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Striped structures of stable and unstable sets of expansive homeomorphisms and a theorem of K. Kuratowski on independent sets

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1. Introduction.

All spaces under consideration are assumed to be metric. By a *compactum*, we mean a compact metric space, and by a *continuum*, a connected nondegenerate compactum. A homeomorphism $f: X \to X$ of a compactum X is called *expansive* if there is a constant $c > 0$ (called an *expansive constant for* f) such that if $x, y \in X$ and $x \neq y$, then there is an integer $n = n(x,y) \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

This property has frequent applications in topological dynamics, ergodic theory and continuum theory [1,3,7,8]. A homeomorphism $f: X \to X$ of a compactum X is *continuum-wise expansive* if there is a constant $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n = n(A) \in \mathbb{Z}$ such that diam $f^n(A) > c$. By definitions, we can easily see that every expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many important examples of homeomorphisms which are continuum-wise expansive homeomorphisms, but not expansive homeomorphisms.

In this note, we show that if $f: X \to X$ is an expansive homeomorphism of a compactum X with dim X > 0, then the decompositions $\{W^s(x) | x \in X\}$ and $\{W^u(x) | x \in X\}$ of X to stable and unstable sets are uncountable respectively, and moreover there is σ (σ = s or u) and a positive number $\rho > 0$ such that the σ-striped set $Z(\sigma,\rho)$ of f is not empty. Hence, by using a theorem of K. Kuratowski on independent sets [6], it is proved that almost every Cantor set C of $Z(\sigma,\rho)$ satisfies the property that for each $x \in C$, $W^\sigma(x)$ contains a nondegenerate subcontinuum containing x and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$. Also, we show that if $f: G \to G$ is a map of a graph G and the shift map $\tilde{f}:(G,f) \to (G,f)$ of f is expansive, then for each $\tilde{x} \in (G,f)$, $W^u(\tilde{x})$ is equal to the arc-component of (G,f) containing $\tilde{x}$, and $W^s(\tilde{x})$ is 0-dimensional.

2. Definitions and preliminaries.

Let $f: X \to X$ be a homeomorphism of a compactum X and let $x \in X$. Then the *stable set* $W^s(x)$ and the *unstable set* $W^u(x)$ are defined as follows:

$$W^s(x) = \{y \in X | \lim_{n\to\infty} d(f^n(x),f^n(y)) = 0\},$$
$$W^u(x) = \{y \in X | \lim_{n\to\infty} d(f^{-n}(x),f^{-n}(y)) = 0\}.$$

Also, the *continuum-wise stable* and *unstable sets* $V^s(x)$, $V^u(x)$ are defined as follows:

$V^s(x) = \{y \in X|$ there is $A \in C(X)$ such that $x, y \in A$ and $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\}$,

$V^u(x) = \{y \in X|$ there is $A \in C(X)$ such that $x, y \in A$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}$.

Clearly, $W^\sigma(x) \supset V^\sigma(x)$, $\{W^\sigma(x)|x \in X\}$ and $\{V^\sigma(x)|x \in X\}$ are decompositions of X for each σ = s and u, i.e., $X = \cup\{W^\sigma(x)|x \in X\}$ (resp. $X = \cup\{V^\sigma(x)|x \in X\}$), and if $W^\sigma(x) \neq W^\sigma(y)$ (resp. $V^\sigma(x) \neq V^\sigma(y)$), then $W^\sigma(x) \cap W^\sigma(y) = \phi$ (resp. $V^\sigma(x) \cap V^\sigma(y) = \phi$).

We are interested in the structures of the decompositions $\{W^\sigma(x)|x \in X\}$ and $\{V^\sigma(x)|x \in X\}$ (σ = s and u) of X. Let $f: X \to X$ be a homeomorphism of a compactum X with $\dim X > 0$. Let $\rho > 0$ be a positive number. Consider the family $\Phi(\sigma) = \{Z|$ Z is a closed subset of X satisfying that (i) for each $x \in Z$ there is a subcontinuum A_x of X such that $\operatorname{diam} A_x \geq \rho$, $x \in A_x \subset W^\sigma(x)$, and (ii) for any neighborhood U of x in X, there is $y \in Z \cap U$ such that $W^\sigma(x) \neq W^\sigma(y)\}$. Clearly, $\Phi(\sigma)$ has the maximal element $Z(\sigma,\rho)$ $(= \mathrm{Cl}(\cup\{Z|\ Z \in \Phi(\sigma)\}))$. The set $Z(\sigma,\rho)$ is said to be a *σ-striped set of* f. Note that if $0 < \rho_1 < \rho_2$, then $Z(\sigma,\rho_1) \supset Z(\sigma,\rho_2)$. Also, note that if $Z(\sigma,\rho) \neq \phi$ for some $\rho > 0$, then X contains an uncountable collection of mutually disjoint, nondegenerate subcontinua of X each of which is contained in a different element of $\{W^\sigma(x)|x \in X\}$.

