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On Global Weak Solutions of the Nostationary Two-phase Navier-Stokes flow

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A global weak solution of the nonstationary two-phase Navier-Stokes flow is constructed for arbitrary given initial phase configuration. Our solution tracks the evolution of the interface after it develops singularities. The theory of viscosity solutions is adapted to solves the interface equation. Surface tension effects are ignored here.

1. Introduction

This paper studies the dynamics of the interface (free boundary) of two immiscible incompressible viscous fluids with same constant density, say one, in a smoothly bounded domain. Each fluid velocity satisfies the Navier-Stokes equations with different viscosities. The interface is assumed to move with the fluid velocities. No surface tension on the interface is considered in this paper. The interface is also assumed to intersect the boundary of the domain perpendicularly. We impose nonzero fluid velocity on the boundary to consider the dynamics of the interface not only in the interior of the domain but also up to the boundary.

Let $\nu_\pm$ be the viscosities of each fluid. Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ with smooth boundary $\partial\Omega$ (at least $\partial\Omega \in C^{2+\mu}, 0 < \mu < 1$) and let $\Omega_\pm(t) \subset \Omega$ be the disjoint open sets occupied with the fluids of viscosities $\nu_\pm$ at time t, respectively. The velocities $u_\pm = u_\pm(t,x)$ and the pressures $\pi_\pm = \pi_\pm(t,x)$ of fluids of viscosities $\nu_\pm$ are assumed to satisfy the incompressible Navier-Stokes system:

$$\partial_t u_\pm - \nu_\pm \Delta u_\pm + (u_\pm \cdot \nabla) u_\pm + \nabla \pi_\pm = 0, \qquad \text{in} \quad (0,T) \times \Omega_\pm(t), \tag{1.1}$$

$$\nabla \cdot u_\pm = 0, \qquad \text{in} \quad (0,T) \times \Omega_\pm(t). \tag{1.2}$$

The complement of the union of $\Omega_+(t)$ and $\Omega_-(t)$ is called the interface and denoted by $\Gamma(t)$. To write down the interface equation we assume that the interface $\Gamma(t)$ is a smooth hypersurface so that $\Gamma(t)$ is the boundary between $\Omega_+(t)$ and $\Omega_-(t)$. We first impose on the interface

$$u_+ = u_-, \qquad \text{on} \quad \Gamma(t), \tag{1.3}$$

$$T_+(u_+, \pi_+) \cdot \mathbf{n} = T_-(u_-, \pi_-) \cdot \mathbf{n}, \qquad \text{on} \quad \Gamma(t), \tag{1.4}$$

where $\mathbf{n}$ denotes the unit normal vector from $\Omega_+(t)$ to $\Omega_-(t)$ and $T_\pm(u_\pm, \pi_\pm) := \nu_\pm D(u_\pm) - \pi_\pm I$ denote the stress tensors with the strain tensor

$$D(u) = (D_{k\ell}(u)) := \frac{\partial u^k}{\partial u^\ell} + \frac{\partial u^\ell}{\partial u^k}.$$

The dynamics of the interface is assumed to be determined by the motion of the fluids with prependicular cross condition on the boundary $\partial\Omega$. Let $V = V(t,x)$ denote the speed of $\Gamma(t)$ at $x \in \Gamma(t)$ in the direction $\mathbf{n}$. Let γ be the unit normal vector field on the boundary $\partial\Omega$. We consider the interface equations for $\Gamma(t)$:

$$V = u_+ \cdot \mathbf{n} \quad \text{on} \quad \Gamma(t) \quad \text{with initial data} \quad \Omega_\pm(0) = \Omega_{\pm 0}, \tag{1.5}$$

$$\gamma \cdot \mathbf{n} = 0, \quad \text{on} \quad \partial\Omega \cap \overline{\Gamma(t)}. \tag{1.6}$$

The above equations (1.5)-(1.6) imply that the interface on the boundary $\partial\Omega$ is transported only by the tangential component of the velocity on $\partial\Omega$. Since we consider the motion of the interface also on the boundary, we impose nonzero Dirichlet condition:

$$u_\pm = h_\pm \qquad \text{on} \quad \partial\Omega_\pm(t) \cap \partial\Omega. \tag{1.7}$$

The initial velocities are assumed to be zero for simlicity:

$$u_\pm(0,x) = 0 \quad \text{in} \quad \Omega_\pm(0). \tag{1.8}$$

Here $0 < \nu_- < \nu_+ < \infty$ and $0 < T \leqslant \infty$.

