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無限次元空間上の順序を保存する作用素の resolvent 方程式について

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1. Introduction

The Perron-Frobenius theorem, concerning the properties of eigenvalues and eigenvectors of square matrices whose components are nonnegative, has been extended and applied in various ways. Recently, in the interest of mathematical economics, nonlinear extensions have been obtained by H. Nikaido [5], M. Morishima [6], T. Fujimoto [7], Y. Oshime [8] [9] [10]. Many of them have been studied in a n-dimensional Euclidean space and the resolvent problem has been discussed in the nonlinear Leontief model.

In this paper, we extend their results to a mapping on an infinite dimensional space. In doing so, we introduced the notion of indecomposability for a nonlinear mapping on an infinite dimensional space. We consider the resolvent problem of an order-preserving mapping on a positive cone(a closed convex cone with vertex at 0) of a strongly ordered Banach space (an ordered Banach space with positive cone having nonempty interior). We define the resolvent mapping and study the solutions of the resolevent equation applying the results on eigenvalue problem which is obtained by the author. Though we deal with a class of positively homogeneous mappings in this paper, we obtain the same results on a more general class of subhomogeneous mappings along the argument of Y. Oshime [9] for a finite dimensional space.

2. Notations and assumptions

Let E be an ordered Banach space, that is, a real Banach space provided with an order cone E_+ (a closed convex cone with vertex at 0 such that

$E_+ \cap (-E_+) = \{0\}$). We assume that E_+ has nonempty interior (we call such a space a strongly ordered Banach space) and $\dim E \geq 2$. Let $(E_+)^i$ denote the interior of E_+ and $(E_+)^b$ the boundary of E_+. Note that $(E_+)^b \setminus \{0\}$ is not empty, which follows from $\dim E \geq 2$.

For $x, y \in E$, we write $x \gg y$ if $x - y \in (E_+)^i$, $x > y$ if $x - y \in E_+ \setminus \{0\}$, and $x \geq y$ if $x - y \in E_+$. For $x \in E$, we say x is strongly positive, positive, nonnegative if and only if $x \gg 0, x > 0, x \geq 0$, respectively.

We assume order-preserving norm on E, namely, $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$.

Fix $e \gg 0$, e-norm can be defined on E as

$$\|x\|_e = \inf\{m \geq 0 \mid -me \leq x \leq me\},$$

because of closedness of E_+

$$\|x\|_e = \min\{m \geq 0 \mid -me \leq x \leq me\}.$$

For each $e \gg 0$, e-norm is well-defined and equivalent to the original norm.

Let T be a mapping from E_+ into itself. We consider the following assumptions:

$A1$:(compact) T is continuous and the image of a bounded set by T is relatively compact,

$A2$:(positively homogeneous)

$$T(\lambda x) = \lambda Tx \qquad \text{for any} \quad \lambda > 0, x \geq 0,$$

$A3$:(order-preserving)

$$x \leq y \qquad \text{implies} \qquad Tx \leq Ty.$$

An order-preserving map T is also called monotone or isotone.

For $x \geq 0$, we denote

$$E_x = \{y \geq 0 \mid \exists \lambda > 0, y \leq \lambda x\}.$$

We can see that $E_x = \{0\}$ if and only if $x = 0$, and $E_x = E_+$ if and only if $x \gg 0$.

$A4$:(indecomposable)

$$\{0\} \subsetneq E_{x-y} \subsetneq E_+ \qquad \text{implies} \qquad Tx - Ty \notin E_{x-y}.$$

This is an infinite dimensional extension of indecomposability defined by M. Morishima [6], where $E = R^n$, $E_+ = R^n_+ = \{(x_1, \cdots, x_n) \in R^n \mid x_i \geq 0, 0 \leq i \leq n\}$. It is also an nonlinear extension of indecomposability found in M. A. Krasnosel'skiĭ [4] and order irreducibility for a linear operator introduced by H. Nikaido [5]. Furthermore, under the assumption A3, when T is a bounded linear operator, T is indecomposable if and only if T is irreducible [2], semi-non-support [3]. Strongly order-preserving mappings($x < y$ implies $Tx \ll Ty$) are indecomposable.

