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FLOATING-POINT NUMBER SOLUTIONS IN A SIMPLE LINEAR EQUATION WITH ADDITION ALGORITHM

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Abstract

A model of a floating-point addition $u \bigoplus v = c$ is established in FORTRAN language. Here the exponent of u is greater than the exponent of v. The two kinds of simple linear equations $y \bigoplus v = c$ and $u \bigoplus x = c$ are solved theoretically . Here u, v and c are given floating-point numbers and y and x are unknown floating-point numbers. The affection of round-off error arising from algorithm is analysed to the error of the solutions for two kinds of the linear equations. It is shown that the solution y is machine precision accuracy and x is not always machine precision accuracy.

MODEL OF A FLOATING-POINT ADDITION AND DOMAIN FOR INPUTS

Let $d[e-1,e-t>$ denote a floating-point number with exponent e which has the positions from $e-1$ to $e-t$ with the bit $d(k)$ at k position for $k=e-1,\ldots,e-t$. Here $d(e-1)=1$ and $d(k)=1$ *or* 0 for $k=e-2,\ldots,e-t$. Let T be a set of floating-point numbers of length $t\geq 3$: $T=\{d[e-1,e-t>;-\infty<e<\infty, d>0\}$. A model algorithm of $u\bigoplus v$ is defined as a mapping from $T\times T$ to T for a pair (u,v) in $T\times T$. The algorithm is defined as follows: (1) v is correctly rounded so as to the last significant position is $e(u)-t$ and this value is denoted as v_c: $v_c=v[e(v)-1,e(u)-t))+v(e(u)-t-1)b^{e(u)-t}$. (2) u and v_c are added: $u+v_c=$

$$u[e(u)-1,e(u)-t>+v[e(v)-1,e(u)-t))+v(e(u)-t-1)b^{e(u)-t}.$$

Here $v[e(v)-1,e(u)-t))$ is the sub power series in positions from $e(v)-1$ to $e(u)-t$. Our algorithm is the case that $e(u+v_c)=e(u)+1$. $u+v_c$ is $t+1$ length, $u+v_c$ is chopped to t length. The floating-point addition is given as

$$u\bigoplus v=u[e(u)-1,e(u)-t+1))+v[e(v)-1,e(u)-t+1))$$
$$+C(u(e(u)-t),v(e(u)-t),v(e(u)-t-1))b^{e(u)-t+1},$$

where $C(x,y,z)$ is the carry to heigher poisition in $x+y+z$ defined as $xy\vee(x\oplus y)z$. Here xy, $x\vee y$ and $x\oplus y$ denote respectively AND, OR and Exclusive-OR operations in Boolean functions.

The algorithm is introduced for one case such that

$$e(u+v_c)=e(u)+1, e(v)+t-2\geq e(u)\geq e(v)+1, uv>0.$$

Let E denote the domain for inputs u and v, and then

$$E=\{(u,v):e(u+v_c)=e+1, 2\leq i\leq t-1, uv>0\},$$

putting $e(u)=e$, $e(c)=e+1$ and $e(v)=e-i+1$.

ROUND-OFF ERROR OF THE FLOATING-POINT ADDITION

The round-off error of the floating-point addition is defined as $u+v-u\bigoplus v$ and denoted as $\delta(u,v)$. $\delta(u,v)$ is calculated as

$$\delta(u,v)=v((e-t,e-i+1-t>$$
$$-C(u(e-t),v(e-t),v(e-t-1))b^{e-t+1}+u(e-t)b^{e-t}.$$

$v((e-t,e-i+1-t>$ denote a sub power series of v in positions from $e-t$ to $e-i+1-t$. Since $\delta(u,v)$ depends only on $u(e-t)$, $\delta(u,v)$ is also denoted as $\delta'(u(e-t),v)$.

TRANSPOSED EQUATION $y \oplus v = c$

The linear equation $y + v = c$ is transposed on a computer as $y \oplus v = c$. The transposed equation $y \oplus v = c$ has two solutions:

$$y[e-1, e-t> = c - v[e-i, e-t+1)) + y(e-t)b^{e-t}$$
$$-C(y(e-t), v(e-t), v(e-t-1))b^{e+1-t},$$

corresponding to the bits $y(e-t) = 0, 1$ in the least significant position of y. Here (v, c) is in the trapezoid

$$S(i, 0) = \{b^{e-i+1} - b^{e-i+1-t} \geq v \geq b^{e-i},$$
$$b^e - b^{e+1-t} + G(0, v) \geq c \geq b^e\}.$$

$G(j, v)$ is a rounding step function in v which is a monotone increasing step function with step width $p = b^{e+1-t}$ for $j = 0, 1$:

$$G(j, v) = v[e-i, e-i-t+1)) + C(j, v(e-t), v(e-t-1))b^{e+1-t}.$$

For (v, c) in the set

$$S(i,1) - S(i,0)$$
$$= \{v(e-t) \oplus v(e-t-1) = 1, c = b^e + v[e-i, e-t+1))\},$$

the equation $u \oplus v = c$ has one solution corresponding to $y(e-t) = 1$.

