Striped structures of stable and unstable sets 
of expansive homeomorphisms and a theorem 
of K. Kuratowski on independent sets

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1. Introduction.

All spaces under consideration are assumed to be metric. By a compactum, we mean a compact metric space, and by a continuum, a connected nondegenerate compactum. A homeomorphism \( f: X \to X \) of a compactum \( X \) is called \textit{expansive} if there is a constant \( c > 0 \) (called an \textit{expansive constant for} \( f \)) such that if \( x, y \in X \) and \( x \neq y \), then there is an integer \( n = n(x,y) \in \mathbb{Z} \) such that
\[
d(f^n(x), f^n(y)) > c.
\]
This property has frequent applications in topological dynamics, ergodic theory and continuum theory [1,3,7,8].

A homeomorphism \( f: X \to X \) of a compactum \( X \) is \textit{continuum-wise expansive} if there is a constant \( c > 0 \) such that if \( A \) is a nondegenerate subcontinuum of \( X \), then there is an integer \( n = n(A) \in \mathbb{Z} \) such that \( \text{diam } f^n(A) > c \). By definitions, we can easily see that every expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many important examples of homeomorphisms which are continuum-wise expansive homeomorphisms, but not expansive homeomorphisms.

In this note, we show that if \( f: X \to X \) is an expansive
homeomorphism of a compactum $X$ with $\dim X > 0$, then the decompositions $\{W^S(x) | x \in X\}$ and $\{W^U(x) | x \in X\}$ of $X$ to stable and unstable sets are uncountable respectively, and moreover there is $\sigma$ ($\sigma = s$ or $u$) and a positive number $\rho > 0$ such that the $\sigma$-striped set $Z(\sigma, \rho)$ of $f$ is not empty. Hence, by using a theorem of K. Kuratowski on independent sets [6], it is proved that almost every Cantor set $C$ of $Z(\sigma, \rho)$ satisfies the property that for each $x \in C$, $W^\sigma(x)$ contains a nondegenerate subcontinuum containing $x$ and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$. Also, we show that if $f: G \to G$ is a map of a graph $G$ and the shift map $\hat{f}: (G, f) \to (G, f)$ of $f$ is expansive, then for each $\hat{x} \in (G, f)$, $W^U(\hat{x})$ is equal to the arc-component of $(G, f)$ containing $\hat{x}$, and $W^S(\hat{x})$ is 0-dimensional.

2. Definitions and preliminaries.

Let $f: X \to X$ be a homeomorphism of a compactum $X$ and let $x \in X$. Then the stable set $W^S(x)$ and the unstable set $W^U(x)$ are defined as follows:

\[
W^S(x) = \{ y \in X | \lim_{n \to \infty} d(f^n(x), f^n(y)) = 0 \},
\]
\[
W^U(x) = \{ y \in X | \lim_{n \to \infty} d(f^{-n}(x), f^{-n}(y)) = 0 \}.
\]

Also, the continuum-wise stable and unstable sets $V^S(x)$, $V^U(x)$ are defined as follows:
$$V^S(x) = \{y \in X| \text{ there is } A \in C(X) \text{ such that } x, y \in A \text{ and } \lim_{n \to \infty} \text{ diam } f^n(A) = 0\},$$

$$V^U(x) = \{y \in X| \text{ there is } A \in C(X) \text{ such that } x, y \in A \text{ and } \lim_{n \to \infty} \text{ diam } f^{-n}(A) = 0\}.$$ 

Clearly, $W^\sigma(x) \supset V^\sigma(x)$, $\{W^\sigma(x)|x \in X\}$ and $\{V^\sigma(x)|x \in X\}$ are decompositions of $X$ for each $\sigma = s$ and $u$, i.e.,

$$X = \bigcup \{W^\sigma(x)|x \in X\} \quad \text{(resp. } X = \bigcup \{V^\sigma(x)|x \in X\}),$$

and if $W^\sigma(x) \neq W^\sigma(y)$ (resp. $V^\sigma(x) \neq V^\sigma(y)$), then $W^\sigma(x) \cap W^\sigma(y) = \emptyset$

(resp. $V^\sigma(x) \cap V^\sigma(y) = \emptyset$).

We are interested in the structures of the decompositions

$\{W^\sigma(x)|x \in X\}$ and $\{V^\sigma(x)|x \in X\}$ ($\sigma = s$ and $u$) of $X$. Let $f: X \to X$ be a homeomorphism of a compactum $X$ with $\dim X > 0$. Let $\rho > 0$ be a positive number. Consider the family $\Phi(\sigma) = \{Z| Z \text{ is a closed subset of } X \text{ satisfying that (i) for each } x \in Z \text{ there is a subcontinuum } A_x \text{ of } X \text{ such that } \text{diam } A_x \geq \rho, x \in A_x \subset W^\sigma(x), \text{ and (ii) for any neighborhood } U \text{ of } x \text{ in } X, \text{ there is } y \in Z \cap U \text{ such that } W^\sigma(x) \neq W^\sigma(y)\}$. Clearly, $\Phi(\sigma)$ has the maximal element $Z(\sigma, \rho) = \text{Cl}(\bigcup\{Z| Z \in \Phi(\sigma)\})$.