We denote

$$\begin{aligned} VP(T) &= \{\lambda \mid \lambda \text{ is an eigenvalue of } T\} \\ &= \{\lambda \mid Tx = \lambda x \text{ for } \exists x > 0\}, \end{aligned}$$

$$\lambda_0(T) = \begin{cases} \sup\{\lambda \mid \lambda \in VP(T)\}, & (VP(T) \neq \emptyset), \\ -\infty, & (VP(T) = \emptyset), \end{cases}$$

$$||T|| = \sup\{||Tx|| \mid x \in B_1\}.$$

$\lambda_0(T)$ is called the spectral radius, as in the case of linear operators.

Lemma 2.1 *Under the assumptions $A2$ and $A3$, $||T|| < +\infty$, $\lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$ exists and satisfies $\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} \leq ||T||$.*

Denote $r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$.

3. Fundamental results

We summarize the results obtained before.

Theprem 3.1 *Suppose that $T{:}E_+ \to E_+$ satisfies the assumptions A1,A2,A3, and $r(T) > 0$. Then $r(T)$ is an eigenvalue of T and $r(T) = \lambda_0(T)$.*

Theorem 3.2 *Let T satisfy the assumptions A1,A2,A3 and A4. Then $r(T)$ is a positive eigenvalue having strongly positive eigenvectors uniquely without positive scalar multiplication. Moreover there is no other eigenvalue.*

Under some conditions, $r(T)$ coincides with the maximal eigenvalue. Now we give a characterization of $r(T)$, which is a key to solve the resolvent problem.

Theorem 3.3 *Let T satisfy the assumptions A1,A2,A3, then*

(i) *$\lambda > r(T)$ if and only if there exists some $x \gg 0$ such that $Tx \ll \lambda x$.*

(ii) *$\lambda \leq r(T)$ if and only if there exists some $x > 0$ such that $Tx \geq \lambda x$.*

Further let T satisfy A4. Then

(iii) *$\lambda \geq r(T)$ if and only if there exists some $x > 0$ such that $Tx \leq \lambda x$.*

(iv) *$\lambda < r(T)$ if and only if there exists some $x \gg 0$ such that $Tx \gg \lambda x$.*

4. Resolvent problem

In this subsection, we treat the resolvent problem. We denote by $R_\lambda(c)$ the set of the nonnegative solutions of the resolvent equation

$$\lambda x = Tx + c, \tag{1}$$

where λ is a nonnegative number and c is an element of E_+, that is,

$$R_\lambda(c) = \{x \geq 0 \mid \lambda x = Tx + c\}.$$

This defines a multivalued mapping R_λ can be defined E_+ into itself:

$$R_\lambda{:}E_+ \twoheadrightarrow E_+,$$

which we call the resolvent mapping of T corresponding to λ. Throughtout this paper the symbol $\twoheadrightarrow$ is used to indicate the domain and the range of a multivalued mapping. When the equation (1) has no solution for some $\lambda \geq 0, c \geq 0$, we define $R_\lambda(c) = \emptyset$.

In Theorem 4.1, it is shown that under the condition $A1$,$A2$,$A3$, $R_\lambda(c) \neq \emptyset$ for all $c \geq 0, \lambda > r(T)$, on the other hand, the resolvent equation (1) has no nonnegative solution for any $\lambda \leq r(T), c \gg 0$.

By definition of R_λ, it is obvious that when $R_\lambda(c) \neq \emptyset$ for all $c \geq 0$ the composed mapping $(\lambda I - T) \circ R_\lambda = I$ and $R_\lambda \circ (\lambda I - T)_{|(\lambda I-T)^{-1}E_+ \cap E_+} \supseteq I_{|(\lambda I-T)^{-1}E_+ \cap E_+}$, where I is the identical mapping on E_+, $T_{|K}$ means the restriction of T on a subset K of E_+.

Theorem 4.1 *Suppose that* $T{:}E_+ \to E_+$ *satisfies* $A1, A2, A3$.
(1) Let $\lambda > r(T)$ *be fixed. Then the equation* (1) *is solvable for all* $c \geq 0$, *that is,* $R_\lambda(c)$ *is not empty.* $R_\lambda(c)$ *is a singleton if* $c \gg 0$ *or c=0. Moreover*

(1.1) R_λ *is positively homogeneous* $(R_\lambda(\alpha c) = \alpha R_\lambda(c) \quad \alpha \geq 0, c \geq 0)$.

(1.2) $R_\lambda(c) \gg 0$ *when* $c \gg 0$.

(1.3) $0 \ll R_\lambda(c) < R_\lambda(c')$ *when* $0 \ll c < c'$.