Let $f: X \to X$ be a map of a compactum X with metric d. Consider the following inverse limit space:

$$(X,f) = \{(x_i)_{i=0}^{\infty} \mid x_i \in X,\ f(x_{i+1}) = x_i \text{ for each } i \geq 0\}.$$

Define a metric $\tilde{d}$ for (G,f) by

$$\tilde{d}(\tilde{x},\tilde{y}) = \Sigma_{i=0}^{\infty} d(x_i,y_i)/2^i \text{ for } \tilde{x} = (x_i)_{i=0}^{\infty},$$
$$\tilde{y} = (y_i)_{i=0}^{\infty} \in (X,f).$$

The space (X,f) is called the *inverse limit of the map* f. Define a map $\tilde{f}: (X,f) \to (X,f)$ by

$$\tilde{f}(x_0,x_1,\ldots,) = (f(x_0),x_0,x_1,\ldots,), \text{ for } (x_i)_{i=0}^{\infty} \in (X,f).$$

Then the map $\tilde{f}$ is a homeomorphism and it is called the *shift map of* f.

(2.1) Example. Let S^1 be the unit circle and let $f: S^1 \to S^1$ be the natural covering map with degree 2. Consider the inverse limit (S^1,f) of f and the shift map $\tilde{f}$: $(S^1,f) \to (S^1,f)$. The continuum (S^1,f) is well-known as the 2-adic solenoid and $\tilde{f}$ is an expansive homeomorphism. In this case, for each $\tilde{x} \in (S^1,f)$, $W^u(\tilde{x}) = V^u(\tilde{x})$ is the arc-component of (S^1,f) containing $\tilde{x}$. Also, $V^s(\tilde{x}) = \{\tilde{x}\} \subsetneq W^s(\tilde{x})$ for each $\tilde{x} \in (S^1,f)$. Then the decomposition $\{W^{\sigma}(\tilde{x}) \mid \tilde{x} \in (S^1,f)\}$ (σ = s and u) is uncountable.

Note that $\dim W^s(\tilde{x}) = 0$, because $W^s(\tilde{x})$ is an F_σ-set and $W^s(\tilde{x})$ does not contain a nondegenerate subcontinuum. Note that the continuum (S^1,f) itself is a u-striped set $Z(\sigma,\rho)$ of $\tilde{f}$ for some $\rho > 0$, but $Z(s,\rho) = \phi$ for each $\rho > 0$.

(2.2) Example. There is an expansive homeomorphism $f: X \to X$ such that $\mathrm{Int}_X W^\sigma(x) \neq \phi$ for some $x \in X$. Let G be the one point union of the unit interval I and a circle S^1, i.e., $(G,*) = (I,1) \vee (S^1,*)$. Define a map $g: G \to G$ such that $g|S^1: S^1 \to S^1$ is the natural covering map with degree 2 and $g(0)=0$, $g(1) = *$ and $g(I) = G$. We can choose $g: G \to G$ so that $\tilde{g}: X=(G,g) \to X=(G,g)$ is expansive. Then $W^u(\tilde{0})$ is a dense open set of X, where $\tilde{0} = (0,0,\ldots,)$. Hence X itself is not a u-striped set of $\tilde{g}$.

A subset E of a space X is called to be an F_σ-*set* in X if E is a union of countable closed subsets F_n of X, i.e., $E = \cup_{n=1}^{\infty} F_n$. A subset E of X is called to be an $F_{\sigma\delta}$-*set* in X if E is an intersection of countable F_σ-sets E_n, i.e., $E = \cap_{n=1}^{\infty} E_n$.

We use a theorem of K. Kuratowski on independent sets [6]. A subset F of X is said to be *independent in* $R \subset X^n$, if for every system $x_1,x_2,\ldots,x_n$ of different points of F the point $(x_1,x_2,\ldots,x_n) \in F^n$ never belongs to R. In [6], K. Kuratowski proved the following theorem.

(2.3) Theorem ([6, Main theorem and Corollary 3]). *If* X *is a complete space and and* $R \subset X^n$ *is an* F_σ*-set of the first category, then the set* $J(R)$ *of all compact subsets* F *of* X *independent in* R *is a dense* G_δ*-set in* 2^X *of all compact subsets of* X. *Moreover, if* X *has no isolated points, then almost every Cantor set of* X *is independent in* R.