TRANSPOSED EQUATION $u \oplus x = c$

The linear eqution $u + x = c$ is transposed on a computer as $u \oplus x = c$. The transposed equation $u \oplus x = c$ has at most 2^{i-t} solutions corresponding to the ways of choosing the bits in $x((e-t, e-i+1-t>$:

$$x[e-i, e-i+1-t> = c[e, e+1-t> - u[e-1, e+1-t))$$
$$-C(u(e-t), x(e-t), x(e-t-1))b^{e-t+1} + x((e-t, e-i+1-t> .$$

The following theorem shows the number of solutions x for $u \oplus x = c$.

Theorem 1 *Let i and n be integers such that*

$$c[e, e-t+1> - u[e-1, e-t+1)) = b^{e-i} + nb^{e-t+1}.$$

Let IN be an integer such that

$$IN = b^{t-2} - u((e-2, e+1-t))b^{-e-1+t} - b^{t-i-1}.$$

Let NUM denote the number of the solutions of $u \oplus x = c$. Then NUM is given as follows.

1. *If $(u, c) \in S_1'(i)$ then $0 \leq n \leq b^{t-i-1}$ and*

$$NUM = (1-j)b^{i-1} + b^{i-2} \textit{ for } n = 0;$$
$$NUM = b^i \textit{ for } 1 \leq n \leq b^{t-i-1} - 1;$$
$$NUM = b^{i-2} + jb^{i-1} \textit{ for } n = b^{t-i-1}.$$

Here $S_1'(i)$ is the trapesoid

$$S_1'(i) = \{b^{e-i+1} \geq c - u[e-1, e-t+1)) \geq b^{e-i},$$
$$b^e - b^{e-t} \geq u \geq b^e - b^{e-i}\}$$

2. *If $(u,c) \in S_2'(i)$ then $IN \leq n \leq b^{t-i-1}$ and*

$$NUM = b^i \ for \ b^{t-i-1} - 1 \geq n \geq IN;$$
$$NUM = b^{i-2} + jb^{i-1} \ for \ n = b^{t-i-1}.$$

Here $S_2'(i)$ is the trapesoid

$$S_2'(i) = \{b^{e-i+1} + u[e-1, e-t+1)) \geq c \geq b^e,$$
$$b^e - b^{e-i} - b^{e-t} \geq u \geq b^e - b^{e-i+1} + b^{e+1-t}\}.$$

3. *If $c = b^e$ and $u = b^e - b^{e-i+1} + jb^{e-t}$ with $j = 0$ or $j = 1$, then $IN = b^{t-i-1}$ and*

$$NUM = b^{i-2} + jb^{i-1}.$$

COMPARISON OF ROUND-OFF ERROR FOR TWO SOLUTIONS FOR $u \bigoplus x = c$ and $y \bigoplus v = c$

Since $u \bigoplus x = c$,

$$\varepsilon(u,x) = (c-u) - x = (u \bigoplus x - u) - x$$
$$= -\delta'(u,x) = -\delta'(u(e-t), x).$$

Since $y \bigoplus v = c$,

$$\varepsilon(y,v) = (c-v) - y = (y \bigoplus v - v) - y$$
$$= -\delta(y,v) = -\delta'(y(e-t), v).$$

The following theorem shows that the maximum round-off error of the solutions for

$$u \bigoplus x = c \ and \ y \bigoplus v = c,$$

is the same $b^{e-t} + b^{e-t-1} - b^{e-t-i+1}$. Let $D'(t)$ be a set defined as

$$D'(t) = \{v[e-i, e-i+1-t>\}.$$

Theorem 2 *1. The error function $\varepsilon(y,v)$ in v and the error function $\varepsilon(u,x)$ in x are expressed as the same functions $-\delta'(j,\cdot)$ if $y(e-t) = u(e-t) = j$.*

2. *The function $\delta'(j,v)$ is a piecewise linear periodic function as follows with period $p = b^{e-t+1}$:*

   (a) *$\delta'(j,v)$ is a periodic function with period p:*

   $$\delta'(j, v+p) = \delta'(j,v).$$

   (b) *The figure of $\delta'(j,v)$ on the initial half-open interval I is given as follows:*

   $$\delta'(j,v) = v - b^{e-i} + jb^{e-t}$$
   $$on \ \{b^{e-i} \leq v \leq s_1(j,i) - b^{e-i+1-t}\};$$
   $$\delta'(j,v) = v - s_1(j,i) - b^{e-t-1}$$
   $$on \ \{s_1(j,i) \leq v \leq b^{e-i} + p - b^{e-i+1-t}\}.$$