(1.4) R_λ *continuously maps* $(E_+)^i \cup \{0\}$ *into itself.*

(1.5) $R_\lambda(c)$ *is not necessarily a singleton when* $c \in (E_+)^b \setminus \{0\}$.

(1.6) $R_\lambda(c)$ *contains the limit of a converging sequence* $\{R_\lambda(c_n)\}$ *which satisfies* $c_n \gg c$ *and* $c_n \to c$ *as* $n \to \infty$, *that is,* $\lim_{c_n \downarrow c} R_\lambda(c_n) \in R_\lambda(c)$. *Moreover, this limit is independent of the choice of the sequence* $\{c_n\}$.

(1.7) $R_\lambda\colon E_+ \twoheadrightarrow E_+$ *is compact-valued and upper semicontinueous.*

(2) If $R_\lambda(c)$ *is not empty for some* $c \gg 0$, *then* $\lambda > r(T)$.

Remark 4.2 *Under the condition of Theorem* 4.1, *for each* $\lambda > r(T)$ *(1.6) shows that* $R_\lambda(c)$ *contains* $\lim_{c_n \downarrow c} R_\lambda(c_n)$. *We denote this limit by* $\overline{R_\lambda}(c)$. *Then a single-valued mapping* $\overline{R_\lambda}\colon E_+ \to E_+$ *has the following properties:*

(1) $R_\lambda(c) = \overline{R_\lambda}(c)$ *when* $c \gg 0$ *or* $c = 0$,

(2) $0 \leq \overline{R_\lambda}(c) < \overline{R_\lambda}(c')$ *when* $0 \leq c < c'$,

(3) $0 \leq \overline{R_\lambda}(c) \ll \overline{R_\lambda}(c')$ *when* $0 \leq c \ll c'$.

Proof of Theorem 4.1 (1). Let $\lambda > r(T)$ be fixed. We begin with the solvability of the resolvent equation (1).

Since $\lambda > r(T)$, from Theorem 3.3 we can choose $x_0 \gg 0$ such that $Tx_0 \ll \lambda x_0$, that is,

$$\|\frac{Tx_0}{\lambda}\|_{x_0} < 1.$$

Then, for each $c \geq 0$, there exists a sufficiently large $M > 0$ such that

$$\|\frac{c}{\lambda M}\|_{x_0} < 1 - \|\frac{Tx_0}{\lambda}\|_{x_0}.$$

Thus

$$\frac{1}{\lambda}(Tx_0 + \frac{c}{M}) \leq x_0. \tag{2}$$

Therefore the compact mapping F defined as

$$Fx = \frac{1}{\lambda}(Tx + \frac{c}{M})$$

leaves the interval $[0, x_0]$ invariant. In fact, $0 \leq x \leq x_0$ implies $0 = T0 \leq Tx \leq Tx_0$, then,

$$0 \leq \frac{c}{\lambda M} \leq \frac{1}{\lambda}(Tx + \frac{c}{M}) \leq \frac{1}{\lambda}(Tx_0 + \frac{c}{M}).$$

Hence, from (2)

$$0 \leq Fx \leq x_0.$$

Since $F([0, x_0])$ is relatively compact, F continuously maps a compact set K into itself where $K = \overline{co}(F([0, x_0]) \subseteq [0, x_0]$. By virtue of Shauder's fixed points theorem, F has fixed points in K. Then there exists $x_0 \in K \subseteq E_+$ such that

$$x_0 = Fx_0 = \frac{1}{\lambda}(Tx_0 + \frac{c}{M}),$$

that is,

$$\lambda(Mx_0) = T(Mx_0) + c.$$

This shows that Mx_0 is a solution of the equation (1).

Next we prove that $R_\lambda(c)$ is a singleton if $c \gg 0$ or $c = 0$. Let $c \gg 0$ and $x, y \in R_\lambda(c)$, where we can see $x \gg 0, y \gg 0$ from the inequalities

$$\lambda x = Tx + c \geq c \gg 0,\ \lambda y = Ty + c \geq c \gg 0. \tag{3}$$

If $\|x\|_y > 1$, then $x \leq \|x\|_y y$ implies

$$Tx \leq \|x\|_y Ty,$$

$$\lambda x - c \leq \|x\|_y(\lambda y - c),$$

$$\begin{aligned} x &\leq \|x\|_y y + \frac{1 - \|x\|_y}{\lambda} c \\ &\leq (\|x\|_y + \frac{1 - \|x\|_y}{\lambda} \cdot \frac{1}{\|y\|_c}) y. \end{aligned}$$

By definition of y-norm to the contrary,

$$\|x\|_y \leq \|x\|_y + \frac{1 - \|x\|_y}{\lambda} \cdot \frac{1}{\|y\|_c} < \|x\|_y.$$

Thus $\|x\|_y \leq 1$, which means $x \leq y$. Similarly $x \geq y$ can be shown, hence $x = y$. Therefore it is proved that the equation (1) has a unique solution.