The set $Z(\sigma, \rho)$ is said to be a $\sigma$-striped set of $f$. Note that if $0 < \rho_1 < \rho_2$, then $Z(\sigma, \rho_1) \supset Z(\sigma, \rho_2)$. Also, note that if $Z(\sigma, \rho) \neq \emptyset$ for some $\rho > 0$, then $X$ contains an uncountable collection of mutually disjoint, nondegenerate subcontinua of $X$ each of which is contained in a different element of $\{W^\sigma(x)|x \in X\}$. 
Let \( f: X \to X \) be a map of a compactum \( X \) with metric \( d \).

Consider the following inverse limit space:

\[
\{(x_1)_1^\infty | x_1 \in X, f(x_{i+1}) = x_i \text{ for each } i \geq 0\},
\]

Define a metric \( \tilde{d} \) for \( (G,f) \) by

\[
\tilde{d}(\tilde{x},\tilde{y}) = \Sigma_{i=0}^\infty d(x_i,y_i)/2^i \text{ for } \tilde{x} = (x_i)_{i=0}^\infty,
\]

\[
\tilde{y} = (y_i)_{i=0}^\infty \in (X,f).
\]

The space \( (X,f) \) is called the inverse limit of the map \( f \).

Define a map \( \tilde{f}: (X,f) \to (X,f) \) by

\[
\tilde{f}(x_0,x_1,...) = (f(x_0),x_0,x_1,...), \text{ for } (x_i)_{i=0}^\infty \in (X,f).
\]

Then the map \( \tilde{f} \) is a homeomorphism and it is called the shift map of \( f \).

(2.1) Example. Let \( S^1 \) be the unit circle and let \( f: S^1 \to S^1 \) be the natural covering map with degree 2.

Consider the inverse limit \( (S^1,f) \) of \( f \) and the shift map \( \tilde{f}: (S^1,f) \to (S^1,f) \). The continuum \( (S^1,f) \) is well-known as the 2-adic solenoid and \( \tilde{f} \) is an expansive homeomorphism.

In this case, for each \( \tilde{x} \in (S^1,f) \), \( W^u(\tilde{x}) = V^u(\tilde{x}) \) is the arc-component of \( (S^1,f) \) containing \( \tilde{x} \). Also,

\[
V^s(\tilde{x}) = \{\tilde{x}\} \cap \bigcup_{\tilde{x}} W^s(\tilde{x}) \text{ for each } \tilde{x} \in (S^1,f).
\]

Then the decomposition \( \{W^s(\tilde{x}) | \tilde{x} \in (S^1,f)\} (\sigma = s \text{ and } u) \) is uncountable.
Note that \( \dim W^S(\tilde{x}) = 0 \), because \( W^S(\tilde{x}) \) is an \( F_\sigma \)-set and \( W^S(\tilde{x}) \) does not contain a nondegenerate subcontinuum. Note that the continuum \((S^1, f)\) itself is a u-striped set \( Z(\sigma, \rho) \) of \( \tilde{x} \) for some \( \rho > 0 \), but \( Z(s, \rho) = \emptyset \) for each \( \rho > 0 \).

(2.2) Example. There is an expansive homeomorphism \( f: X \to X \) such that \( \text{Int}_X W^S(x) \neq \emptyset \) for some \( x \in X \). Let \( G \) be the one point union of the unit interval \( I \) and a circle \( S^1 \), i.e., \( (G, \ast) = (I, 1) \vee (S^1, \ast) \). Define a map \( g: G \to G \) such that \( g|S^1: S^1 \to S^1 \) is the natural covering map with degree 2 and \( g(0) = 0 \), \( g(1) = \ast \) and \( g(I) = G \). We can choose \( g: G \to G \) so that \( \tilde{g}: X = (G, g) \to X = (G, g) \) is expansive. Then \( W^U(\emptyset) \) is a dense open set of \( X \), where \( \emptyset = (0, 0, \ldots) \). Hence \( X \) itself is not a u-striped set of \( \tilde{g} \).

A subset \( E \) of a space \( X \) is called to be an \( F_\sigma \)-set in \( X \) if \( E \) is a union of countable closed subsets \( F_n \) of \( X \), i.e., \( E = \bigcup_{n=1}^{\infty} F_n \). A subset \( E \) of \( X \) is called to be an \( F_{\sigma\delta} \)-set in \( X \) if \( E \) is an intersection of countable \( F_\sigma \)-sets \( E_n \), i.e., \( E = \bigcap_{n=1}^{\infty} E_n \).