   Here the initial switching point is

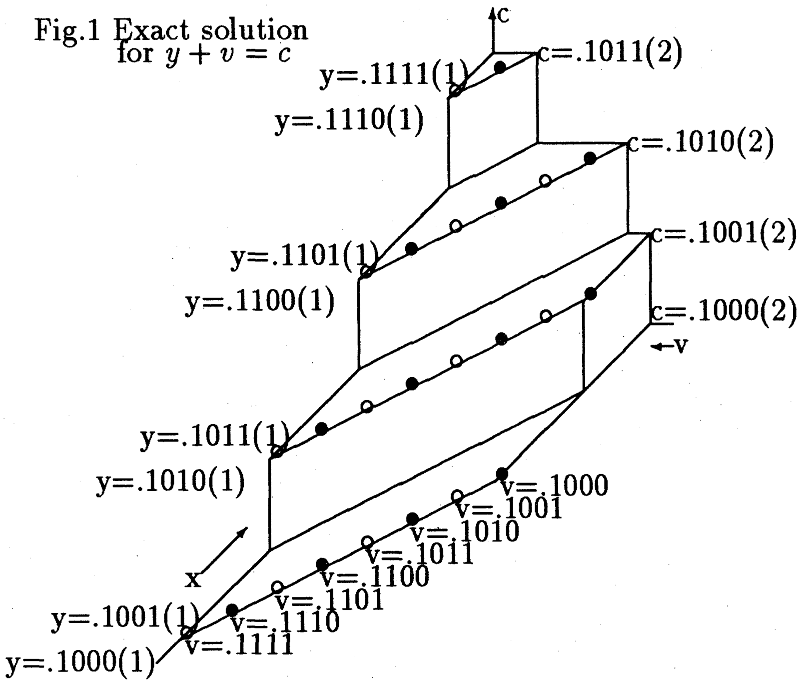
   $$s_1(j,i) = b^{e-i} + b^{e-t-1} + jb^{e-t}.$$

3. *Let $e(v) = e(x) = e - i$. The maximum of $\mid \delta'(j,v) \mid$ in v is given as follows:*

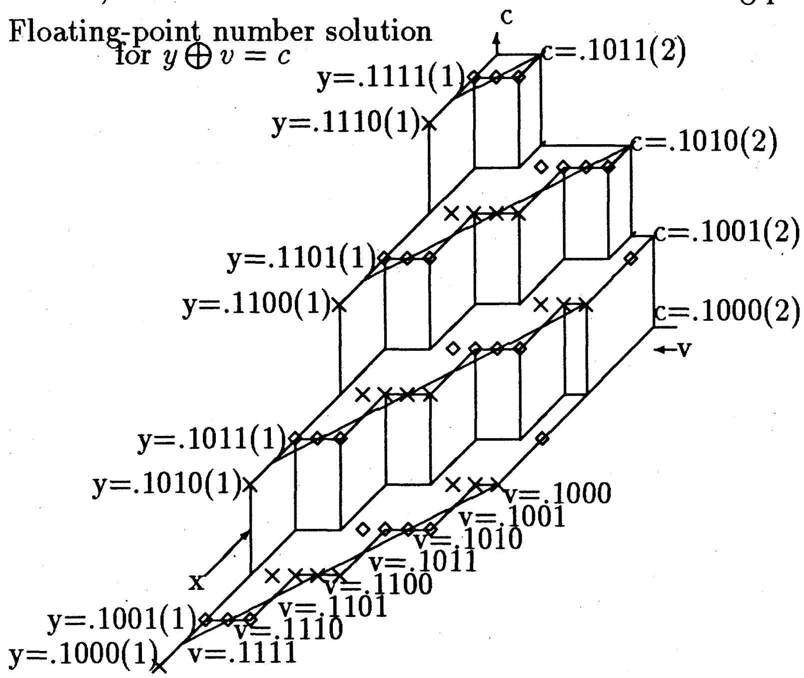
   $$max_{v \in D'(t)} \mid \delta'(j,v) \mid = b^{e-t} + b^{e-t-1} - b^{e-t-i+1},$$

   which is attained at $v = s_n(j,i) - b^{e-t-i+1}$. Here the switching points are

   $$s_n(j,i) = s_1(j,i) + (n-1)p.$$



o:Exact solution, •: Exact solution which coincides with floating-point number solution.



◇ Floating-point number solution (bit is 1 at -3 position), × Floating-point number solution (bit is 0 at -3 position)

By theorem 2, the following results are obtained.

Theorem 3 *1. The maximum error is* $b^{e-t} + b^{e-t-1} - b^{e-t-i+1}$.

2. The maximum error is monotone increasing from b^{e-t} *to* $b^{e-t} + b^{e-t-1} - b^{e-2t+2}$ *as* i *increases from* $i = 2$ *to* $i - t - 1$.

3. The ISP is $s_1(j,i) = b^{e-i} + b^{e-t-1} + jb^{e-t}$. *The distance of ISP and the initial point* $v = b^{e-i}$ *is independent in* i.

4. The period is b^{e-t+1} *and is independent in* i.

5. The number of oscillations in $D'(t)$ *is* b^{t-i-1} *and decreases as* i *increases.*

MACHIN PRECISION ACCURACY

The machine precision accuracy means that the error takes the positions less than the least significant position $e(X) - t + 1$ *for the exact solution* X.

