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GLOBAL DYNAMICAL BEHAVIOR FOR LOTKA-VOLTERRA SYSTEMS

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ABSTRACT

Recently, Redheffer[13] obtained a global stability result of a positive equilibrium point for a class of Lotka-Volterra systems with reducible interaction matrices.

The present paper, following his proving method, extends his result in some sense and shows that qualitative stability of an interaction matrix implies global stability of the system.

1 INTRODUCTION

It is known that global asymptotic behavior of the solutions of the general n-dimensional Lotka-Volterra system

$$\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij}(x_j - x_j^*), \quad i = 1, 2, ..., n, \tag{1}$$

where $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ is a positive equilibrium point of the system, largely depends upon the structure of the Jacobian or the interaction matrix $A = (a_{ij})_{n\times n}$ of the system. A lot of results about the relationship between the structure of the interaction matrix and global asymptotic behavior of solutions of system (1) are quoted by Hofbauer and Sigmund[4].

In this paper, we consider a basic and interesting question in the study of Lotka-Volterra systems, i.e., what structure of the interaction matrix can imply global stability of a positive equilibrium point of the system. Hereafter, we call system (1) globally stable if and only if so is the positive equilibrium point. We define similarly global stability for subsystems of the system.

In Section 2, we discuss so called D-stability of a matrix and some related conjectures. We will prove that a special kind of D-stability, i.e., qualitative stability of the interaction matrix, implies global stability of system (1) in Section 3. Finally, a brief dicussion is given.

2 SOME CONJECTURES

The following is a well-known concept for a matrix:

Definition 1. A matrix is (i) *semistable* if the real parts of its eigenvalues are all nonpositive; (ii) *quasistable* if it is semistable and no eigenvalue with zero real part is repeated in its minimal polynomial; (iii)*stable* if the real parts of its eigenvalues are all negative.

Definition 2. A is D-stable if for every positive vector x^*, diag$(x^*)A$ is stable.

In [4], Hofbauer and Sigmund proposed the following

Hofbauer-Sigmund Conjecture: If the interaction matrix A is D-stable, then system (1) is globally stable.

They[4] have also indicated that their conjecture is a special case of the well-known Jacobian conjecture stated below.

Let's now consider a general n-dimensional system of ordinary differential equations in R^n:

$$\dot{x} = f(x), \tag{2}$$

where $f(x)$ has continuous first-order partial derivatives. Then we have the following[10]:

Jacobian Conjecture: Suppose that

(i) 0 is a fixed point, i.e., $f(0) = 0$,

(ii) the Jacobian is a stable matrix at every point $x \in R^n$. Then 0 is globally stable.

Jacobian conjecture has not been completely solved even in two-dimensional case yet. Meisters and Olech[11] proved that in two-dimensional case the conjecture is true for the class of polynomial $f : R^2 \to R^2$, i.e., for maps $f = (f_1, f_2)$ such that f_1 and f_2 are real polynomials in two variables. Recently, Gasull, Llibre and Sotomayor[2] extended Meisters and Olech's result to a larger class of analytic vector fields.

In the next section, we shall consider system (1) with an interaction matrix satisfying different stability concept, motivated from qualitative economics[15], i.e., qualitative stability, which implies D-stability[4]. We will prove that qualitative stability of the matrix implies global stability of Lotka-Volterra system. This can serve as an affirmative answer to a special case of Hofbauer-Sigmund conjecture.

3 MAIN RESULTS

Let Q(A) denote the convex cone consisting of all $n \times n$ matrices $\tilde{A} = (\tilde{a}_{ij})$ that have the same sign pattern $(+, -, 0)$ as A so that sgn$\tilde{a}_{ij}$ =sgna_{ij} for all i and j. Then we have

Definition 3. A matrix A is *qualitative semistable* (*qualitative quasistable, qualitative stable*) if each member of Q(A) is semistable (quasistable, stable).

The digraph G(A) of an $n \times n$ matrix A consists of n vertices representing n populations together with edges representing the interrelations between populations. Black dot $\bullet$ (when $a_{ii} < 0$) or white dot $\circ$ ($a_{ii} = 0$) will be put on each vertex. If, in general discussions, we will not be concerned in the concrete sign of a_{ii}, $\oplus$ will be used. An edge between i and j is called a *stronge-edge* whenever $a_{ij} \neq 0$ and $a_{ji} \neq 0$. A p-cycle of G(A) is a nonvanishing product of $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_p i_1}$ for a sequence of distinct indices $i_1, i_2, \ldots, i_p$.

To explain these, we use a system quoted from Jeffries[5].

Example 1. Consider system (1) with the interaction matrix A_f as follows

$$A_f = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} & 0 \\ a_{31} & 0 & 0 & 0 & a_{35} \\ 0 & a_{42} & 0 & 0 & 0 \\ 0 & 0 & a_{53} & 0 & 0 \end{pmatrix}, \tag{3}$$

where $a_{11} < 0$, $a_{ij}a_{ji} < 0$ for $i \neq j$. Then the G(A_f) is expressed as

$$\circ - - - \circ - - - \bullet - - - \circ - - - \circ.$$

The following result was established by Quirk and Ruppert[12] on the basis of Hurwitz Theorem and Liapunov function technique.

