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<th>Title</th>
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</tr>
</thead>
<tbody>
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Kyoto University
ON THE INSTABILITY OF THE UNIFORM 
ROTATION OF A BODY WITH LIQUID INSIDE

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1. Introduction

The present work deals with the stability of a uniformly rotating body with a cavity filled with an incompressible viscous fluid. Various statements of the problem of the motion of a rigid body with a cavity filled with fluid have been discussed in the literature [1-5]. In the present work we shall follow [4-5].

Given a rigid body $G$ with a cavity $\Omega$ entirely filled with an incompressible viscous liquid. Let $\bar{O}$ be the center of mass of the entire system "body+liquid", $Ox_1x_2x_3$ be an orthogonal system of coordinates rigidly connected with the body. We assume that the undisturbed motion of the system relative to the point $\bar{O}$ be a uniform rotation of whole system with constant angular velocity $\omega_0e_3$ around the axis $Ox_3$. We shall examine the perturbed motion assuming that its deviations from the unperturbed one are small. Then in the rotating system of coordinates $Ox_1x_2x_3$ the linearization of the Navier-Stokes equations and the equations of moments with respect to the point $\bar{O}$ can be written as follows [4-5]

$$\frac{\partial u}{\partial t} + 2\omega_0e_3 \times u + \frac{d\mathbf{w}}{dt} \times \mathbf{r} = -\frac{1}{\rho_l} \nabla p + \nu \Delta u + f$$

(2)

$$\text{div } \mathbf{u} = 0$$

(3) $\mathbf{J} \cdot \frac{d\mathbf{w}}{dt} + \omega_0 \mathbf{w} \times \mathbf{J} \cdot e_3 + \omega_0 e_3 \times \mathbf{J} \cdot \mathbf{w} + \rho_l \int_{\Omega} \mathbf{r} \times \frac{\partial \mathbf{u}}{\partial t} d\Omega + \omega_0 e_3 \times \left( \rho_l \int_{\Omega} \mathbf{r} \times \mathbf{u} d\Omega \right) = \mathbf{M}$

where

$$\mathbf{J} \cdot \mathbf{w} = \rho_l \int_{\Omega} \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) d\Omega + \rho_b \int_{\bar{G}} \mathbf{r} \times (\mathbf{w} \times \mathbf{r}) d\bar{G}$$

is the inertia tensor of the entire system with respect to the point $\bar{O}$. Here $t$ denotes time; $\mathbf{r} = (x_1, x_2, x_3)$; $\rho_l, \rho_b$ are the constant densities of the liquid and the body; $\nu$ is the kinematic viscosity of the liquid; $\mathbf{u}$ is a relative velocity of the liquid; $\mathbf{w}$ is a relative
angular velocity of the body; \( p \) is a relative pressure; \( \mathbf{M}(t) \) is a central moment of exterior forces with respect to \( O \); \( \mathbf{f}(t, x) \) is a field of external force.

To complete the system \((1)-(3)\) we consider the following boundary and initial conditions

\[
\mathbf{u}|_{\partial\Omega} = 0
\]

\[
\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \mathbf{w}(0) = \mathbf{w}_0
\]

As in [5] the closure in the norm of \( L_2(\Omega) \) of the set of all smooth solenoidal fields \( \mathbf{u} \), satisfying the condition \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( \partial\Omega \), will be referred to as the Hilbert space \( J_0(\Omega) \).

Then equations \((1),(2),(4)\) can be rewritten in the operator form \([4-5]\)

\[
\frac{d\mathbf{u}}{dt} + \mathbf{P}_0 \left( \frac{d\mathbf{w}}{dt} \times \mathbf{r} \right) + \nu \mathbf{A} \mathbf{u} + 2i\rho_0 \mathbf{T}\mathbf{u} = \mathbf{P}_0 \mathbf{f}
\]

where \( \mathbf{P}_0 \) is the operator of the orthogonal projection of space \( L_2(\Omega) \) onto \( J_0(\Omega) \),

\[
\mathbf{T}\mathbf{u} = i\mathbf{P}_0 (\mathbf{u} \times \mathbf{e}_3), \quad \mathbf{T} = \mathbf{T}^*, \quad \sigma(\mathbf{T}) = [-1,1]
\]

and \( \mathbf{A} \) is the Friedrichs extension \([5]\) of the following symmetric operator

\[
\mathbf{A}_0 = -\mathbf{P}_0 \Delta, \quad D(\mathbf{A}_0) = \{ \mathbf{u} \in W_2^2(\Omega) \mid \text{div} \mathbf{u} = 0; \mathbf{u}|_{\partial\Omega} = 0 \}
\]

The operator \( \mathbf{A} \) is usually called Stokes operator \([5]\).