Definition 1 *The error* ε *of the solution is called "machine precision accuracy", if the error* ε *satisfies*

$$\mid \varepsilon \mid < b^{e(X)-t+1} \text{ for the exact solution } X.$$

Theorem 4 *1. The error of the solution* y *is machine precision accuracy for any* v *and* c: $max_{x \in D'(t)} \mid \varepsilon(j,v) \mid < b^{e-t+1}$.

2. The error of the solution x *is not always machine precision accuracy for given* c *and* u *. The maximum of the round-off error is*

$$max_{x \in D'(t)} \mid \varepsilon(j,x) \mid \geq b^{e-i-t+2} = b^{e(\hat{x})-t+1}$$

for the exact solution $\hat{x}$.

MAXIMUM RELATIVE ERROR

The relative error function $r(y,v)$ *in* v *is defined as*

$$r(y,v) = \frac{-\varepsilon(y,v)}{c - v}.$$

The relative error function $r(u,x)$ *in* x *is defined as*

$$r(u,x) = \frac{-\varepsilon(u,x)}{c - u}.$$

In order to use the linearity of the function $\delta'(j,x)$ *in* x, $r(u,x)$ *is rewritten as*

$$r(u,x) = \frac{\delta'(j,x)}{x - \delta'(j,x)}.$$

Theorem 5 *1. The relative error function* $r(y,v)$ *has the following properties:*

(a) $r(y,v_1) < r(y,v_2)$ *for* $v_2 = v_1 + p$.

(b) On $I(n)$, *the relative error function* $r(y,v)$ *is a piecewise monotone increasing convex function given as*

$$r(y,v) = \frac{v - (n-1)p - b^{e-i} + jb^{e-t}}{c - v} \text{ on } I_0^j(n);$$

$$r(y,v) = \frac{v - s_n(j,i) - b^{e-t-1}}{c - v} \text{ on } I_1^j(n);$$

2. *The relative error function $r(u,x)$ has the following properties:*
   (a) $r(u,x_1) > r(u,x_2)$ *for* $x_2 = x_1 + p$.
   (b) *On $I(n)$, the relative error function $r(u,x)$ is a piecewise increasing linear function given as*

$$r(u,x) = \frac{x-(n-1)p-b^{e-i}+jb^{e-t}}{(n-1)p+b^{e-i}-jb^{e-t}} \text{ on } I_0^j(n);$$

$$r(u,x) = \frac{x-s_n(j,i)-b^{e-t-1}}{s_n(j,i)+b^{e-t-1}} \text{ on } I_1^j(n);$$

The maximum of the relative error function $r(y,v)$ with respect to v is defined as

$$mr(y) = max._{v\in D'(t)} \mid r(y,v) \mid .$$

The maximum of relative error function $r(u,x)$ with respect to x is defined as

$$mr(u) = max._{x\in D'(t)} \mid r(u,x) \mid .$$

In the following theorem, two maximum of relative errors $mr(y)$ and $mr(u)$ are compared.

Theorem 6 *1. The maximum relative error of the solution y is attained at $v = s_N(j,i) - b^{e-i+1-t}$. Here $s_N(j,i)$ is the last switching point and $N = b^{t-i-1}$.*

2. *The maximum relative error of y is evaluated as*

$$\begin{aligned} mr(y) &= r(y, s_N - b^{e-i+1-t}) \\ &= \frac{b^{e-t}+b^{e-t-1}-b^{e-t-i+1}}{c-s_1(j,i)-(N-1)p+b^{e-i+1-t}} \\ &< b^{-t+1}(1-b^{-t+j}) \text{ for } j = 0,1. \end{aligned}$$

3. *The maximum relative error of x is attained at $x = s_1(j,i) - b^{e-i+1-t}$ for the initial switching point $s_1(j,i)$.*
4. *The maximum relative error of x is evaluated as*

$$mr(u) = r(u, s_1 - b^{e-i+1-t}) = \frac{b^{i-t}(1+b^{-1})-b^{-t+1}}{1-jb^{i-t}}.$$

The following theorem shows that the maximum relative error $mr(u)$ is monotone increasing as the difference $i-1$ of exponents $e(u) = e$ and $e(x) = e-i+1$ increases.

Theorem 7 *Put the maximum relative error of x as mr*

$$mr = \frac{b^{i-t}(1+b^{-1})-b^{-t+1}}{1-jb^{i-t}}.$$

Then mr is monotone increasing from

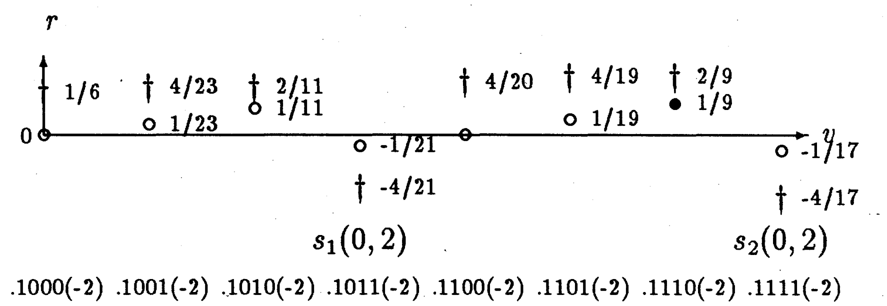
$$\frac{b^{2-t}}{1-jb^{2-t}} \text{ to } \frac{b^{-1}+b^{-2}-b^{-t+1}}{1-jb^{-1}}$$

as i increases from $i = 2$ to $i = t-1$.

