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A Lower Bound of the Expected Maximum Number of Edge-disjoint s-t Paths on Probabilistic Graphs

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Abstract

For a probabilistic graph $(G = (V, E, s, t), p)$, where $G$ is an undirected graph with specified source vertex $s$ and sink vertex $t$ ($s \neq t$) in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p = (p(e_1), ..., p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges $e_i$'s in $E$, we consider the problem of computing the expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s-t paths. It has been known that this computing problem is NP-hard even if $G$ is restricted to several classes like planar graphs, s-t out-in bitrees and s-t complete multi-stage graphs. In this paper, for a probabilistic graph $(G = (V, E, s, t), p)$, we propose a lower bound of $\Gamma_{(G,p)}$ and show the necessary and sufficient conditions by which the lower bound coincides with $\Gamma_{(G,p)}$. Furthermore, we also give a method of computing the lower bound of $\Gamma_{(G,p)}$ for a probabilistic graph $(G = (V, E, s, t), p)$.

1 Introduction

We consider a probabilistic graph $(G = (V, E, s, t), p)$, where $G$ is an undirected graph with specified source vertex $s$ and sink vertex $t$ ($s \neq t$) in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p = (p(e_1), ..., p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges $e_i$'s in $E$. The expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s-t paths (namely, s-t paths having no edge in common) in a probabilistic graph $(G, p)$ is useful for network reliability analysis. Note that the problem of computing s-t-connectedness [1,3], namely, probability that there exists at least one operative s-t path, is a special case of computing $\Gamma_{(G,p)}$ in a probabilistic graph $(G, p)$.

However, it is known that the problem of computing $\Gamma_{(G,p)}$ in a probabilistic graph $(G, p)$ is NP-hard, even if $G$ is restricted to several classes, e.g., planar graphs, s-t out-in bitrees and s-t complete multi-stage graphs [2]. Thus, for estimating $\Gamma_{(G,p)}$, it is interesting for us to find its lower bound in a probabilistic graph $(G, p)$.

In this paper, we define a lower bound of $\Gamma_{(G,p)}$ using an s-t path number function of $G$ for a probabilistic graph $(G, p)$, and give the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$ and a method of computing this lower bound. This paper is organized as follows:

Graph theoretic terminologies used throughout this paper are described in section 2. A lower bound of $\Gamma_{(G,p)}$ in a probabilistic graph $(G,p)$ is defined in section 3. Section 4 shows the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$. Furthermore, we suggest a method of computing the lower bound in section 5.
2 Preliminaries

2.1 Graph Theoretic Terminologies

A two-terminal undirected graph $G = (V, E, s, t)$ consists of a finite vertex set $V$ and a set $E$ of pairs of vertices, called edges, where $s$ and $t$, called source and sink, respectively, are two specified distinct vertices of $V$. For an edge $(u,v)$, the two vertices $u$ and $v$ are said to be end vertices of $(u,v)$, and $(u,v)$ is said to be incident to $u$ and $v$.

In $G = (V, E, s, t)$, an $x$-$y$ path $\pi$ of length $k$ from vertex $x$ to vertex $y$ is an alternating sequence of vertices $v_i \in V$ ($0 \leq i \leq k$) and edges $(v_{i-1}, v_i) \in E$ ($1 \leq i \leq k$),

$$\pi : (x =) v_0, (v_0, v_1), v_1, ..., v_{k-1}, (v_{k-1}, v_k), v_k (= y),$$

where vertices $v_i$'s ($0 \leq i \leq k$) are distinct. i.e., a path denotes a simple path throughout this paper. For short, we also denote an $x$-$y$ path $\pi$ by

$$\pi : (x =) v_0, v_1, ..., v_k (= y).$$

The vertices $v_1, ..., v_{k-1}$ are called its internal vertices and the vertices $v_0 (= s), v_k (= t)$ are called its end vertices. Let $V(\pi)$, $E(\pi)$ denote the set of all vertices and the set of all edges on an $x$-$y$ path $\pi$, respectively. The set of all $x$-$y$ paths in $G$ is denoted by $P_{xy}(G)$. Paths $\pi_1, ..., \pi_r$ are called internal vertex-disjoint paths if they have no vertex in common except their end vertices. s-t paths $\pi_1, ..., \pi_r$ are called edge-disjoint s-t paths if any two of them have no edge in common, and the maximum number of edge-disjoint s-t paths in $G$ is denoted by $\lambda_{st}(G)$.