LEMMA 1. A matrix A is qualitative semistable if and only if it satisfies:

i) each 1-cycle in G(A) is nonpositive;

ii) each 2-cycle in G(A) is nonpositive;

iii) G(A) has no p-cycle for $p \geq 3$.

When A is irreducible these conditions imply qualitative quasistability.

It is easy to check, based on Lemma 1, that matrix A_f in (3) is qualitative quasistable.

The following *stronge community* notation are quoted from Jeffries[5] and Jeffries et al[6], in their cases, they called it as *predation community* and *strong component* respectively.

Definition 4. Associate with a fixed vertex all the other vertices, if any, to which it is connected by strong edges. Then associate with these vertices all additional vertices connected by strong edges, and so on. The maximal set of all such vertices so connected to the first vertex is called the *strong community* containing the first vertex.

By introducing the *color test* notation, Jeffries gave a complete solution for a matrix to be qualitative stable[5].

Definition 5. A strong community G(A) can pass the color test if each vertex in it may be colored black and white with the result that

i) each vertex with $a_{ii} < 0$ is black;

ii) there is at least one white dot;

iii) each white dot is connected by a strong edge to at least one other white dot;

iv) each black dot connected by a strong edge to one white dot, is connected by a strong edge to at least one other white dot.

The following theorem is due to Quirk and Ruppert[12] in the case where $a_{ii} < 0$ for all i and to Jeffries[5] in general case.

JQR THEOREM. A matrix $A = (a_{ij})_{n\times n}$ is qualitative stable if and only if it satisfies:

i) each 1-cycle in $G(A)$ is nonpositive;

ii) each 2-cycle in $G(A)$ is nonpositive;

iii) $G(A)$ has no p-cycle for $p \geq 3$;

iv) each strong community in $G(A)$ can not pass the color test;

v) $\det A \neq 0$.

It is easy to check that matrix A_f (3) is not qualitatively stable, since $G(A_f)$ can pass the color test.

In [8], we have considered system (1) with the following lower-triangle interaction matrix

$$A = (a_{ij})_{n \times n} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \times & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \cdots & A_k \end{pmatrix}, \tag{4}$$

where each submatrix $A_i (i = 1, ..., k)$ is irreducible, all elements in the upper-right blocks are zero and all matrices $\times$ in the left-lower have any elements and obtained:

THEOREM 1[8]. If system (1) with A in (4) satisfies

i) each $A_i \in \bar{S}_w$ $(i = 2, ..., k)$;

ii) each $\mathrm{diag}(x_i^*)A_i$ is stable $(i = 1, ..., k)$;

iii) each subsystem $\dot{x}_i = \mathrm{diag}(x_i)A_i(x_i - x_i^*)$ is globally stable $(i = 1, ..., k-1)$, then $\Omega(x) \subseteq \mathrm{E} = \{(x_1^*; ...; x_{k-1}^*; x_k) \in R_+^n | x_k \in \mathrm{M}\}$, where $\Omega(x)$ is the ω-limit set of system (1) and M is the LaSalle's invariant set of the subsystem $\dot{x}_k = \mathrm{diag}(x_k)A_k(x_k - x_k^*)$. Here $x = (x_1; x_2; ...; x_k)^T = (x_{11}, ..., x_{1i_1}; x_{21}, ..., x_{2i_2}; ...; x_{k1}, ..., x_{ki_k})^T$, $x^* = (x_1^*; x_2^*; ...; x_k^*)^T = (x_{11}^*, ...x_{1i_1}^*; x_{21}^*, ..., x_{2i_2}^*; ...; x_{k1}^*, ..., x_{ki_k}^*)^T$ and $i_1 + i_2 + ... + i_k = n$ and a matrix $B \in \bar{S}_w$ if and only if there exists a positive diagonal matrix C such that $CB + B^TC$ is negative semi-definite.

Combining Theorem 1 and JQR Theorem, we have

THEOREM 2[9]. If the interaction matrix A of system (1) is qualitative stable, then the system is globally stable for any given positive equilibrium point x^*.

Remark. By using Theorem 1 in [13] and JQR Theorem, we can also prove Theorem 2.

4 DISCUSSION

We have proved that qualitative stability of the interaction matrix of a Lotka-Volterra system implies global stability of the system. Since a qualitative stable matrix must be D-stable[4], our Theorem is an affirmtive answer to a special case of Hofbauer-Sigmund conjecture.

A number of results [1,3,7,14,16,17] given before in the literature are special cases of Theorem 2 in Section 3, since in these cases, the interaction matrix of system (1) is qualitative stable.

The remaining question is the relationship between the weaker qualitative properties such as qualitative semistability or qualitative quasistability of matrix A and global stability of the corresponding system (1). But semistability is not so interesting in the sense that it cannot guarantee even the boundedness of the solutions of a corresponding linear system[6].

It seems that there is no general known condition ensuring a matrix to be qualitatively quasistable. From Lemma 1, we know that a qualitative semistable matrix is qualitative quasistable whenever it is irreducible. Furthermore a qualitative quasistable matrix ensures the boundedness for the solutions of a corresponding linear system. Hence, from this subclass of qualitative quasistable (qualitative semistable and irreducible) matrices we can expect to find out some interesting structures of a LaSalle's invariant set which is not identical with just a unique positive equilibrium point.

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