Our goal is to construct global weak solutions of (1.1)-(1.8) for arbitrary given initial phase configuration $\Omega_{\pm 0}$ when ν_+ and ν_- are close and $h_\pm$ is smooth and small. To construct global solutions we have to overcome an intrinsic difficulty that the interface may develops singularities in a finite time. Giga and Takahashi [GT] first constructed a global solution for the two-phase Stokes system with periodic boundary conditions when ν_+ and ν_- are close. We improve their arguments.

We first introduce a weak formulation of the transport equation (1.5)-(1.6). Since the interface $\Gamma(t)$ may not be regular, we consider a generalized evolution of (1.5)-(1.6) through a level set of an auxiliary function. Although such a generalized evolution for (1.5) is constructed on a torus by [GT], we consider also the boundary condition (1.4). Since our velocity field u is merely continuous, one cannot expect the uniqueness of transpotation. Also the Lebesgue measure of the zero-level set may be positive, so our interface may be thick.

We next introduce a step function ν to give a weak formulation of (1.1)-(1.4) with (1.7)-(1.8). The region occupied with high (resp. low) viscous fluid corresponds to the phace where ν takes the value ν_+(resp. ν_-). The interface corresponds to the jump discontinuity of ν. The velocity u is defined by $u = u_\pm$ on $\Omega_\pm$ and the pressure π is defined in the same manner. The system (1.1) and (1.4) is formally equivalent to

$$u_t - \nabla \cdot (\nu D(u)) + u \cdot \nabla u + \nabla \pi = 0, \quad \text{in} \quad (0,T) \times \Omega \tag{1.9}$$

(cf. [GT, Introduction]).

To construct a solution we seek a fixed point of the mapping difined as follows. For a continuous function v we solve (1.5)-(1.6) with $u_+ = v$ and construct generalized evolutions $\Omega_\pm^v$. Let $\nu = \nu_v$ be a step function defined by $\nu = \nu_\pm$ on $\Omega_\pm^v$ and $\nu = (\nu_+ + \nu_-)/2$ outside $\Omega_\pm^v$. We next solve (1.9) with $\nabla \cdot u = 0, u|_{\partial\Omega} = h$ and $u|_{t=0} = 0$, and obtain a mapping $S : v \mapsto u$. Since S may not be continuous, unfortunately, Leray-Schauder's fixed point theory does not apply. We extend mapping S to a set-valued mapping introduced by [GT] so that we can apply Kakutani's fixed point theory. Although a solution obtained in such a way no longer satisfies (1.9) in the whole of $(0,T) \times \Omega$, we can verify it satisfies (1.9) outside the interface.

To apply Kakutani's theory we need a compactness which follows from a priori L^p estimates. We first vanish the boundary value $u|_{\partial\Omega} = h$ with preserving divergence free. We next apply a priori estimates of the Stokes system obtained by M. Giga, Y. Giga and H. Sohr.

2. Interface Equations

This section establishes a (global-in-time) generalized evolution of interface equations. Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ with $\partial\Omega \in C^2$. Let $\Omega_\pm$ be disjoint open sets in $M = [0,\infty) \times \Omega$ and let Γ denote the complement of the union of Ω_+ and Ω_- in M. Physically, $\Gamma(t)$ is called an interface at time t bounding two phases $\Omega_\pm(t)$. Here $W(t)$ denotes the cross-section of $W \subset M$ at time t, i.e., $W(t) = \{x \in \Omega; (t,x) \in W\}$. Suppose that $\Gamma(t)$ is a smooth hypersurface, $\mathbf{n}$ is the unit normal vector field from $\Omega_+(t)$ to $\Omega_-(t)$ and that γ is the unit inner normal vector field on the boundary $\partial\Omega$ of Ω. Let $V = V(t,x)$ denote the speed of $\Gamma(t)$ at $x \in \Gamma(t)$ in the direction $\mathbf{n}$. Suppose that $u : \overline{Q} \to \mathbb{R}^n$ is a continuous vector field, i.e., $u \in C(\overline{Q})$ where $Q = (0,T) \times \Omega$ $(0 < T \leqslant \infty)$ and that $\overline{Q}$ denotes the closure of Q in M. Here and hereafter we do not distinguish the space of real, vector or tensor valued functions. The equation for $\Gamma(t)$ we consider here is

$$V = u \cdot \mathbf{n}, \quad \text{on} \quad \Gamma(t), \tag{2.1}$$

$$\mathbf{n} \cdot \gamma = 0, \quad \text{on} \quad \overline{\Gamma(t)} \cap \partial\Omega, \tag{2.2}$$

where denotes the standard inner product in $\mathbb{R}^n$ and $\overline{\Gamma(t)}$ denotes the closure of $\Gamma(t)$ in $\mathbb{R}^n$.