A graph $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E, s, t)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$ hold. If $G_1$ is a subgraph of $G$, other than $G$ itself, then $G_1$ is a proper subgraph of $G$. For a subset $E' \subseteq E$, the subgraph derived from $G$ by deleting all edges of $E'$ is denoted by $G - E'(= (V, E - E', s, t))$. A subset $E'((\subseteq E)$ is called an s-t edge-cutset if $G - E'$ has no s-t path. An s-t path $\pi$ is an s-t edge-cut-path if $E(\pi)$ is an s-t edge-cutset. An s-t edge-cutset with the minimum cardinality among s-t edge-cutsets of $G$ is said to be minimum. By well-known Menger's theorem [4], $\lambda_{st}(G)$ is equal to the cardinality of a minimum s-t edge-cutset of $G$ for any $G$.

2.2 Probabilistic Graph

A probabilistic graph, denoted by $(G = (V, E, s, t), p)$, or $(G, p)$, for short, is defined as follows:

(i) $G = (V, E, s, t)$ is a two-terminal graph, where each edge $e$ of $E$ is in either of the following two states: failed or operative (not failed), having known independent failure probability $p(e)$, $0 \leq p(e) \leq 1$ (or operative probability $q(e) = 1 - p(e)$), and each vertex is assumed to be failure-free.

(ii) $p$ is a vector consisting of all edge failure probabilities $p(e)$'s in $E$.

For a probabilistic graph $(G = (V, E, s, t), p)$, let a subgraph $G - U(\subseteq E)$ correspond to an event $\mathcal{E}_U$ that all edges of $U$ are failed and all edges of $E - U$ are operative. Clearly, the probability $\rho(G - U)$ of arising a subgraph $G - U(\subseteq E)$ is computed by the following formula.

$$\rho(G - U) = \prod_{e \in U} p(e) \prod_{e \in E - U} q(e) (= 1 - p(e)).$$

Furthermore, $\sum_{U \subseteq E} \rho(G - U) = 1$ holds.

Now, we define the expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s-t paths in a probabilistic graph $(G = (V, E, s, t), p)$ as follows:

$$\Gamma_{(G,p)} \equiv \sum_{U \subseteq E} \lambda_{st}(G - U) \rho(G - U). \quad (1)$$
It is known that the problem of computing $\Gamma_{(G,p)}$ for a probabilistic graph $(G,p)$ is NP-hard, even if $G$ is restricted to several special classes like planar graphs, s-t out-in bitrees and s-t multi-stage complete graphs, etc. [2]. Thus, it is interesting for us to consider a lower bound of $\Gamma_{(G,p)}$ for estimating it.

3 A Lower Bound of $\Gamma_{(G,p)}$

We define a lower bound of the expected maximum number of edge-disjoint s-t paths in a probabilistic graph.

An s-t path number function $f$ of $G = (V, E, s, t)$ is a one-to-one integral function $f : P_{st}(G) \rightarrow \{1, \ldots, l\}$. The s-t path $\pi$ with $f(\pi) = k$ is said to be the s-t path of number $k$, and denoted by $\pi_k$. The s-t path with the minimum number in $G - E'(\subseteq E)$ with respect to $f$ is denoted by $\pi_{m(G-E',f)}$.

First, we give the following procedure FEDP to find edge-disjoint s-t paths in $G = (V, E, s, t)$.

Procedure FEDP

**Input** A graph $G = (V, E, s, t)$ and an s-t path number function $f$ of $G$.

**Output** The set of edge-disjoint s-t paths $FEDP(G, f)$.

BEGIN

$G' := G; \ FEDP(G, f) := \phi$;
WHILE $P_{st}(G') \neq \phi$ DO
BEGIN

Find $\pi_{m(G',f)}$ from $P_{st}(G')$;
$FEDP(G, f) := FEDP(G, f) \cup \{\pi_{m(G',f)}\};$
$G' := G' - E(\pi_{m(G',f)})$
END;

Output $FEDP(G, f)$

END.

It is clear that $FEDP(G, f)$ obtained by FEDP is a set of edge-disjoint s-t paths in $G$. Namely, the following formula holds.

$$|FEDP(G, f)| \leq \kappa_{st}(G), \ for \ any \ G, \ f. \ \ (2)$$

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s-t path number function $f$ of $G$, we now define the value $\underline{\Gamma}_{(G,f,p)}$ as follows:

$$\underline{\Gamma}_{(G,f,p)} \equiv \sum_{U \subseteq E} |FEDP(G - U, f)|p(G - U). \ \ (3)$$

By formulas (1),(2),(3), $\underline{\Gamma}_{(G,f,p)}$ is a lower bound of $\Gamma_{(G,p)}$, namely, the following formula holds.

$$\underline{\Gamma}_{(G,f,p)} \leq \Gamma_{(G,p)}, \ for \ any \ G, \ f, \ p.$$
4.1 A Necessary and Sufficient Condition of an s-t Path Number Function

By formulas (1),(2),(3), the following Theorem 4.1 immediately holds.