When $c = 0$, $x = 0$ is a trivial solution. If there exists a positive solution $x' > 0$,

$$\lambda x' = Tx'$$

means that λ is an eigenvalue of T. This implies $\lambda \leq r(T)$, which contradicts that $\lambda > r(T)$. Therefore $x = 0$ is a unique solution.

When $c \gg 0$ or $c = 0$ we use the convenient notation

$$R_\lambda(c) = \{x_c\}.$$

Now we prove several properties of R_λ (1.1)–(1.7).

Positively homogenuity of R_λ is evident from the fact that

$$\lambda x = Tx + c$$

implies

$$\lambda \alpha x = T(\alpha x) + \alpha c \quad (\forall \alpha > 0).$$

The statement of (1.2) have been noted by (3).

We show the statement (1.3). Let $0 \ll c < c'$, then, as we have shown, $R_\lambda(c), R_\lambda(c')$ are singletons and strongly positive. Further, it is clear that $R_\lambda(c) \neq R_\lambda(c')$. Assume that $x_c < x_{c'}$ is not true. Then, $\|x_c\|_{x_{c'}} > 1$ implies

$$\begin{aligned}
\lambda x_c = Tx_c + c &\leq \|x_c\|_{x_{c'}} Tx_{c'} + c \\
&< \|x_c\|_{x_{c'}} Tx_{c'} + c' \\
&\ll \|x_c\|_{x_{c'}} Tx_{c'} + \|x_c\|_{x_{c'}} c' \\
&= \|x_c\|_{x_{c'}} (\lambda x_{c'}).
\end{aligned}$$

Thus

$$x_c \ll \|x_c\|_{x_{c'}} x_{c'},$$

which contradicts to definition of $x_{c'}$-norm. Hence we obtain

$$0 \ll x_c < x_{c'}.$$

We prove the continuity of R_λ on $(E_+)^i \cup \{0\}$. Let $c_0 \in (E_+)^i \cup \{0\}$ and $\{c_n\} \subseteq (E_+)^i \cup \{0\}$ such that $c_n \to c_0$ $(n \to \infty)$. Fix $e \gg 0$. Since $\{c_n\}$ is bounded, $\{c_n\}$ is also bounded in e-norm. Then, for a sufficiently large $M > 0$,

$$c_n < Me \qquad (\forall n \in N).$$

From (1.3) $x_{c_n} \leq R_\lambda(Me)$, which shows that $\{x_{c_n}\}$ is also bounded. Since $\{Tx_{c_n}\}$ is a relatively compact subset in E_+, there exists a subsequence $\{Tx_{c_{n_j}}\} \subseteq \{Tx_{c_n}\}$, the limit of which being denoted by z_0, then

$$Tx_{c_{n_j}} \to z_0 \quad (n_j \to \infty). \tag{4}$$

Hence, as $n_j \to \infty$,

$$\lambda x_{c_{n_j}} = Tx_{c_{n_j}} + c_{n_j} \to z_0 + c,$$

which means

$$x_{c_{n_j}} \to \frac{1}{\lambda}(z_0 + c).$$

Therefore

$$Tx_{c_{n_j}} \to T(\frac{1}{\lambda}(z_0 + c)) \quad (n_j \to \infty).$$

From this and (4) it must be that $z_0 = T(\frac{1}{\lambda}(z_0 + c))$, that is,

$$\lambda(\frac{1}{\lambda}(z_0 + c)) = z_0 + c = T(\frac{1}{\lambda}(z_0 + c)) + c.$$

Therefore

$$x_{c_{n_j}} \to \frac{1}{\lambda}(z_0 + c) = R_\lambda(c).$$

This shows $\lim_{n_j \to \infty} R_\lambda(c_{n_j}) = R_\lambda(c)$.

As $R_\lambda(c)$ is a singleton, we have

$$\lim_{n \to \infty} R_\lambda(c_n) = R_\lambda(c).$$

Hereafter we consider the case $c \geq 0$.