FIGURES OF RELATIVE ERROR FUNCTION

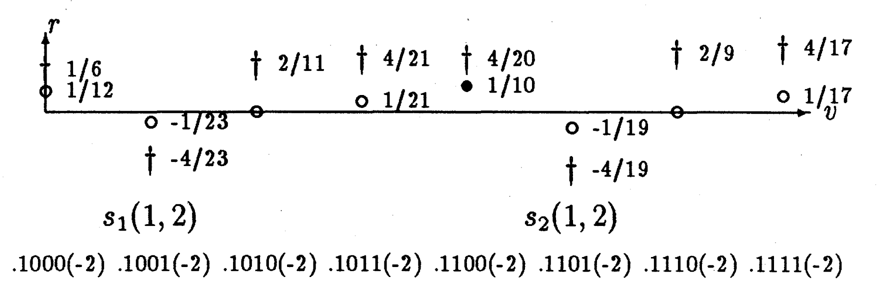
The following figures Fig.3, Fig.4, Fig.5 and Fig. 6 show the relative error functions $r(j,v)$ *of the solution* y *for the equation* $y \oplus v = c$ *for* $j = 0, 1$, $i = 2, 3$ *and* $t = 4$. *The results in theorems 4, 5, and 6 are visualized by the figures.*

1. *The relative error functions are the piecewise monotone increasing convex functions.*
2. *The switching points are coincide with those of the round-off error functions (see the figures of round-off error functions in [?]).*
3. *The maximum of relative error is taken in the last interval* $I(N)$ *with period* p. *Here the point of the maximum relative error is denoted by "•".*
4. *The point which attains the maximum relative error is the left side point adjacent to the switching point* $s_N(i,j)$.
5. *The round-off errors are all machine precision accuracy. Here the relative error such that the round-off error is machine precision accuracy, is denoted by "†".*



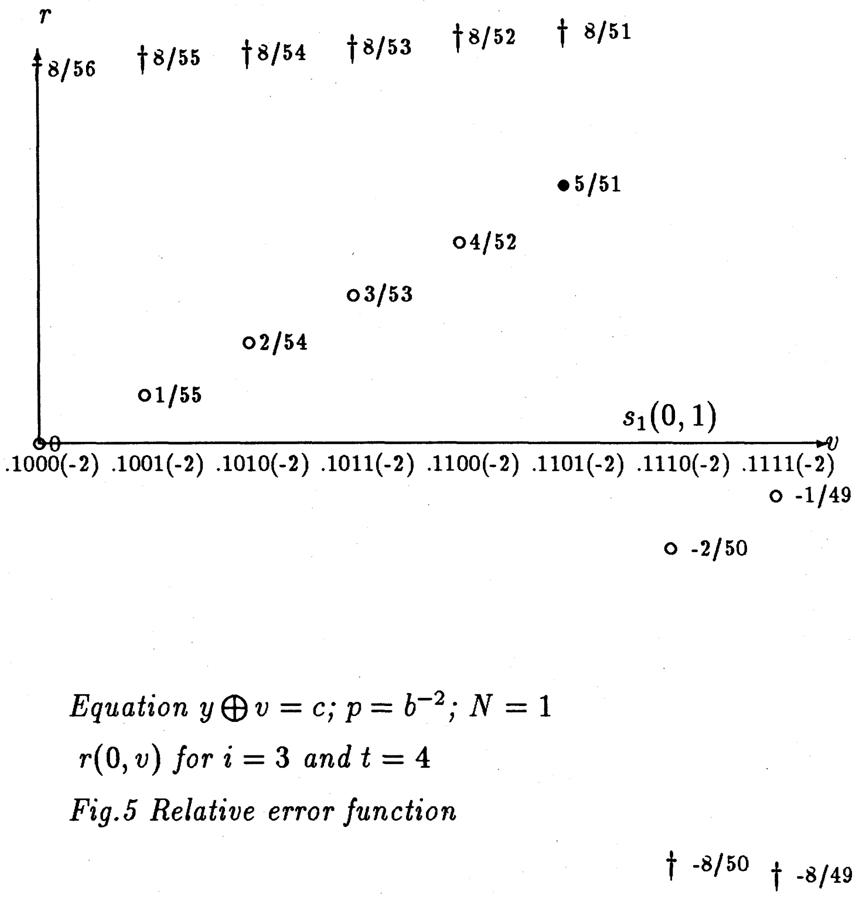
Equation: $y \oplus v = c$*;Period* $p = b^{-3}$*;*$N = 2$