Following \([5]\) we denote

\[
\mathbf{v}(t) = (\mathbf{u}(t, x), \mathbf{w}(t)) \in \mathbf{H} = J_0(\Omega) \oplus \mathbb{R}^3
\]

Then in the Hilbert space \( \mathbf{H} \) system \((1)-(5)\) can be written in the form of abstract Cauchy problem \([5]\)

\[
\frac{d}{dt} \mathbf{N}\mathbf{v} + (\mathbf{C} + \mathbf{B})\mathbf{v} = \mathbf{g}(t) \quad \mathbf{g}(t) = ((\mathbf{P}_0 \mathbf{f})(t), \mathbf{M}(t))
\]

\[
\mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{v}_0 = (\mathbf{u}_0(x), \mathbf{w}_0)
\]

where

\[
\mathbf{N}\mathbf{v} = \left( \mathbf{u} + \mathbf{P}_0 (\mathbf{w} \times \mathbf{r}), \rho_1 \int_\Omega \mathbf{r} \times \mathbf{u}\,d\Omega + \mathbf{J} \cdot \mathbf{w} \right)
\]

\[
\mathbf{C} = \text{diag}(\nu \mathbf{A}, \mathbf{I})
\]

\[
\mathbf{B}\mathbf{v} = \left( 2i\rho_0 \mathbf{T}\mathbf{u}, \mathbf{w}_0 \mathbf{e}_3 \times \left( \rho_1 \int_\Omega \mathbf{r} \times \mathbf{u}\,d\Omega \right) + \mathbf{w}_0 (\mathbf{e}_3 \times \mathbf{J} \cdot \mathbf{w} + \mathbf{w} \times \mathbf{J} \cdot \mathbf{e}_3) - \mathbf{w} \right)
\]
It is shown in [5] that if \( g(t) \) satisfies Hölder condition then for any \( v_0 \in H \) there exists unique solution \( v(t) \in H \) of the problem (9),(10).

Consider the spectral problem corresponding to the operator equation (9). Let \( v(t) = \exp(-\lambda t)V, \ V \in H ; g(t) \equiv 0. \) Then \( V \) should satisfy

\[
(C + B)V = \lambda NV, \quad V \in H
\]

Using the properties of the operators \( C, B, N \) it has been proved in [5] that:

1. Problem (11) has discrete spectrum \( \{\lambda_j\}_{j=1}^{\infty} \). Each eigenvalue \( \lambda_j, j \in \mathbb{N} \) has finite multiplicity.
2. There exist positive constants \( C_1, C_2 > 0 \) such that

\[
\Re \lambda_j \geq -C_1 \quad ; \quad |\Im \lambda_j| \leq C_2, \quad j \in \mathbb{N}
\]

3. The sequence of eigenfunctions and associated functions corresponding to the eigenvalues \( \{\lambda_j\}_{j=1}^{\infty} \) is complete in \( H \).

In the present paper we shall consider the following problem: under what conditions there exists an eigenvalue \( \lambda_{j_0} \) such that

\[
\Re \lambda_{j_0} < 0
\]

If there exists \( \lambda_{j_0} \) satisfying (13) then using results of [6] it can be shown that the uniform rotation with the fixed angular velocity \( w_0 e_3 \) of the body \( G \) with the fluid-filled cavity \( \Omega \) is unstable.

### 2. Symmetry assumption

We assume that the boundary \( \partial \Omega \) is smooth enough and domains \( G, \Omega \) are invariant with respect to the turn to the angle \( \pi \) about the axis \( Ox_3 \). Exactly speaking, it means that the following symmetry condition holds:

\[
(x_1, x_2, x_3) \in \Omega \iff (-x_2, x_1, x_3) \in \Omega
\]

\[
(x_1, x_2, x_3) \in G \iff (-x_2, x_1, x_3) \in G
\]

Consider the inertia tensor \( J \). From (S) it follows that for any \( w = (w_1, w_2, w_3) \)

\[
J \cdot w = (a_0 w_1, a_0 w_2, b_0 w_3) = \text{diag}(a_0, a_0, b_0) \cdot w
\]

where

\[
a_0 = \rho \iota \int_{\Omega} (x_1^2 + x_3^2) dx_1 dx_2 dx_3 + \rho_b \int_{G} (x_1^2 + x_3^2) dx_1 dx_2 dx_3
\]

\[
b_0 = 2 \rho \iota \int_{\Omega} x_1^2 dx_1 dx_2 dx_3 + 2 \rho_b \int_{G} x_1^2 dx_1 dx_2 dx_3
\]
Denote
\[c_0 = b_0 - a_0 = \rho_1 \int_{\Omega} (x_1^2 - x_3^2) \, dx_1 \, dx_2 \, dx_3 + \rho_2 \int_{G} (x_1^2 - x_3^2) \, dx_1 \, dx_2 \, dx_3\]