See Remark 1 of Y. Osime [10] for the detail of the statement (1.5).

We prove the statement (1.6). Take a sequence $\{c_n\}$ such that $c_n \gg c$ which converges to c. Since $\{c_n\}$ is bounded, from (1.3) we can see that $\{x_{c_n}\}$ is also bounded and hence $\{T(x_{c_n})\}$ is a relatively compact set in E_+. Then there exists a subsequence $\{T(x_{c_{n_j}})\} \subseteq \{T(x_{c_n})\}$ with the limit $z_0 \in E_+$, namely,

$$Tx_{c_{n_j}} \to z_0 \in E_+ \quad (n_j \to \infty).$$

In a similar fashion to that in the proof of the continuity of R_λ, the equality

$$\lambda(\frac{z_0 + c}{\lambda}) = z_0 + c = T(\frac{z_0 + c}{\lambda}) + c,$$

holds, that is,

$$\lim_{c_{n_j} \downarrow c} R_\lambda(c_{n_j}) = \frac{1}{\lambda}(z_0 + c) \in R_\lambda(c).$$

This limit does not depend on the choice of $\{c_{n_j}\}$. In fact, if we have two subsequences $\{c_{n_j}\}, \{c_{n_{j'}}\}$ where

$$\lim_{c_{n_j}\downarrow c} R_\lambda(c_{n_j}) = x_0, \quad \lim_{c_{n_{j'}}\downarrow c} R_\lambda(c_{n_{j'}}) = x_0' \in R_\lambda(c). \tag{5}$$

Let us make a new subsequence $\{c_{n_{j''}}\}$ of $\{c_n\}$ combining them such that $c_{n_{j''}} \in \{c_{n_j}\}$ when $n_{j''}$ is odd, $c_{n_{j''}} \in \{c_{n_{j'}}\}$ when $n_{j''}$ is even and $c_{n_{j''}} > c_{n_{j''}+1} > c_{n_{j''}+2} \gg c$. Then, using (1.3), we can note

$$x_{c_{n_{j''}}} > x_{c_{n_{j''}+1}} > x_{c_{n_{j''}+2}} > \cdots.$$

On the other hand, from (5)

$$x_{n_{j''}} \to x_0 \; (n_{j''} \text{ is odd}), \quad x_{n_{j''}} \to x_0' \; (n_{j''} \text{ is even}).$$

Therefore, $x_0 = x_0'$ follows from the closedness of E_+ and $E_+ \cap (-E_+) = \{0\}$. We have proved $x_{c_n} \to \frac{1}{\lambda}(z_0 + c) \in R_\lambda(c)$.

Moreover, in the same way as above, we can show that $\lim\limits_{c_n\downarrow c} R_\lambda(c)$ is independent of the choice of $\{c_n\}$ which satisfies $c_n \gg c$ and $c_n \to c$ as $n \to \infty$.

Thus we have completed the proof of (1.6).

Finally we show (1.7). It is clear that R_λ is compact-valued and upper semicontinueous on $(E_+)^i \cup \{0\}$ from (1.4). Fix $e \gg 0$. Then, in the same way as in the proof of (1.3),

$$x_c \leq R_\lambda(c+e) \quad (\forall x_c \in R_\lambda(c))$$

can be obtained. This shows that $R_\lambda(c)$ is bounded, and

$$R_\lambda(c) = \frac{1}{\lambda} T R_\lambda(c) + \frac{c}{\lambda}$$

implies that $R_\lambda(c)$ is a relatively compact set. Moreover $R_\lambda(c)$ is closed, which follows from the fact that T is continuous. Therefore $R_\lambda(c)$ is compact. Next we prove that $R_\lambda: E_+ \twoheadrightarrow E_+$ is upper semicontinuous.

Since R_λ is compact-valued, it is sufficient to show that if $c_n \to c$ and $x_n \in R_\lambda(c_n)(\forall n)$, there exists some converging subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with the limit in $R_\lambda(c)$. Let $c_n \to c$ and $x_n \in R_\lambda(c_n)$. Fix a sufficiently large $e \gg 0$ such that $c_n \leq c+e(\forall n)$. Repeating the above argument leads

$$x_n = \frac{Tx_n}{\lambda} + \frac{c_n}{\lambda} \leq R_\lambda(c+e) \quad (\forall n),$$

which implies that $\{x_n\}$ is relatively compact. Then $\{x_n\}$ contains a converging subsequence $\{x_{n_j}\}$ with the limit in $R_\lambda(c)$. In fact, let denote the limit by x_0, then, as $n_j \to \infty$,