For the proof of the main theorem of this note, we need the following.

(2.4) Proposition. *Let* $f: X \to X$ *be a homeomorphism of a compactum* X. *Then* $W^\sigma(x)$ *is an* $F_{\sigma\delta}$*-set in* X ($\sigma = s,u$).

(2.5) Proposition. *Let* $f: X \to X$ *be an expansive homeomorphism of a compactum* X. *Then* $W^\sigma(x)$ *is an* F_σ*-set in* X ($\sigma = s,u$).

(2.6) Proposition. *Let* $f: X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X. *Then* $V^\sigma(x)$ *is an* F_σ*-set in* X ($\sigma = s,u$).

3. Striped structures of stable and unstable sets.

In this section, we study striped structures of stable and unstable sets of expansive homeomorphisms and continuum-wise expansive homeomorphisms. The main result of this section is the following theorem.

(3.1) Theorem. *Let* $f\colon X \to X$ *be an expansive homeomorphism of a compactum* X *with* $\dim X > 0$. *Then the decomposition* $\{W^{\sigma}(x) \mid x \in X\}$ ($\sigma = s$ *and* u) *of* X *is uncountable. Moreover, there exists* σ ($\sigma = s$ or u) *and a positive number* $\rho > 0$ *such that the* σ*-striped set* $Z(\sigma, \rho)$ *is not empty. In particular, almost every Cantor set* C *of* $Z(\sigma, \rho)$ *satisfies the property that for any* $x \in C$, *there exists a nondegenerate subcontinuum* A_x *of* X *such that* $x \in A_x \subset W^{\sigma}(x)$, and *if* $x, y \in C$ *and* $x \neq y$, *then* $W^{\sigma}(x) \neq W^{\sigma}(y)$.

To prove (3.1), we need the following facts. The next lemma is obvious.

(3.2) Lemma. *Let* $f\colon X \to X$ *be a map of a compactum* X *and let* $N \geq 1$ *be a natural number. Suppose that there is* $\gamma > 0$ *such that* $d(f^{iN}(x), f^{iN}(y)) \geq \gamma$ *for each* $i = 0, 1, 2, \ldots$. *Then there is a positive number* $\eta > 0$ *such that* $d(f^{i}(x), f^{i}(y)) \geq \eta$ for each $i = 0, 1, 2, \ldots$

(3.3) Lemma ([4,(2.3)]). *Let* $f\colon X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with an expansive constant* $c > 0$ *and let* $0 < \varepsilon < c/2$. *Then thare is* $\delta > 0$ *such that if* A *is any nondegenerate subcontinuum of* X *such that* $\operatorname{diam} A \leq \delta$ *and* $\operatorname{diam} f^{m}(A) \geq \varepsilon$ *for some integer* $m \in \mathbb{Z}$, *then one of the following conditions holds:*

(a) *If* $m \geq 0$, *then* $\operatorname{diam} f^n(A) \geq \delta$ *for each* $n \geq m$. *More precisely, there is a subcontinuum* B *of* A *such that* $\operatorname{diam} f^j(B) \leq \varepsilon$ *for* $0 \leq j \leq n$ *and* $\operatorname{diam} f^n(B) = \delta$.

(b) *If* $m < 0$, *then* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each* $n \geq -m$. *More precisely, there is a subcontinuum* B *of* A *such that* $\operatorname{diam} f^{-j}(B) \leq \varepsilon$ *for* $0 \leq j \leq n$ *and* $\operatorname{diam} f^{-n}(B) = \delta$.

(3.4) Lemma ([4,(2.4)]). *Let* f, c, ε, δ *be as in* (3.3). *Then for any* $\gamma > 0$, *there is* $N > 0$ *such that if* $A \in C(X)$ *and* $\operatorname{diam} A \geq \gamma$, *then* $\operatorname{diam} f^n(A) \geq \delta$ *for each* $n \geq N$ *or* $\operatorname{diam} f^{-n}(A) \geq \delta$ *for each* $n \geq N$.