**Theorem 4.1.** Given \((G = (V, E, s, t), p)\), then \(\Gamma(G, f, p) = \Gamma(G, p)\) holds iff \(G\) has an s-t path number function \(f\) satisfying the following formula.

\[ |FEDP(G - U, f)| = \lambda_{st}(G - U), \text{ for any } U \subseteq E. \tag{4} \]

\[ \square \]

**Definition 4.1.** An s-t path number function \(f\) of \(G\) is called *exact* if \(f\) satisfies formula (4). \(\square\)

A graph \(G = (V, E, s, t)\) is said to be an s-t \(k\)-edge-connected if \(\lambda_{st}(G) = k\) holds. A graph \(G\) is called to be \(\pi\)-edge-cut if \(\pi\) is an s-t edge-cut-path in \(G\). A graph \(G\) is said to be \(\pi\)-edge-cut s-t 2-edge-connected if \(\pi\) is an s-t edge-cut-path of \(G\) and \(G\) is s-t 2-edge-connected. A \(\pi\)-edge-cut s-t 2-edge-connected graph \(G = (V, E, s, t)\) is minimal, if \(G - \{e\}\) for any \(e \in E - E(\pi)\) is not \(\pi\)-edge-cut s-t 2-edge-connected. For example, the graph \(G\) shown in Fig.1 is a \(\pi\)-edge-cut s-t 2-edge-connected graph, where \(\pi : v_0(=s), v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9(=t)\). But it is not minimal as \(G - \{e\}\) is \(\pi\)-edge-cut s-t 2-edge-connected. Furthermore, the set of all \(\pi\)-edge-cut s-t 2-edge-connected subgraphs of an s-t path \(\pi\) of \(G\) is denoted by \(W(G, \pi)\). For example, in the graph \(G\) given in Fig.1, \(W(G, \pi) = \{G - \{e = (u_1, u_2)\}, (G - \{(u_1, v_4), (u_2, v_5), (v_3, v_5)\}\). Clearly, the following Lemma 4.1 holds.

**Lemma 4.1.** If \(\lambda_{st}(G) \geq 2\) holds and an s-t path \(\pi\) of \(G\) is an s-t edge-cut-path, then \(W(G, \pi) \neq \phi\) holds. \(\square\)

**Lemma 4.2.** In a graph \(G = (V, E, s, t)\), if there exists an s-t path \(\pi\) satisfying \(W(G, \pi) = \phi\), then the following formula holds.

\[\lambda_{st}(G - E(\pi)) = \lambda_{st}(G) - 1.\]

**Proof.** Clearly, \(\lambda_{st}(G - E(\pi)) \leq \lambda_{st}(G) - 1\) holds. Assume that \(\lambda_{st}(G - E(\pi)) < \lambda_{st}(G) - 1\) holds. By this assumption, there exists a minimum s-t edge-cutset \(E^*\) in \(G - E(\pi)\) that satisfies \(|E^*| \leq \lambda_{st}(G) - 2\) by Menger's Theorem [4]. Consider graph \(G - E^*\), and it is clear that all s-t paths in \(G - E^*\) share at least one edge of \(E(\pi)\), i.e., \(\pi\) is an s-t edge-cut-path of \(G - E^*\). Furthermore, let \(E'\) be a minimum s-t edge-cutset of \(G - E^*\). As \(E' \cup E^*\) is an s-t edge-cutset of \(G\), \(|E' \cup E^*| = |E'| + |E^*| \geq \lambda_{st}(G)\) holds. By \(|E^*| \leq \lambda_{st}(G) - 2\), we obtain \(|E'| = \lambda_{st}(G - E^*) \geq 2\), contradicting the fact that \(W(G, \pi) \neq \phi\) holds by Lemma 4.1. \(\square\)

We now prove the following Theorem 4.2.
Theorem 4.2. In a graph $G = (V, E, s, t)$, an s-t path number function $f$ of $G$ is exact iff for any $U \subseteq E$ with $P_{st}(G - U) \neq \phi$, $W(G - U, \pi_{m(G-U,f)}) = \phi$ holds.