Fig.3 Relative error function $r(0,v)$ *for* $i = 2$ *and* $t = 4$



y:solution of $y \oplus v = c$*;* $p = b^{-3}$*;*$N = 2$

Fig.4 Relative error function $r(1,v)$ *for* $i = 2$ *and* $t = 4$



Equation $y \oplus v = c$; $p = b^{-2}$; $N = 1$

$r(0, v)$ *for* $i = 3$ *and* $t = 4$

Fig.5 Relative error function

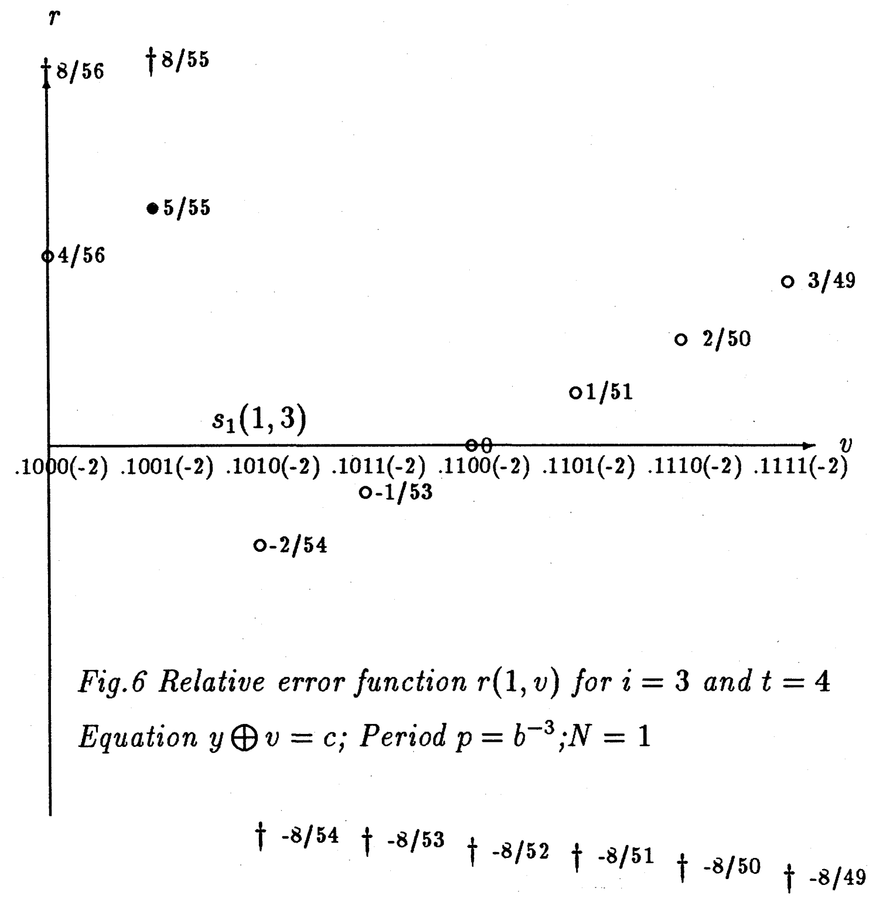


Fig.6 Relative error function $r(1,v)$ for $i=3$ and $t=4$

Equation $y \bigoplus v = c$; Period $p = b^{-3}$;$N = 1$

The following figures Fig.7, Fig.8 , Fig. 9 and Fig. 10 show the relative error functions $r(j,x)$ of the solution x for the equation $x \bigoplus v = c$ for $j = 0,1$, $i = 2,3$ and $t = 4$. The results in theorems 4, 5, 6 and 7, are visualized by the figures.

1. *The relative error functions are the piecewise increasing linear functions.*
2. *The switching points are coincide with those of the round-off error functions (see the figures of round-off error functions in [?]).*
3. *The maximum of relative error is taken in the initial interval $I(1)$ with period p.*
4. *The point which attains the maximum relative error is the point of left side adjacent to the initial switching point $s_1(i,j)$ by one.*
5. *The round-off errors are not always machine precision accuracy. Here the relative error such that the round-off error is machine precision accuracy, is expressed by the line "- - - - ".*
6. *The maximum relative error $mr(u)$ is monotone increasing as the difference $i-1$ increases. The maximum relative error 1/4 in Fig. 7 increases to 5/8 in Fig. 9 as i increases from 2 to 3. The maximum relative error 1/3 in Fig. 8 increases to 5/4 in Fig. 10 as i increases from 2 to 3.*