We use a theorem of K. Kuratowski on independent sets [6]. A subset \( F \) of \( X \) is said to be independent in \( R \subset X^n \), if for every system \( x_1, x_2, \ldots, x_n \) of different points of \( F \) the point \((x_1, x_2, \ldots, x_n) \in F^n \) never belongs to \( R \). In [6], K. Kuratowski proved the following theorem.
(2.3) Theorem ([6, Main theorem and Corollary 3]).
If \( X \) is a complete space and and \( R \subset X^n \) is an \( F_\sigma \)-set of the first category, then the set \( J(R) \) of all compact subsets \( F \) of \( X \) independent in \( R \) is a dense \( G_\delta \)-set in \( 2^X \) of all compact subsets of \( X \). Moreover, if \( X \) has no isolated points, then almost every Cantor set of \( X \) is independent in \( R \).

For the proof of the main theorem of this note, we need the following.

(2.4) Proposition. Let \( f: X \to X \) be a homeomorphism of a compactum \( X \). Then \( W^\sigma(x) \) is an \( F_{\sigma_\delta} \)-set in \( X \) \((\sigma = s,u)\).

(2.5) Proposition. Let \( f: X \to X \) be an expansive homeomorphism of a compactum \( X \). Then \( W^\sigma(x) \) is an \( F_\sigma \)-set in \( X \) \((\sigma = s,u)\).

(2.6) Proposition. Let \( f: X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \). Then \( V^\sigma(x) \) is an \( F_\sigma \)-set in \( X \) \((\sigma = s,u)\).

In this section, we study striped structures of stable and unstable sets of expansive homeomorphisms and continuum-wise expansive homeomorphisms. The main result of this section is the following theorem.
(3.1) Theorem. Let $f: X \to X$ be an expansive homeomorphism of a compactum $X$ with $\dim X > 0$. Then the decomposition $\{W^\sigma(x) | x \in X\}$ ($\sigma = s$ and $u$) of $X$ is uncountable. Moreover, there exists $\sigma$ ($\sigma = s$ or $u$) and a positive number $\rho > 0$ such that the $\sigma$-striped set $Z(\sigma, \rho)$ is not empty. In particular, almost every Cantor set $C$ of $Z(\sigma, \rho)$ satisfies the property that for any $x \in C$, there exists a nondegenerate subcontinuum $A_x$ of $X$ such that $x \in A_x \subset W^\sigma(x)$, and if $x, y \in C$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$.

To prove (3.1), we need the following facts. The next lemma is obvious.

(3.2) Lemma. Let $f: X \to X$ be a map of a compactum $X$ and let $N \geq 1$ be a natural number. Suppose that there is $\gamma > 0$ such that $d(f_1^{1N}(x), f_1^{1N}(y)) \geq \gamma$ for each $i = 0, 1, 2, \ldots$. Then there is a positive number $\eta > 0$ such that $d(f_1^1(x), f_1^1(y)) > \eta$ for each $i = 0, 1, 2, \ldots$.

(3.3) Lemma ([4, (2.3)]). Let $f: X \to X$ be a continuum-wise expansive homeomorphism of a compactum $X$ with an expansive constant $c > 0$ and let $0 < \varepsilon < c/2$. Then there is $\delta > 0$ such that if $A$ is any nondegenerate subcontinuum of $X$ such that $\text{diam} A \leq \delta$ and $\text{diam} f^m(A) \geq \varepsilon$ for some integer $m \in \mathbb{Z}$, then one of the following conditions holds:
(a) If \( m \geq 0 \), then \( \text{diam } f^n(A) \geq \varepsilon \) for each \( n \geq m \).

More precisely, there is a subcontinuum \( B \) of \( A \) such that \( \text{diam } f^j(B) \leq \varepsilon \) for \( 0 \leq j \leq n \) and \( \text{diam } f^n(B) = \varepsilon \).

(b) If \( m < 0 \), then \( \text{diam } f^{-n}(A) \geq \varepsilon \) for each \( n \geq -m \).

More precisely, there is a subcontinuum \( B \) of \( A \) such that \( \text{diam } f^{-j}(B) \leq \varepsilon \) for \( 0 \leq j \leq n \) and \( \text{diam } f^{-n}(B) = \varepsilon \).

(3.4) Lemma ([4,(2.4)]). Let \( f, c, \varepsilon, \delta \) be as in (3.3). Then for any \( \gamma > 0 \), there is \( N > 0 \) such that if \( A \in C(X) \) and \( \text{diam } A \geq \gamma \), then \( \text{diam } f^n(A) \geq \delta \) for each \( n \geq N \) or \( \text{diam } f^{-n}(A) \geq \delta \) for each \( n \geq N \).