Proof. Necessity: Assume that an s-t path number function $f$ of $G$ is exact and that for some $U \subseteq E$ with $P_{st}(G - U) \neq \phi$, $W(G - U, \pi_{m(G-U,f)}) \neq \phi$ holds. By $W(G - U, \pi_{m(G-U,f)}) \neq \phi$, $G - U$ has a subgraph $G' \in W(G - U, \pi_{m(G-U,f)})$. $\lambda_{st}(G') = 2$ holds by the definition of $W(G - U, \pi_{m(G-U,f)})$. As $\pi_{m(G-U,f)}$ is the s-t path with the minimum number of $G'$ and an s-t edge-cut-path of $G'$, we have $FEDP(G', f) = \{\pi_{m(G-U,f)}\}$ by FEDP. Hence, $|FEDP(G', f)| (\neq 1) < \lambda_{st}(G') (\neq 2)$ holds, contradicting the fact that $f$ is exact.

Sufficiency: Assume that for any $U \subseteq E$ with $P_{st}(G - U) \neq \phi$, $W(G - U, \pi_{m(G-U,f)}) = \phi$ holds. Then it is easy to prove that for any $U \subseteq E$, $|FEDP(G - U, f)| = \lambda_{st}(G - U)$ holds by iteratively applying Lemma 4.2. \qed

4.2 A Necessary and Sufficient Condition of s-t Paths

Definition 4.2. (Prohibitive s-t Path Set)

Let $P(\subseteq P_{st}(G))$ be a subset of the set of all s-t paths of $G$. If, for each s-t path $\pi$ of $P$, there is a $\pi$-edge-cut s-t 2-edge-connected subgraph $G_{\pi} \in \mathcal{W}(G, \pi)$ in $G$ that satisfies $P_{st}(G_{\pi}) \subseteq P$, then $P$ is called a prohibitive s-t path set.

Procedure TEST

Input: A graph $G = (V, E, s, t)$.

Output: Either an s-t path number function $f$ of $G$ or a subset $P$ of $P_{st}(G)$.

BEGIN

$P := P_{st}(G); \ i := 1; \ Q := \{\pi \in P_{st}(G) | W(G, \pi) = \phi\};$

WHILE $Q \neq \phi$ DO

BEGIN

$P := P - Q;$

REPEAT

Select an s-t path $\pi$ from $Q$;

$f(\pi) := i; \ i := i + 1; \ Q := Q - \{\pi\}$

UNTIL $Q = \phi$;

$Q := \{\pi \in P | P_{st}(G_{\pi}) \subseteq P, \ for \ all \ G_{\pi} \in \mathcal{W}(G, \pi)\}$

END;

IF $P = \phi$ THEN output $f$ ELSE output $P$

END.

Clearly, the following Lemma 4.3 holds by Definitions 4.1 and 4.2.

Lemma 4.3. If TEST outputs an s-t path number function $f$ of $G$, then $f$ is exact, when a graph $G = (V, E, s, t)$ is input. If TEST outputs a subset $P$ of $P_{st}(G)$, then $P$ is a prohibitive s-t path set, when a graph $G = (V, E, s, t)$ is input. \qed

If there is a prohibitive s-t path set $P(\subseteq P_{st}(G))$ where $G = (V, E, s, t)$, then there does not exist any exact s-t path number function $f$. Otherwise, if $G$ has an exact s-t path number function $f$, and suppose $\pi_{m}$ be the s-t path of the minimum number with respect to $f$ among $P$. By Definition 4.2,
there is $G_{\pi_m} \in \mathcal{W}(G, \pi_m)$ in $G$ that satisfies $P_{st}(G_{\pi_m}) \subseteq P$. Thus, $\pi_m$ is also the $s$-$t$ path of the minimum number with respect to $f$ in $G_{\pi_m}$. Therefore, by FEDP, $FEDP(G_{\pi_m}, f) = 1 < \lambda_{st}(G_{\pi_m}) = 2$ holds. This leads to a contradiction that $f$ is an exact $s$-$t$ path number function of $G$. Hence, by Theorem 4.2 and Lemma 4.3, the following Theorem 4.3 holds.

**Theorem 4.3.** In a graph $G = (V, E, s, t)$, $G$ has an exact $s$-$t$ path number function iff it contains no prohibitive $s$-$t$ path set as its $s$-$t$ path subset. $\square$

### 4.3 Characterization of Graph Having a Prohibitive $s$-$t$ Path Set

A graph is connected if there is a path connecting each pair of vertices and otherwise disconnected.