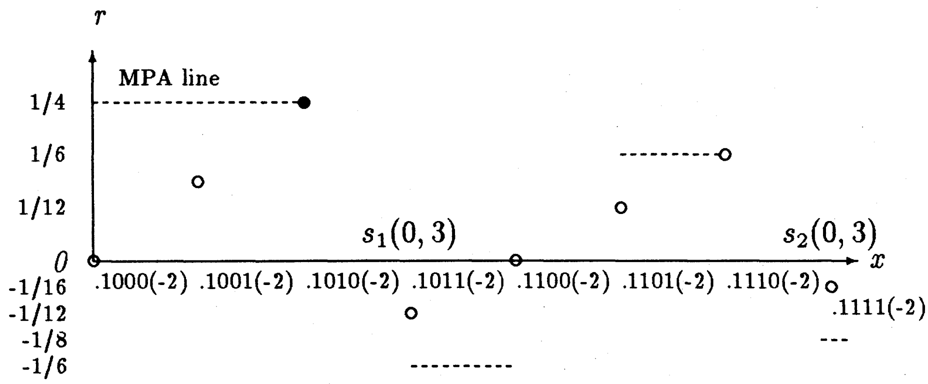


Fig.7 Relative error function $r(0,x)$ for $i=2$ and $t=4$

x:solution of $u \oplus x = c$; $p = b^{-3}$; $N = 2$

The above graph shows that 6 points are machine precision accuracy and the other are not.

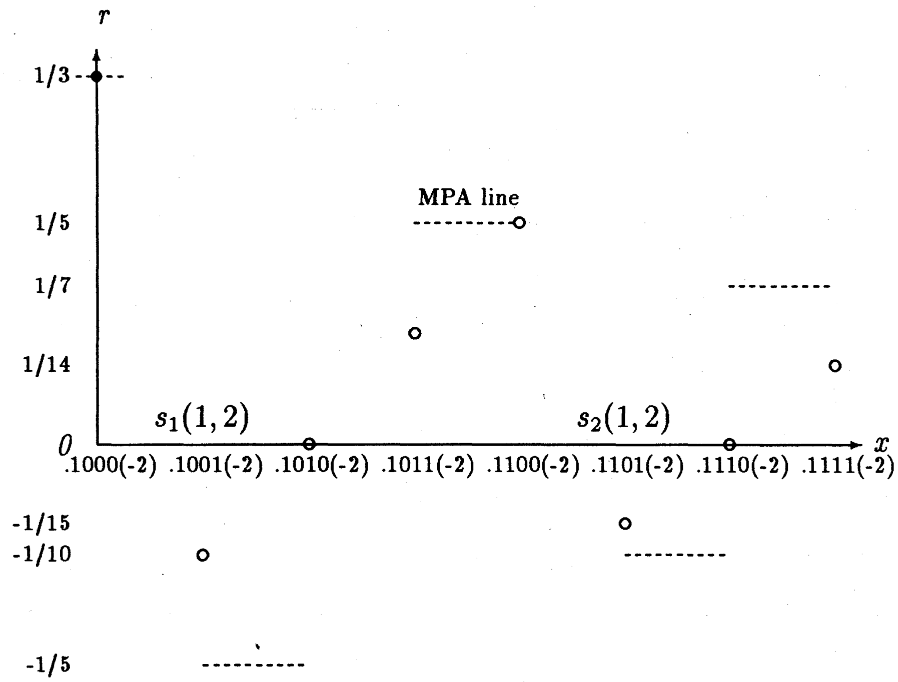


Fig.8 Relative error function $r(1,x)$ for $i=2$ and $t=4$

x:solution of $u \oplus x = c$. $p = b^{-3}$, $N = 2$

The above graph shows that 6 points are machine precision accuracy and the other are not.

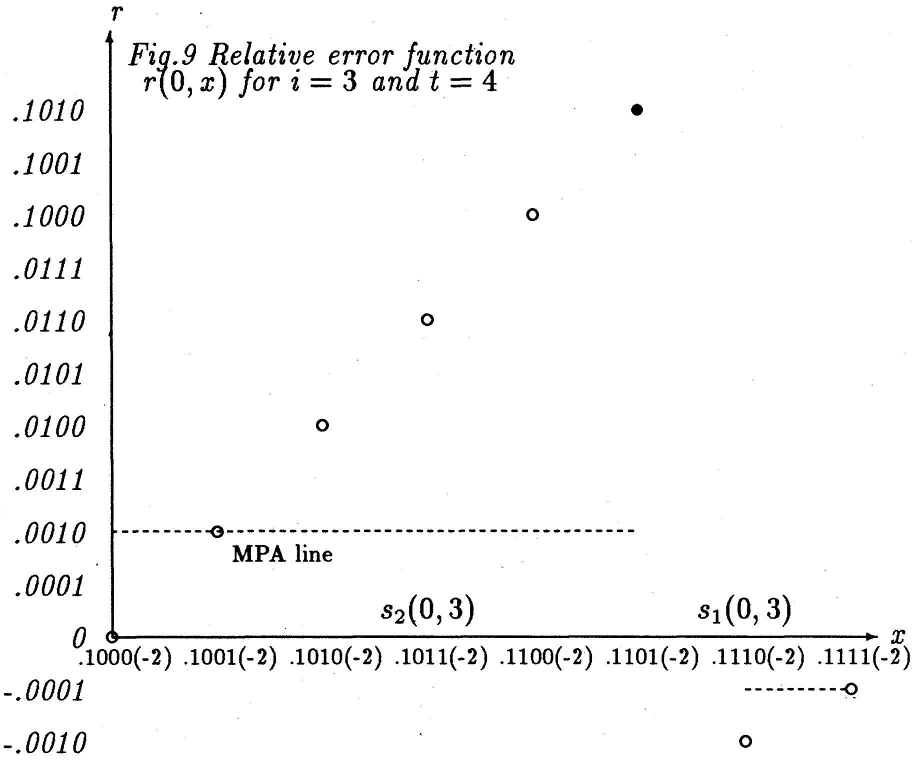


Fig.9 Relative error function $r(0,x)$ for $i=3$ and $t=4$

x:solution of $u \oplus x = c$. $p = b^{-2}$, $N = 1$

The above graph shows that only initial point is machine precision accuracy and the other are not.