For the case of continuum-wise expansive homeomorphism, we have

(3.5) Theorem. Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Then the decompositions \( \{V^\sigma(x) | x \in X \} \) (\( \sigma = s \) and \( u \)) are uncountable. Moreover, there is \( \sigma (\sigma = s \) or \( u \)) and a positive number \( \rho > 0 \) such that there is a nonempty closed set \( Z' \) of \( X \) satisfying that (i) for each \( x \in Z' \) there is a subcontinuum \( A_x \) of \( X \) satisfying that \( \text{diam } A_x \geq \rho \), \( x \in A_x \subset V^\sigma(x) \), (ii) for any neighborhood \( U \) of \( x \) in \( X \), there is \( y \in Z' \cap U \) such that \( V^\sigma(x) \neq V^\sigma(y) \). In particular, almost every Cantor set \( C \) of \( Z(\sigma) \) satisfies the property that for any \( x \in C \), there is a nondegenerate subcontinuum \( A_x \) of \( X \) with \( x \in A_x \subset V^\sigma(x) \), and if \( x, y \in C \) and \( x \neq y \), then
$V^\sigma(x) \neq V^\sigma(y)$.

(3.6) Theorem. Let $X$ be a locally connected continuum (= Peano continuum). If $f: X \to X$ is an expansive homeomorphism (resp. a continuum-wise expansive homeomorphism) of $X$, then there is an uncountable subset $Z$ of $X$ such that $\text{Cl}(Z) = X$, and (1) for each $x \in Z$ and $\sigma = s$ and $u$, there is a nondegenerate subcontinuum $A_x \in V^\sigma$ with $x \in A_x$ and $\text{diam } A_x \geq \delta$ for some $\delta > 0$, (2) if $x, y \in Z$ and $x \neq y$, then $W^\sigma(x) \neq W^\sigma(y)$ (resp. $V^\sigma(x) \neq V^\sigma(y)$) for each $\sigma = s$ and $u$.

To prove (3.6), we need the following.

(3.7) Lemma ([5,(1.6)]). Let $f: X \to X$ be a continuum-wise expansive homeomorphism of a Peano continuum $X$. Then there is $\delta > 0$ such that for each $x \in X$, there are two subcontinua $A_x$ and $B_x$ such that $x \in A_x \cap B_x$, $A_x \in V^S$, $B_x \in V^U$, $\text{diam } A_x = \delta$ and $\text{diam } B_x = \delta$. In particular, $\text{Int}_X(W^\sigma(x)) = \emptyset$ for each $x \in X$ and $\sigma = s, u$.

For the case of inverse limits of graphs, we have the following theorem.

(3.8) Theorem. Let $f: G \to G$ be a map of a graph $G$ (= finite connected 1-dimensional polyhedron). Suppose that the shift map $T: (G,f) \to (G,f)$ is expansive. Then for each $\tilde{x} \in (G,f)$, (a) $W^U(\tilde{x})$ is equal to the arc-component $A(\tilde{x})$ of $\pi_1(G,f)$
containing $\bar{x}$, and (b) $W^S(\bar{x})$ is 0-dimensional.

To prove (3.8), we need the following notations.
Let $A$ be a closed subset of a compactum $X$. A map $f: X \to X$ is called positively expansive on $A$ if there is a positive number $c > 0$ such that if $x, y \in A$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > c$. If a map $f: X \to X$ is positively expansive on the total space $X$, we say $f$ is positively expansive. Let $\mathcal{A}$ be a finite closed covering of $X$. A map $f: X \to X$ is positively pseudo-expansive with respect to $\mathcal{A}$ if the following conditions hold:

$(P_1)$ $f$ is positively expansive on $A$ for each $A \in \mathcal{A}$.
$(P_2)$ For each $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$, one of the following two conditions holds: $(*)$ $f$ is positively expansive on $A \cup B$. $(**)$ If $f$ is not positively expansive on $A \cup B$, then there is a natural number $k \geq 1$ such that for any $A', A'' \in \mathcal{A}$ with $A' \cap A'' \neq \emptyset$, $f^k(A' \cup A'') \cap (A - B) = \emptyset$ or $f^k(A' \cup A'') \cap (B - A) = \emptyset$.

(3.9) Theorem. Let $G$ be a graph and let $f: G \to G$ be an onto map. Then the shift map $\bar{f}: (G,f) \to (G,f)$ is expansive if and only if $f$ is positively pseudo-expansive map with respect to $\mathcal{A}$, where $\mathcal{A} = \{e| e$ is an edge of some simplicial complex $K$ with $|K| = G\}$. 
References