A connected component of $G$ is a maximal connected subgraph, which is simply called a component. If there exist vertices $x$ and $y$, $x \neq v$ and $y \neq v$ such that all the paths connecting $x$ and $y$ have $v$ as an internal vertex, then $v$ is an *articulation vertex*. A two-terminal connected graph is said to be $s,t$ *non-separable* if its subgraph obtained by removing $s,t$ is connected. In the following discussion, we assume that $G$ is an $s,t$ non-separable two-terminal connected graph, unless otherwise specified.

**Definition 4.3.** *(s-t 2-edge-connected Articulation Vertex)*

A vertex $v$ is said to be an *s-t 2-edge-connected articulation vertex* of $G$, if $v$ is an s-t articulation vertex of $G$ and there exist both two edge-disjoint $s$-$v$ paths and two edge-disjoint $v$-$t$ paths in $G$. $\square$

For example, in the graph illustrated in Fig.2(a), vertices $u, v, w$ are s-t 2-edge-connected articulation vertices of $G$.

![Diagram](https://via.placeholder.com/150)

**Definition 4.4.** *(Separation of $G$ at an s-t 2-edge-connected Articulation Vertex)*

![Diagram](https://via.placeholder.com/150)

Fig.2 An illustration of separation of $G$ at an s-t 2-edge-connected articulation vertex.
Assume that $G$ has an s-t 2-edge-connected articulation vertex $v$. The following sequence of operations is said to be separation of $G$ at an s-t 2-edge-connected articulation vertex $v$.

(i) The two components $C_1$ and $C_2$ are obtained by removing $v$ from $G$.
(ii) $v$ is connected to $C_1$ (or $C_2$) with all edges $(u,v)$'s of $G$ having one end vertex $u$ in $C_1$ (or $C_2$).
(iii) Note that $C_1$ contains either of $s,t$. If $C_1$ contains $s$ (or $t$) then let $s$ (or $t$) be $s_1$ (or $t_1$) and let $v$ be $t_1$ (or $s_1$). $s_2$ and $t_2$ are similarly defined for $C_2$. □

For example, the two graphs illustrated in Fig.2(b),(c) are obtained by separation of the graph given in Fig.2(a) at an s-t 2-edge-connected articulation vertex $v$.

\textbf{Definition 4.5. (Prohibitive Graph)}
A graph $G$ is said to be a prohibitive graph, if $G$, or one of the graphs derived from $G$ by separations of $G$ at all s-t 2-edge-connected articulation vertices in $G$ is homeomorphic to the graph shown in Fig.3. □

The two graphs illustrated in Fig.2(a),(b) are both prohibitive graphs. But the graph given in Fig.2(d), although it contains a subgraph homeomorphic to the graph shown in Fig.3, is not a prohibitive graph as the vertex $u$ is not its s-t 2-edge-connected articulation vertex and it is not homeomorphic to the graph shown in Fig.3. It is easy to verify that for a prohibitive graph $G$, $P_{st}(G)$ is a prohibitive s-t path set. Thus, we immediately obtain the following Lemma 4.4.

\begin{center}
\begin{tikzpicture}
\node [shape=circle,draw=black] (s) at (-1,0) {$s$};
\node [shape=circle,draw=black] (t) at (1,0) {$t$};
\node [shape=circle,draw=black] (v) at (0,1) {};\node [shape=circle,draw=black] (w) at (0,-1) {};
\node [shape=circle,draw=black] (u) at (-1,1) {};\node [shape=circle,draw=black] (x) at (1,1) {};
\node [shape=circle,draw=black] (y) at (-1,-1) {};\node [shape=circle,draw=black] (z) at (1,-1) {};
\draw (s) -- (v) -- (t) -- (x) -- (s) -- (w) -- (t) -- (y) -- (s) -- (u) -- (x) -- (t) -- (z) -- (s) -- (v) -- (y) -- (s) -- (w) -- (x);\end{tikzpicture}
\end{center}

Fig.3 A prohibitive graph.

\textbf{Lemma 4.4.} If $G$ contains a prohibitive graph as its subgraph, then it also has a prohibitive s-t path set as its s-t path subset. □

Now, we show that if $G$ has a prohibitive s-t path set as its s-t path subset, then it contains a prohibitive graph as its subgraph. For our aim, we need more definitions.