It is easy to check that if \(\lambda = 0\) then \((u, w) \equiv \alpha(0, e_3)\) satisfies (11) for any \(\alpha \in \mathbb{C}\). Therefore \(\lambda = 0\) is eigenvalue of the spectral problem (11). Let \(\lambda \neq 0\). Then (11) can be rewritten as follows [7]:
\[
\nu' Au + 2i Tu + m \gamma \left[ \frac{\gamma + i}{\gamma + i k} P_1 u + \frac{\gamma - i}{\gamma - i k} P_2 u + d P_3 u \right] = \gamma u
\]
where
\[
\gamma = \frac{\lambda}{\omega_0}, \quad k = -\frac{c_0}{a_0}, \quad \nu' = \frac{\nu}{w_0}, \quad m = \frac{\rho_1 ||c_1||_{L_2(\Omega)}^2}{2a_0}, \quad d = \frac{a_0}{b_0} \frac{||c_3||_{L_2(\Omega)}^2}{||c_1||_{L_2(\Omega)}^2}
\]

It is easy to check that the following inequalities hold [7]
\[
-1 < k < 1, \quad 0 < m < 1, \quad 0 < md < 1
\]
Since \(\omega_0 > 0\) then the spectral problem (11) has an eigenvalue \(\lambda \in \mathbb{C}^- = \{ \omega \in \mathbb{C} | \text{Re} \, \omega < 0 \}\) if and only if (18) has an eigenvalue \(\gamma \in \mathbb{C}^-\).

### 3. Necessary condition

Let \(\gamma \in \mathbb{C}^-\) be an eigenvalue of (69), \(u_\gamma\) be a corresponding eigenfunction. Then (18) implies
\[
\nu'(Au_\gamma, u_\gamma)_{L_2} + 2i(Tu_\gamma, u_\gamma)_{L_2} + m(1 - k) \left[ \frac{\gamma k + i \gamma^2}{\gamma^2 + k^2} ||P_1 u_\gamma||_{L_2(\Omega)}^2 + \frac{\gamma k - i \gamma^2}{\gamma^2 + k^2} ||P_2 u_\gamma||_{L_2(\Omega)}^2 \right]
\]
\[
= \gamma \left( ||u_\gamma||_{L_2(\Omega)}^2 - m ||P_1 u_\gamma||_{L_2(\Omega)}^2 - m ||P_2 u_\gamma||_{L_2(\Omega)}^2 - md ||P_3 u_\gamma||_{L_2(\Omega)}^2 \right)
\]
Since \(A \gg 0, T = T^*, \nu' > 0\) then using (22) and Lemma 1 we obtain
\[
\nu'(Au_\gamma, u_\gamma)_{L_2} + m(1 - k) \text{Re} \gamma \left[ \frac{1}{||P_1 u_\gamma||_{L_2}^2} + \frac{1}{||P_2 u_\gamma||_{L_2}^2} \frac{1}{||P_3 u_\gamma||_{L_2}^2} \right]
\]
\[
= \text{Re} \gamma \left( ||u_\gamma||_{L_2(\Omega)}^2 - m ||P_1 u_\gamma||_{L_2(\Omega)}^2 - m ||P_2 u_\gamma||_{L_2(\Omega)}^2 \right)
\]
\[
\leq \text{Re} \gamma (1 - \max(m, md)) ||u_\gamma||_{L_2(\Omega)}^2 < 0
\]
As far as \(k \in (-1, 1), m > 0\) then condition \(\text{Re} \gamma < 0\) implies
\[
k > 0
\]
which is a necessary condition for the existence an eigenvalue \(\gamma \in \mathbb{C}^-\). Henceforth we shall assume \(k \in (0, 1)\).
4. Instability criterion

We denote

\[ D(\gamma, k) = \left[ \frac{\gamma + i}{\gamma + ik} P_1 + \frac{\gamma - i}{\gamma - ik} P_2 + dP_3 \right] \]

three-dimensional operator in \( J_0(\Omega) \). Then (18) can be written as follows:

\[
\nu'A u + 2i Tu + m\gamma D(\gamma, k)u = \gamma u
\]