Classical solutions of (2.1)-(2.2) may not exist even for a short time for merely continuous vector field u and they are not uniquely determined by initial data even if they exist. So we consider largest and smallest solutions and show uniqueness of them.

Let $u \in C(\overline{Q})$ and a $\in C(\overline{\Omega})$. We say $\psi : Q \to \mathbb{R}$ is a *subsolution* of

$$\psi_t + (u \cdot \nabla)\psi = 0, \qquad \text{in} \quad Q, \tag{2.3}$$

$$\partial\psi/\partial\gamma = 0, \qquad \text{on} \quad \partial\Omega \tag{2.4}$$

$$\psi(0, x) = a(x), \tag{2.5}$$

if ψ is a viscosity subsolution of (2.3)-(2.4) on Q and $\psi_*(0, x) = a(x)$, where ψ_* denotes the lower semicontinuous envelope of ψ. We say ψ is a *supersolution* of (2.3)-(2.5) if $-\psi$ is a subsolution of (2.3)-(2.4) and $(-\psi)_*(0, x) = -a(x)$. We simply say ψ is a *solution* of (2.3)-(2.5) if ψ is both super- and subsolution. For general theory of viscosity solution see [CIL].

Lions [L] and Giga and Sato [GS] established comparison theorems on solutions of the Neumann problem (2.3)-(2.5) provided that u is uniformly Lipschitz continuous. However, for general $u \in C(\overline{Q})$ there is no uniqueness of solutions of (2.3)-(2.5).

Lions [L] and Sato [S, Proposition 3.10] constructed the existence of solutions of the Neumann problem (2.3)-(2.5) even on a nonconvex domain. The following lemmas and theorems are parallel results to those in [GT] and can be proved in the same manner with applying the above results in [L], [GS] and [S]. To prove them we note that for viscosity sub-(super)solution ψ of (2.3)-(2.4) there exists $\zeta \in C^2(Q)$

satisfying

(2.6) $$\max_Q(\psi-\zeta)=(\psi-\zeta)(t_0,x_0) \quad \text{for some} \quad (t_0,x_0)\in Q,$$

(2.7) $$\zeta_t+(u\cdot\nabla)\zeta \leqslant 0 \quad \text{at} \quad (t_0,x_0),$$
$$(\text{resp. } \geqslant)$$

(2.8) $$\gamma(x)\cdot\nabla\zeta(x)\leqslant 0 \quad \text{for} \quad x\in\partial\Omega,$$
$$(\text{resp. } \geqslant)$$

while ζ does not have to satisfy (2.8) in order that ψ is a sub-(super)solution of only (2.3).

So we state the followings without proofs. We say solution λ (resp. γ) of (2.3)-(2.5) is *largest* (resp. *smallest*) if $\lambda \geqslant \psi$ (resp. $\sigma \leqslant \psi$) for all other solutions ψ of (2.3)-(2.5).

Lemma. *2.1. Suppose that $u\in C(\overline{Q})$ and $a\in C(\overline{\Omega})$. There are unique largest and smallest solutions λ and σ of (2.3)-(2.5) which are bounded on every compact set in $\overline{Q}$. Moreover, λ and σ are expressed as*

(2.9) $$\lambda(t,x)=\sup\{\psi(t,x);\ \text{is a subsolution of (2.3)-(2.5)}\},$$

(2.10) $$\sigma(t,x)=\inf\{\psi(t,x);\psi\ \text{is a supersolution of (2.3)-(2.5)}\}.$$

Lemma. *2.2 (Uniqueness of level sets). The set*

(2.11) $$\Omega_+=\{(t,x)\in[0,T)\times\overline{\Omega};\ \sigma_*(t,x)>0\}$$

(2.12) $$(\textit{resp. } \Omega_-=\{(t,x)\in[0,T)\times\overline{\Omega};\ \lambda^*(t,x)<0\})$$

is completely determined by the initial data $\Omega_+(0)$ (resp. $\Omega_-(0)$) and u, and is independent of choice of a defining $\Omega_\pm(0)=\{x\in\Omega;\ a(x)\gtrless 0\}$. Here $\lambda^*=-(-\lambda)_*$.

For the proof of Lemma 2.2 see [GT, Lemma 2.3] and [S, Proposition 3.6]. We call Ω_+ (resp. Ω_-) $+$ (resp. $-$)*generalized evolution with speed $u\in C(\overline{Q})$ and initial data* $\Omega_+(0)$ (resp. $\Omega_-(0)$) on $[0,T)$. For any open set Ω_{+0} (resp. Ω_{-0}) in Ω and any $u\in C(\overline{Q})$, there exists a unique $+$ (resp. $-$) generalized evolution with speed u and initial data $\Omega_\pm(0)=\Omega_{\pm 0}$.