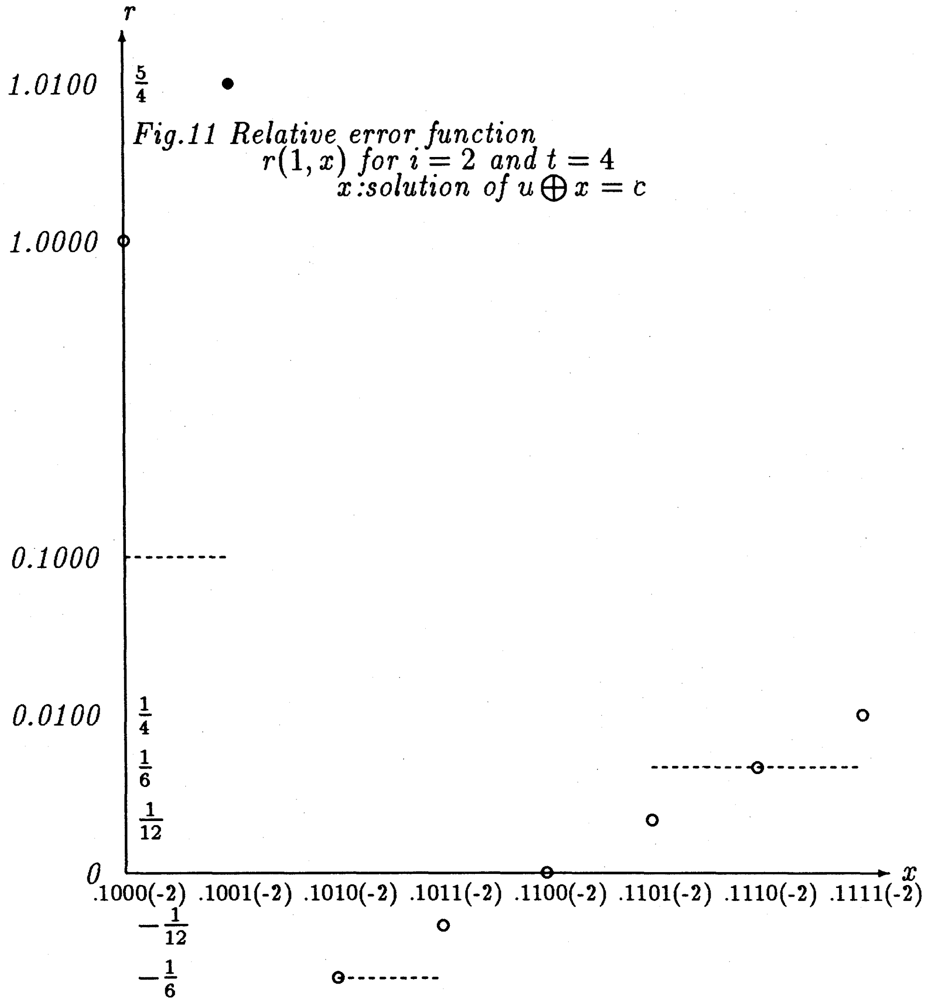


Fig.11 Relative error function $r(1, x)$ for $i = 2$ and $t = 4$
x:solution of $u \oplus x = c$

In Fig.10, the maximum relative error is the vaue 5/4 which is more than 1. This phenomenon is analysed as follows. In this case, for given

(1) $$u[-1, -4>= .1111 \text{ and } c[0, -3>= 1.000,$$

$u \oplus x = c$ is solved as

$$x[-3, -6>= .1001(-2),$$

since

$$\begin{aligned} u \oplus x &= u[-1, -3)) + x[-3, -3)) + C(u(-4), x(-4), x(-5))b^{-3} \\ &= .111 + .100(-2) + C(1, 0, 0)b^{-3} = 1.000. \end{aligned}$$

For given u and c in (1), $u + x = c$ is solved as $\hat{x} = b^{-4}$. The error is $\hat{x} - x[-3, -6>= -.101(-3)$ and the relative error is $\frac{.101(-3)}{b^{-4}} = 5/4$. The exact solution $\hat{x}$ is extraordinally *small. In the calculation of $c - u$, the catastrophic cancellation occurs. In the calculation of $u + x$, the carry propagates from the least significant position to the leading position of c. Thus the round-off error becomes more than the exact solution.*

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