\textbf{Definition 4.6. (Attachment Vertex [5][6])}
An attachment vertex of a subgraph $G_1$ in $G$ is a vertex of $G_1$ incident in $G$ with some edge not belonging to $G_1$. □

\textbf{Definition 4.7. (Bridges [5],[6])}
Let $J$ be a fixed subgraph of $G$. A subgraph $G_1$ of $G$ is said to be $J$-detached in $G$ if all its attachment vertices are in $J$. We define a bridge of $J$ in $G$ as any subgraph $B$ that satisfies the following three conditions:
(i) $B$ is not a subgraph of $J$.
(ii) $B$ is J-detached in $G$.
(iii) No proper subgraph of $B$ satisfies both (i) and (ii). □

\textbf{Definition 4.8. (Degenerate and Proper Bridges. Nucleus of a Bridge [5],[6])}
An edge $e = (u,v)$ of $G$ not belonging to $J$ but having both end vertices in $J$ is referred to as a degenerate bridge.
Let $G^-$ be the graph derived from $G$ by deleting the vertices of $J$ and all edges incident to them.
Let $C$ be any component of $G^-$. Let $B$ be the subgraph of $G$ obtained from $C$ by adjoining to it each edge of $G$ having one end vertex in $C$ and the other end vertex in $J$ and adjoining also the end vertices in $J$ of all such edges. The subgraph $B$ satisfies the conditions (i),(ii),(iii) in Definition 4.7 and is a bridge. Such a bridge is called to be proper. The component $C$ of $G^-$ is the nucleus of $B$. □

For the graph $G$ shown in Fig.4, let $J$ be an s-t path $\pi : v_0(=s), v_1, v_2, v_3, v_4, v_5, v_6(=t)$, then all vertices on $\pi$ other than $v_4$ are all attachment vertices of $\pi$ in $G$. $B_1$, $B_2$, $B_3$ are proper bridges of $\pi$ in $G$ and $B_4$ is a degenerate bridge of $\pi$ in $G$. By Definitions 4.6,4.7, the following Lemma 4.5 obviously holds.

**Lemma 4.5.** Let $\pi$ be an s-t path of $G$. If there is a proper bridge $B$ of $\pi$ in $G$, then any two vertices $u, v$ in $B$ are connected by a path consisting of edges and vertices only in the nucleus of $B$. □

Let $\gamma : v_0, v_1, ..., v_{k-1}, v_k$ be a path from $v_0$ to $v_k$ of $G$. If $0 \leq i < j \leq k$, then the sequence $v_i, v_{i+1}, ..., v_{j-1}, v_j$ is a subpath of $\gamma$, and denoted by $\gamma[v_i, v_j]$.

**Definition 4.9. (Path Avoiding s-t Path $\pi$)**
Let $\pi$ be an s-t path of $G$. For two vertices $v_i, v_j$ in $V(\pi)$, a path between $v_i$ and $v_j$ consisting of edges not in $E(\pi)$ and vertices not in $V(\pi)$ except $v_i, v_j$ is said to be avoiding $\pi$. □

For example, the path $v_1, u_1, u_2, u_3$ is avoiding the s-t path $\pi$ in the graph $G$ illustrated in Fig.1.

**Definition 4.10. (Order Relation with Respect to an s-t Path $\pi$)**
Let $\pi : v_0(=s), v_1, ..., v_{k-1}, v_k(=t)$ be an s-t path of $G$. We define an order relation $<_\pi$ on $V(\pi)$ with respect to $\pi$ as follows: For any $v_i, v_j$ $(0 \leq i, j \leq k)$, $v_i <\pi v_j$ holds iff $i < j$ holds. If $v_i <\pi v_j$, $v_i$ ($v_j$) is said to be to the left (right) of $v_j$ ($v_i$). □

**Definition 4.11. (Intersection Vertex of Two Paths $\pi, \alpha$)**
Let $\pi, \alpha$ be two paths of $G$. A vertex $v$ is called an intersection vertex of $\pi, \alpha$ if $\pi$ and $\alpha$ have at least three distinct edges incident to $v$. The set of all intersection vertices of $\pi, \alpha$ is denoted by $V_{\pi,\alpha}$. □

In the graph $G$ given in Fig.1, for two s-t paths $\pi$ and $\alpha : v_0(=s), v_1, u_1, u_2, v_6, v_7, v_9(=t)$, we have $V_{\pi,\alpha} = \{v_1, v_6, v_7, v_9\}$.