$$c_{n_j} \to c, \quad x_{n_j} \to x_0, \quad Tx_{n_j} \to Tx_0,$$

which implies

$$\lambda x_0 = Tx_0 + c.$$

This shows $x_0 \in R_\lambda(c)$. We have completed the proof. $\square$

Proof of Theorem 4.1 (2). Let the condition be satisfied, that is, we have $c \gg 0$ and $x \geq 0$ such that

$$\lambda x = Tx + c.$$

Then $\lambda x \geq c \gg 0$ implies $x \gg 0$ and $\lambda x = Tx + c \gg Tx$. Applying Theorem 3.3, we obtain $\lambda > r(T)$. $\square$

Proof of Remark 4.2. The statement (1) is clear because $R_\lambda(c)$ is a singleton if $c \gg 0$ or $c = 0$. We prove the statements (2),(3). It is clear that each $x \geq 0$ cannot be a solution of the equation (1) for different c, $c' \geq 0$. This means $\overline{R_\lambda}(c) \neq \overline{R_\lambda}(c')$ when $c \neq c'$. Let $0 \leq c < c'$. Let $e \gg 0$ be fixed, then, for each $\epsilon > 0$,

$$0 \ll c + \epsilon e < c' + \epsilon e.$$

From (1.3) we have

$$0 \ll R_\lambda(c + \epsilon e) < R_\lambda(c' + \epsilon e) \quad (\forall \epsilon > 0).$$

Because $c \neq c'$ implies $\overline{R_\lambda}(c) \neq \overline{R_\lambda}(c')$, as $\epsilon \downarrow 0$ we obtain

$$0 \leq \overline{R_\lambda}(c) < \overline{R_\lambda}(c').$$

Let now $0 \leq c \ll c'$. For a sufficiently small $\epsilon > 0$,

$$0 \leq c < (1-\epsilon)c' \ll c'.$$

From (1.1) and the fact $R_\lambda(c') \gg 0$, we can prove

$$0 \leq \overline{R_\lambda}(c) < (1-\epsilon)R_\lambda(c') \ll R_\lambda(c'). \qquad \square$$

Besides we inquire into the problem when T is indecomposable. For each $\lambda > r(T)$, the resolvent mapping R_λ which is defined as a single-valued mapping is continuous, positively homogeneous, strongly order-preserving ($x < y$ implies $R_\lambda(x) \ll R_\lambda(y)$) and hence indecomposable. The resolvent equation (1) has no nonnegative solution for any $\lambda \leq r(T), c > 0$.

Theorem 4.3 *In addition to the assumptions in Theorem 4.1, assume further that A4 holds.*
(1) Let $\lambda > r(T)$ be fixed. Then $R_\lambda(c)$ is a singleton for all $c \geq 0$, and $R_\lambda : E_+ \to E_+$ is also an indecomposable and continuous mapping.
(1.1) $R_\lambda(c) \gg 0$ when $c > 0$.
(1.2) $R_\lambda(c) \ll R_\lambda(c')$ when $0 \leq c < c'$.
(2) If $R_\lambda(c)$ is not empty for some $c > 0$, then $\lambda > r(T)$.

Proof of Theorem 4.3 (1). Let $\lambda > r(T)$ be fixed. It is clear that $R_\lambda(c)$ is a singleton if $c \gg 0$ or $c = 0$ from the preceding theorem. We consider the case $c \in (E_+)^b \setminus \{0\}$. Assume that there exist $x, y \in R_\lambda(c)$, then it is clear that x, y are linearly independent. Since

$$\lambda x = Tx + c, \quad \lambda y = Ty + c$$

implies $\lambda x \geq c > 0, \lambda y \geq c > 0$, we have $x, y > 0$. Further, $\lambda x \geq Tx, \lambda y \geq Ty$ means $E_x = E_y = E_+$ by the definition of indecomposability. Then we

can note $x, y \gg 0$.
Hence, we have $x \leq \|x\|_y y$ which satisfies $\{0\} \subsetneq E_{\|x\|_y y - x} \subsetneq E_+$. From the assumption $A4$

$$\begin{aligned} E_{\|x\|_y y-x} &\not\ni \|x\|_y Ty - Tx \\ &= \|x\|_y(\lambda y - c) - (\lambda x - c) \\ &= \lambda(\|x\|_y y - x) + (1 - \|x\|_y)c. \end{aligned}$$

This shows $1 - \|x\|_y > 0$, that is, $x \ll y$. we can similarly obtain the opposite inequality $y \ll x$. This is a contradiction. Thus $R_\lambda(c)$ is single-valued for each $c \geq 0$.

