define, inductively, \( \mathcal{G}_\beta \in \Omega \), \( u_\beta \in \omega^* \) and a layer \( T_\alpha \) of \( M(\mathcal{G}_\alpha, u_\alpha) \) for \( \alpha \geq 0 \) satisfying that

(a) \( u_\alpha \) is a P-point for all \( \alpha > 0 \);

(b) \( M(\mathcal{G}_\alpha, u_\alpha) \subset T_\beta \) for \( \alpha < \beta \).

Our induction process will stop at some \( \lambda \) if \( \beta[0, \omega) - [0, \omega) = \bigcup \{ M(\mathcal{G}_\alpha, u_\alpha) : \alpha < \lambda \} \). Suppose that we have defined \( \mathcal{G}_\beta \), \( u_\beta \) and \( T_\beta \) for all \( \beta < \alpha \) satisfying (a) and (b). If \( \alpha = 0 \) or \( \gamma + 1 \), then, by Lemma 7, we can easily define \( \mathcal{G}_\alpha \), \( u_\alpha \) and \( T_\alpha \) satisfying (a) and (b). Assume that \( \alpha \neq 0 \) is a limit and \( \beta[0, \omega) - [0, \omega) \) is not covered by \( \{ M(\mathcal{G}_\beta, u_\beta) : \beta < \alpha \} \). Note that by (b) \( \bigcup \{ M(\mathcal{G}_\beta, u_\beta) : \beta < \alpha \} = \)
\( \bigvee \{ T_\beta : \beta < \alpha \} \) since \( \alpha \) is a limit. Take \( x, y \in T_0 \) and \( y \in T_\beta \) for all \( \beta < \alpha \). By NCF, there is a proper subcontinuum \( C \) of \( \beta(0, \infty) - (0, \infty) \), containing both \( x \) and \( y \). By Lemma 6, \( T_\beta \subseteq C \) for all \( \beta < \alpha \). By Lemma 7, there is \( s_\alpha \in \Omega \), \( u_\alpha \in \omega^* \) and a layer \( T_\alpha \) of \( M(s_\alpha, u_\alpha) \) such that \( C \subseteq T_\alpha \) and \( u_\alpha \) is a P-point. This completes our inductive construction. Since \( \{ M(s_\alpha, u_\alpha) : \alpha \geq 0 \} \) is a strictly increasing sequence, our induction process can not go over \( |R^*| \) steps. This completes the proof of Theorem 2'.

References