**Definition 4.12. (Interlacing Subpaths)**
Suppose that $G$ has an s-t path $\pi: v_0(=s), v_1, \ldots, v_k(=t)$ satisfying $\mathcal{W}(G, \pi) \neq \emptyset$. Let $G_* \in \mathcal{W}(G, \pi)$ be a minimal $\pi$-edge-cut s-t 2-edge-connected subgraph of $G$. Let $\alpha, \beta$ be two edge-disjoint s-t paths of $G_*$. Let $V_{\alpha} = \{x_1, x_2, \ldots, x_p\} (\subseteq V(\pi))$ be the set of all intersection vertices of $\pi$, $\alpha$, where $x_1 <_\pi x_2 <_\pi \cdots <_\pi x_p$. Let $V_{\beta} = \{y_1, y_2, \ldots, y_q\} (\subseteq V(\pi))$ be the set of all intersection vertices of $\pi$, $\beta$, where $y_1 <_\pi y_2 <_\pi \cdots <_\pi y_q$. Let $V_{\alpha\beta} = \{z_1, z_2, \ldots, z_r\} (\subseteq V(\pi))$ be the set of all vertices which $\pi$, $\alpha$, $\beta$ have in common, where $z_1 <_\pi z_2 <_\pi \cdots <_\pi z_r$. Subpaths $\alpha[x_i, x_{i+1}]$ of $\alpha$ avoiding $\pi$ and $\beta[y_j, y_{j+1}]$ of $\beta$ avoiding $\pi$, where either $x_i <_\pi y_j$ or $y_j <_\pi x_i$, are said to be interlacing subpaths, if the subpath $\pi[x_i, y_{j+1}](\pi[y_j, x_{i+1}])$ contains no vertex of $V_{\alpha\beta}$ when $z_i <_\pi y_j (y_j <_\pi x_i)$. $\square$

In the graph $G$ given in Fig.1, for two edge-disjoint s-t paths; 
$\alpha: v_0(=s), v_1, v_4, v_5, u_2, v_6, v_7, v_9(=t)$, $\beta: v_0(=s), u_1, v_2, v_3, v_5, v_8, v_9(=t)$, we have $V_{\alpha} = \{v_1, v_4, v_5, v_7, v_9\}$, $V_{\beta} = \{v_0, v_2, v_3, v_5, v_8, v_9\}$. And subpaths $\alpha[v_1, v_4]$ and $\beta[v_0, v_2]$ are interlacing subpaths, and $\alpha[v_7, v_9]$ and $\beta[v_8, v_9]$ are also interlacing paths. But $\alpha[v_1, v_4]$ and $\beta[v_5, v_8]$ are not interlacing subpaths as $v_5, v_8 \notin V_{\alpha\beta}$ are on $\pi[v_0, v_9]$.

In order to show that if graph $G$ has a prohibitive s-t path set $P(\subseteq P_{st}(G))$, then $G$ must contain a prohibitive graph as its subgraph, we can prove the following Lemma 4.6 and Lemma 4.7.

**Lemma 4.6.** Suppose that $G$ has a prohibitive s-t path set $P$. Then there is an s-t path $\pi$ of $P$ whose proper bridge $B$ in $G$ contains two interlacing subpaths $\alpha[x_i, x_{i+1}]$ of $\alpha$ and $\beta[y_j, y_{j+1}]$ of $\beta$ with respect to $\pi$ in $G_*$, where $G_*$ is a minimal $\pi$-edge-cut s-t 2-edge-connected subgraph of $G$, and $\alpha, \beta$ are two edge-disjoint s-t paths in $G_*$. 

**Sketch of Proof.** Let $P$ be a prohibitive s-t path set of $G$. We can find the s-t path $\pi$ of $P$ satisfying the following condition I by using the following procedure I.

**Condition I:** There is a proper bridge $B$ of $\pi$ in $G$ such that $B$ contains interlacing subpaths $\alpha[x_i, x_{i+1}]$ of $\alpha$ and $\beta[y_j, y_{j+1}]$ of $\beta$ with respect to $\pi$ in $G_*$, where $G_*$ is a minimal $\pi$-edge-cut s-t 2-edge-connected subgraph of $G$, and $\alpha, \beta$ are two edge-disjoint s-t paths in $G_*$. 

**Procedure I:** Let $\pi$ be an s-t path of $P$. Let $B$ be a proper bridge of $\pi$ in $G$. We do the following loop iteratively.

**Loop:** If $\pi$ satisfies Condition I then end. Otherwise, we can find an s-t path $\pi'$ of $P$ such that there is a bridge $B'$ of $\pi'$ in $G$ whose nucleus contains the nucleus of $B$ and there are more vertices in the nucleus of $B'$ than in the nucleus of $B$. Let $B, \pi$ be $B', \pi'$, respectively.