Theorem. *2.3 (Stability). Suppose that $T < \infty$ and $u_j \to u$ in $C(\overline{Q})$ as $j \to \infty$. Let Ω_{+j} be the + generalized evolution with speed $u_j \in C(\overline{Q})$ and initial data $\Omega_{+j}(0) = \Omega_{+0}$ on $[0,T)$ for $j = 1, 2, \cdots$. Let Ω_+ be the + generalized evolution on $[0,T)$ with u and $\Omega_+(0) = \Omega_{+0}$. Let K be a compact set in Ω_+. Then K is also contained in Ω_{+j} for sufficiently large j. The same holds for $-$ evolution.*

Remark 2.4. . If we can construst a global solution of (2.3) with nonzero Neumann condition $\partial\psi/\partial\gamma = b(t,x)$ for some givenfunction b, we can consider the motion of the interface on the boundary with the non-slip condition.

3. Global Existence of Weak Solutions

We introduce a weak formulation of the problem (1.1)-(1.4) with (1.7). Let $\Omega_\pm$ be two disjoint open sets in $[0,T) \times \Omega$. Let ν be a step function such that $\nu = \nu_\pm$ in $\Omega_\pm$ and $\nu = (\nu_+ + \nu_-)/2$ outside $\Omega_+ \cup \Omega_-$, where $0 < \nu_- < \nu_+$. Let $h \in C([0,T] \times \partial\Omega)$ satisfy $h = h_\pm$ in $\overline{\Omega_\pm} \cap \partial\Omega$ and

$$(3.1) \qquad h(0,x) = 0 \quad \text{on} \quad \partial\Omega, \;\; h(t,x)\cdot\gamma(x) = 0 \quad \text{on } [0,T] \times \partial\Omega.$$

Here γ is the inner unit normal vector on the boundary $\partial\Omega$. We say u is a *weak solution* of (1.1)-(1.4) with (1.7) for $\Omega_\pm$ in $Q = (0,T) \times \Omega$ if $u \in C(\overline{Q})$ with $\nabla u \in L^q(Q)$ (for some $1 < q < \infty$) and it solves

$$(3.2) \qquad \begin{cases} u_t - \nabla\cdot(\nu D(u)) + u\cdot\nabla u + \nabla\pi = \nabla\cdot\zeta, & \text{in} \quad Q, \\ \nabla\cdot u = 0, & \text{in} \quad Q, \\ u|_{\partial\Omega} = h, & \end{cases}$$

in the sense of distribution with some π and some tensor field ζ whose support spt ζ is contained in $\Gamma = \overline{Q}\backslash(\Omega_+ \cup \Omega_-)$.

If the Lebesgue measure of the interface Γ is zero, then (3.2) yields (1.1)-(1.2) and (1.7) by interpreting $u = u_\pm$ in $\Omega_\pm$. If $\{\Gamma(t)\}_{t\ 0}$ is a smooth family of smooth hypersufaces, (1.4) is contained in (3.2). The condition (1.3) is automatic since $u \in C(\overline{Q})$.

We now state our main result in this paper.

Theorem. *3.1. Let $p > 2(n+1)$. Let Ω be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ with $\partial\Omega \in C^{2+\mu}(0 < \mu < 1)$. Assume that $\Omega_{\pm 0}$ are two disjoint open sets in Ω and that $h \in C([0,T] \times \partial\Omega)$ satisfies (3.1). Then there exists a positivee constant $\varepsilon = \varepsilon(n, p, \Omega)$ such that if*

$$(3.3) \quad \frac{1}{\nu_+}\left\{\frac{||\nabla h_t + h_t||_{r_0}}{\nu_+} + ||(\nabla^2 + \nabla)h||_{\frac{n+2}{2}} + ||\nabla h + h||_{n+2} + (\nu_+ - \nu_-)\right\} < \varepsilon$$

with $\frac{1}{r_0} = \frac{1}{n} + \frac{2}{n+2}$, then there exist $u \in C(\overline{Q})$ with $\nabla u \in L^p(Q)$ and $\Omega_\pm \subset \overline{Q}$ such that u is a weak solution of (1.1)-(1.4) with (1.7)-(1.8) for $\Omega_\pm$ and that $\Omega_\pm$ are generalized evolutions with the speed u and initial data $\Omega_{\pm 0}$. Moreover, ζ in (3.2) can be taken as an element of $L^p((0,T_0) \times \Omega)$ for all finite $T_0 \leqslant T$. Here T is allowed to be infinite.

Here we simply denote $|| \;||_p = || \;||_{L^p(Q)}$.

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