For the case of continuum-wise expansive homeomorphism, we have

(3.5) Theorem. *Let* $f: X \to X$ *be a continuum-wise expansive homeomorphism of a compactum* X *with* $\dim X > 0$. *Then the decompositions* $\{V^\sigma(x) | x \in X\}$ (σ = s *and* u) *are uncountable. Moreover, there is* σ (σ = s *or* u) *and a* positive *number* $\rho > 0$ *such that there is a nonempty closed set* Z' *of* X *satisfying that* (i) *for each* $x \in Z'$ *there is a subcontinuum* A_x *of* X *satisfying that* $\operatorname{diam} A_x \geq \rho$, $x \in A_x \subset V^\sigma(x)$, (ii) *for any neighborhood* U *of* x *in* X, *there is* $y \in Z' \cap U$ *such that* $V^\sigma(x) \neq V^\sigma(y)\}$. *In particular, almost every Cantor set* C *of* $Z(\sigma)$ *satisfies the property that for any* $x \in C$, *there is a nondegenerate subcontinuum* A_x *of* X *with* $x \in A_x \subset V^\sigma(x)$, *and if* $x, y \in C$ *and* $x \neq y$, *then*

$V^{\sigma}(x) \neq V^{\sigma}(y)$.

(3.6) Theorem. *Let* X *be a locally connected continuum* (= *Peano continuum*). *If* $f: X \to X$ *is an expansive homeomorphism* (*resp. a continuum-wise expansive homeomorphism*) *of* X, *then there is an uncountable subset* Z *of* X *such that* $Cl(Z) = X$, *and* (1) *for each* $x \in Z$ *and* $\sigma = s$ *and* u, *there is a nondegenerate subcontinuum* $A_x \in V^{\sigma}$ *with* $x \in A_x$ *and* $\text{diam } A_x \geq \delta$ *for some* $\delta > 0$, (2) *if* $x, y \in Z$ *and* $x \neq y$, *then* $W^{\sigma}(x) \neq W^{\sigma}(y)$ (*resp.* $V^{\sigma}(x) \neq V^{\sigma}(y)$) *for each* $\sigma = s$ *and* u.

To prove (3.6), we need the following.

(3.7) Lemma ([5,(1.6)]). *Let* $f: X \to X$ *be a continuum-wise expansive homeomorphism of a Peano continuum* X. *Then there is* $\delta > 0$ *such that for each* $x \in X$, *there are two subcontinua* A_x *and* B_x *such that* $x \in A_x \cap B_x$, $A_x \in V^s$, $B_x \in V^u$, $\text{diam } A_x = \delta$ and $\text{diam } B_x = \delta$. *In particular,* $\text{Int}_X(W^{\sigma}(x)) = \phi$ *for each* $x \in X$ *and* $\sigma = s, u$.

For the case of inverse limits of graphs, we have the following theorem.

(3.8) Theorem. *Let* $f: G \to G$ *be a map of a graph* G (= *finite connected* 1-*dimensional polyhedron*). *Suppose that the shift map* $\tilde{f}: (G,f) \to (G,f)$ *is expansive. Then for each* $\tilde{x} \in (G,f)$, (a) $W^u(\tilde{x})$ *is equal to the arc-component* $A(\tilde{x})$ *of* $X = (G,f)$

containing $\tilde{x}$, *and* (b) $W^s(\tilde{x})$ *is 0-dimensional.*

To prove (3.8), we need the following notations. Let A be a closed subset of a compactum X. A map f: X → X is called *positively expansive on* A if there is a positive number c > 0 such that if x, y ∈ A and x ≠ y, then there is a natural number n ≥ 0 such that $d(f^n(x), f^n(y)) > c$. If a map f: X → X is positively expansive on the total space X, we say f is *positively expansive*. Let $\mathcal{A}$ be a finite closed covering of X. A map f: X → X is *positively pseudo-expansive with respect to* $\mathcal{A}$ if the following conditions hold:

(P_1) f is positively expansive on A for each A ∈ $\mathcal{A}$.

(P_2) For each A, B ∈ $\mathcal{A}$ with A ∩ B ≠ ∅, one of the following two conditions holds: (*) f is positively expansive on A ∪ B. (**) If f is not positively expansive on A ∪ B, then there is a natural number k ≥ 1 such that for any A', A" ∈ $\mathcal{A}$ with A' ∩ A" ≠ ∅, $f^k(A' \cup A'') \cap (A - B) = \emptyset$ or $f^k(A' \cup A'') \cap (B - A) = \emptyset$.

(3.9) Theorem. *Let* G *be a graph and let* f: G → G *be an onto map. Then the shift map* $\tilde{f}$: (G,f) → (G,f) *is expansive if and only if* f *is positively pseudo-expansive map with respect to* $\mathcal{A}$, *where* $\mathcal{A}$ = {e | e *is an edge of some simplicial complex* K *with* |K| = G}.

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