SInce $R_\lambda(c) \gg 0$ provided $c > 0$ as we have shown as above, the statement (1.1) is true.

Next we prove (1.2). Let $0 \leq c < c'$. From Theorem 4.1 it is clear that

$$0 \leq R_\lambda(c) < R_\lambda(c').$$

We prove $R_\lambda(c) \ll R_\lambda(c')$. Assume that $\{0\} \subsetneq E_{x_{c'} - x_c} \subsetneq E_+$, where x_c, $x_{c'}$ are convinient notations of $R_\lambda(c)$, $R_\lambda(c')$ defined in the proof of the preceding theorem, respectively. Then, by the assumption $A4$, we have

$$\begin{aligned} E_{x_{c'}-x_c} &\not\ni Tx_{c'} - Tx_c \\ &= (\lambda x_{c'} - c') - (\lambda x_c - c) \\ &= \lambda(x_{c'} - x_c) - (c' - c) \\ &\leq \lambda(x_{c'} - x_c). \end{aligned}$$

This is a contradiction.

We can see that R_λ is indecomposable by the property (1.2). In fact, let $\{0\} \subsetneq E_{y-x} \subsetneq E_+$. Then from (1.2) we have $R_\lambda(x) \ll R_\lambda(y)$, which shows

$$0 \ll R_\lambda(y) - R_\lambda(x) \notin E_{y-x}.$$

Since R_λ is single-valued and upper semicontinuous in the sense of a multi-valued mapping from Theorem 4.1 (1.7), R_λ is clearly continuous regarded as a single-valued mapping. □

Proof of Theorem 4.3 (2). Let the condition be satisfied, that is, we have $c > 0$ and $x \geq 0$ such that

$$\lambda x = Tx + c.$$

Then, since $\lambda x \geq c > 0$, it must be that $x > 0$, and since $\lambda x > Tx$, it must be that $x \gg 0$ by the indecomposability of T.

From Theorem 3.2 it is shown that $r(T)$ is an eigenvalue of T with a strongly positive eigenvector. Let $x_0 \gg 0$ be the eigenvector. If x and x_0 are linearly dependent, we have

$$\lambda x > Tx = r(T)x,$$

which implies $\lambda > r(T)$. If x and x_0 are linearly independent, we have

$$x_0 \leq \|x_0\|_x x,$$

where $\{0\} \subsetneq E_{\|x_0\|_x x - x_0} \subsetneq E_+$. From the indecomposability of T,

$$\begin{aligned} E_{\|x_0\|_x x - x_0} &\not\ni \|x_0\|_x Tx - Tx_0 \\ &\leq \|x_0\|_x \lambda x - r(T)x_0 \\ &= \lambda(\|x_0\|_x x - \frac{r(T)}{\lambda} x_0). \end{aligned}$$

Therefore, we obtain $\frac{r(T)}{\lambda} < 1$, namely, $r(T) < \lambda$. □

Next we consider the relation between λ and $R_\lambda(c)$, where $c \geq 0$ is given.

Theorem 4.4 *Let T satisfy the assumptions A1,A2,A3, and c be an arbitrarily fixed element in E_+. Then the multivalued mapping $R_\lambda(c)$ on $[0, \infty)$ is compact-valued and upper semicontinuous on $(r(T), \infty)$.*

In case $c \gg 0$, the following statements hold:

(i) $\lambda \leq r(T)$ *implies* $R_\lambda(c) = \emptyset$,

(ii) $r(T) < \lambda' < \lambda$ *implies* $0 \ll R_\lambda(c) < R_{\lambda'}(c)$,

(iii) $\lim_{\lambda \downarrow r(T)} \|R_\lambda(c)\| = \infty$.

Theorem 4.5 *Let T satisfy the assumptions A1,A2,A3,A4, and c be an arbitrarily fixed element in E_+. Then $R_\lambda(c)$ is single-valued and continuous on $(r(T), \infty)$.*

In case $c > 0$, the following statements hold:

(i) $\lambda \le r(T)$ *implies* $R_\lambda(c) = \emptyset$,

(ii) $r(T) < \lambda' < \lambda$ *implies* $0 \ll R_\lambda(c) \ll R_{\lambda'}(c)$,

(iii) $\lim_{\lambda \downarrow r(T)} \|R_\lambda(c)\| = \infty$.