Note that, in each loop, the nucleus of $B$ increases at least by one vertex. Thus the loop will end in at most $|V|$ times, where $V$ is the set of vertices in $G$.

![Fig.5 An illustration of the proof of Lemma 4.7.](image_url)

**Lemma 4.7.** Suppose that $G$ has an s-t path $\pi$ satisfying $\mathcal{W}(G, \pi) \neq \emptyset$. Let $\alpha, \beta$ be two edge-disjoint s-t paths of $G_* \in \mathcal{W}(G, \pi)$. Let $V_{\alpha} = \{x_1, x_2, \ldots, x_p\}$, $V_{\beta} = \{y_1, y_2, \ldots, y_q\}$ and $V_{\alpha\beta} = \{z_1, \ldots, z_r\}$
be defined as in Definition 4.12. If a bridge $B$ of $\pi$ in $G$ contains interlacing subpaths $\alpha[x_i, x_{i+1}]$ of $\alpha$ and $\beta[y_j, y_{j+1}]$ of $\beta$ in $G_\pi$ with respect to $\pi$, then $G$ contains a prohibitive graph as its subgraph.

**Sketch of Proof.** By the known conditions given in this lemma, we construct a prohibitive graph as its subgraph.

By Lemma 4.5, there is a path $\pi_{uv}$ between an internal vertex $u$ on $\alpha[x_i, x_{i+1}]$ and an internal vertex $v$ on $\beta[y_j, y_{j+1}]$ consisting of edges and vertices only in the nucleus of bridge $B$, i.e., $\pi_{uv}$ is vertex-disjoint path with $\pi$ except $u, v$. See Fig.5. Thus, we can also find a prohibitive graph as subgraph of $G$ independently of the way how the path $\pi_{uv}$ is traced.

By Theorem 4.3 and Lemmas 4.5, 4.6, 4.7, the following Theorem 4.4 holds.

**Theorem 4.4.** In a probabilistic graph $(G, p)$, $\underline{\Gamma}(G, f, p) = \Gamma(G, p)$ holds iff $G$ contains no prohibitive graph as its subgraph. \hfill \Box

## 5 A Method of Computing the Lower Bound

Given a probabilistic graph $(G, p)$ and an s-t path number $f$ of $G$, we show a method of computing the lower bound $\underline{\Gamma}(G, f, p)$. We first wish to recall the procedure FEDP and the definition of $\underline{\Gamma}(G, f, p)$ in section 3.

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s-t path number function $f$ of $G$, let $U_{f, \pi_i}$ denote the set of all $U \subseteq E$ for which s-t path $\pi_i$ is selected as a member of edge-disjoint s-t paths $\text{FEDP}(G - U, f)$. Let $p(E_U)$ be the probability of the event $E_U$ that all edges of $U$ are failed and all edges of $E - U$ are operative, and $p(E_{f, \pi_i})$ is the probability of the event that at least one event $E_{U}$, for all $U \in U_{f, \pi_i}$, arises in $(G, p)$. Thus, we have

$$\underline{\Gamma}(G, f, p) = \sum_{U \subseteq E} |\text{FEDP}(G - U, f)| p(G - U)$$

$$= \sum_{i=1}^{n} \sum_{U \in U_{f, \pi_i}} p(G - U)$$

$$= \sum_{i=1}^{n} p(E_{f, \pi_i})$$

We can compute the lower bound $\underline{\Gamma}(G, f, p)$ by formula (5) instead of formula (3).

## 6 Concluding Remarks

For a probabilistic graph, we proposed a lower bound for estimating the expected maximum number of edge-disjoint s-t paths. The necessary and sufficient conditions with respect to both s-t path number function and graph construction, where this lower bound coincides with the expected maximum number of edge-disjoint s-t paths, are clarified. A method of computing this lower bound is also given, although by this computing method the lower bound does not seem to be efficiently computed for a general probabilistic graph.
However, for a probabilistic one-layered s-t graph, (a two-terminal graph where the subgraph obtained by deleting its s, t is exactly a simple path. Fig.6 illustrates an example of one-layered s-t graph.) as it satisfies the necessary and sufficient conditions and the number of all its s-t paths is a polynomial function in the number of its vertices, the lower bound based on its exact s-t path number function can efficiently be computed by the computing method shown in section 5, i.e., the expected maximum number of edge-disjoint s-t paths in a probabilistic one-layered s-t graph can efficiently be computed. Detailed description of these proofs is lengthy and to be reported elsewhere.

![Fig.6 A one-layered s-t graph.](image)

References