Using properties of \( A \) and applying usual arguments [5] it is easy to show that the operator \( \nu'A + 2i T \) has discrete spectrum \( \{\lambda_j(\nu')\}_{j=1}^{\infty} \). Since \( A^* = A \gg 0 \), \( T = T^* \) and \( A^{-1} \) is compact then \( \text{Re} \lambda_j(\nu') \geq \nu'\lambda_1(A) \). Therefore for any \( \nu' > 0 \), \( \gamma \in \mathbb{C}^- \) there exists

\[ R(\gamma, \nu') = (\nu'A + 2i T - \gamma I)^{-1} \]

Thus for \( \gamma \in \mathbb{C}^- \) equation (24) can be written in the form

\[
u'u + m\gamma R(\gamma, \nu') \circ D(\gamma, k)u = 0
\]

Applying the orthogonal projector \( P = P_1 + P_2 + P_3 \) we can rewrite (25) as follows:

\[
Pu + m\gamma P \circ R(\gamma, \nu') \circ D(\gamma, k) \circ Pu = 0
\]

\[
(I - P)u + m\gamma(I - P) \circ R(\gamma, \nu') \circ D(\gamma, k) \circ Pu = 0
\]

It is obvious that the system (26),(27) is solvable if and only if the equation (27) is solvable. Since \( P + m\gamma P \circ R(\gamma, \nu') \circ D(\gamma, k) \circ P \) is a linear three-dimensional operator mapping \( PJ_0(\Omega) \) into \( PJ_0(\Omega) \) then (27) is solvable if and only if

\[
\det \left\| \left( c_j, c_n \right)_{L_2(\Omega)} + m\gamma \left( P \circ R(\gamma, \nu') \circ D(\gamma, k) c_j, c_n \right)_{L_2(\Omega)} \right\|_{j,n=1,2,3} = 0
\]

We denote for \( \nu > 0, \gamma \in \mathbb{C}^- \)

\[
b_{jn}(\gamma, \nu) = (R(\gamma, \nu)c_j, c_n)_{L_2(\Omega)}, \quad j, n = 1, 2, 3
\]

Lemma 2 [7]. For any \( \gamma \in \mathbb{C}^-, \nu > 0, \), \( j, n \in \{1, 2, 3\}, j \neq n \)

\[
b_{jn}(\gamma, \nu) = 0
\]

From (29),(30) it follows that equation (28) can be rewritten as follows:

\[
0 = \left[ \left\| c_1 \right\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma + i}{\gamma + ik} b_{11}(\gamma, \nu') \right] \cdot \left[ \left\| c_2 \right\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma - i}{\gamma - ik} b_{22}(\gamma, \nu') \right] \cdot \left[ \left\| c_3 \right\|_{L_2(\Omega)}^2 + md\gamma b_{33}(\gamma, \nu') \right]
\]
We denote for any $\nu > 0$, $m, k \in (0, 1)$, $\gamma \in \mathbb{C}^-$

\[
\begin{cases}
  f_1(\gamma, \nu, m, k) = \|c_1\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma + i}{\gamma + ik} b_{11}(\gamma, \nu) \\
  f_2(\gamma, \nu, m, k) = \|c_2\|_{L_2(\Omega)}^2 + m\gamma \frac{\gamma - i}{\gamma - ik} b_{22}(\gamma, \nu) \\
  f_3(\gamma, \nu, m, k) = \|c_3\|_{L_2(\Omega)}^2 + m\gamma b_{33}(\gamma, \nu)
\end{cases}
\]  

(32)

Functions $f_j(\gamma, \nu, m, k)$, $j = 1, 2, 3$ are analytic in $\gamma \in \mathbb{C}^-$ for any $\nu > 0$, $m, k \in (0, 1)$. Since

\[\lim_{|\gamma| \to \infty} |\gamma| \cdot \|R(\gamma, \nu)\| = 1\]

then for any $\nu > 0$, $m \in (0, 1)$ there exists $r(\nu, m) > 0$ such that

\[f_j(\gamma, \nu, m, k) \neq 0, \quad \gamma \in \mathbb{C}^-, \quad |\gamma| \geq r(\nu, m)
\]

for any $j = 1, 2, 3$, $k \in (0, 1)$.

Thus we obtain the following instability criterion:

spectral problem (18) has an eigenvalue $\gamma \in \mathbb{C}^-$ if and only if

\[f_1(\gamma, \nu', m, k) \cdot f_2(\gamma, \nu', m, k) \cdot f_3(\gamma, \nu', md, k) = 0
\]

All the eigenvalues $\gamma \in \mathbb{C}^-$ are situated inside the half-disk

\[\{z \in \mathbb{C}^- \mid |z| \leq \max(r(\nu', m), r(\nu', md))\}\]

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