Proof of Theorem 4.5. The proof is similar to that of Theorem 4.4. □

Proof of Theorem 4.4. It is clear that $R_\lambda(c)$ is a nonempty compact set for each $\lambda > r(T)$, $c \ge 0$, and a singleton for each $\lambda > r(T)$, $c \gg 0$ from Theorem 4.1 (1).

Fix $c \gg 0$. Then the statement (i) is true by Theorem 4.1 (2). Let $\{x_\lambda\} = R_\lambda(c)$, $\{x_{\lambda'}\} = R_{\lambda'}(c)$ where $r(T) < \lambda' < \lambda$. Suppose that the statement (ii) is not true. Then we have $\|x_\lambda\|_{x_{\lambda'}} > 1$, since $x_\lambda, x_{\lambda'} \gg 0$. On the other hand, $x_\lambda \le \|x_\lambda\|_{x_{\lambda'}} x_{\lambda'}$ implies

$$Tx_\lambda \le \|x_\lambda\|_{x_{\lambda'}} Tx_{\lambda'}$$

$$\lambda x_\lambda - c \le \|x_\lambda\|_{x_{\lambda'}} (\lambda' x_{\lambda'} - c).$$

Thus

$$\begin{aligned} x_\lambda &\le \|x_\lambda\|_{x_{\lambda'}} \frac{\lambda'}{\lambda} x_\lambda + \frac{(1 - \|x_\lambda\|_{x_{\lambda'}})}{\lambda} c \\ &\le \|x_\lambda\|_{x_{\lambda'}} x_\lambda + \frac{(1 - \|x_\lambda\|_{x_{\lambda'}})}{\lambda} c \\ &\ll \|x_\lambda\|_{x_{\lambda'}} x_\lambda, \end{aligned}$$

which is a contradiction. We proved the statement (ii).

Now we show that the compact-valued mapping $\lambda \mapsto\!\!\!\to R_\lambda(c)$ is upper semicontinuous on $(r(T), \infty)$ for any $c \geq 0$. Let $c \geq 0$ be fixed and $\lambda_n \to \lambda_0$ where $\lambda_n, \lambda_0 \in (r(T), \infty)$. As we mentioned in the proof of Theorem 4.1 (1), for any fixed $e \gg 0$ and $\lambda > r(T)$,

$$x < R_\lambda(c+e) \quad (\forall x \in R_\lambda(c)).$$

By the statement (ii), for a sufficiently small $M > r(T)$ such that $\lambda_n > M\,(\forall n)$,

$$x < R_{\lambda_n}(c+e) < R_M(c+e) \quad (\forall x \in R_{\lambda_n}(c)).$$

Then, in the same way as in the proof of Theorem 4.1 (1), we can choose a converging subsequence $\{x_{n_j}\}$ of $\{x_n\}$ where $x_n \in R_{\lambda_n}(c)$, the limit of which is contained in $R_{\lambda_0}(c)$. Thus we can show that $R_\lambda(c)$ is upper semicontinuous on $(r(T), \infty)$.

Last we prove that the statement (iii). Fix $c \gg 0$ and assume that there exists a sequence $\{R_{\lambda_n}(c)\}$ such that $\|R_{\lambda_n}(c)\|$ is bounded and $\lambda_n \downarrow r(T)$. Since T is compact, $\{TR_{\lambda_n}(c)\}$ contains a converging subsequence $\{TR_{\lambda_{n_j}}(c)\}$, the limit of which we denote by x_0. Then $\lambda_n R_{\lambda_n}(c) = TR_{\lambda_n}(c) + c$ implies, as $n_j \to \infty$,

$$\lambda_{n_j} R_{\lambda_{n_j}}(c) \to x_0 + c \gg 0.$$

Thus we can see that $r(T) > 0$ and

$$R_{\lambda_{n_j}}(c) \to \frac{x_0 + c}{r(T)} \quad (n_j \to \infty).$$

Hence $T(\dfrac{x_0+c}{r(T)}) = x_0$, that is, $T(\dfrac{x_0+c}{r(T)}) + c = r(T) \cdot \dfrac{x_0+c}{r(T)}$. This means $\dfrac{x_0+c}{r(T)} \in R_{r(T)}(c)$, which contradicts to Theorem 4.1 (2). We have completed the proof. □

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