Church-Rosser Property and Unique Normal Form Property of Non-Duplicating Term Rewriting Systems
– DRAFT*–

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1 Introduction

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer [4], Klop and De Vrijer [7] for giving a simpler proof of Chew’s theorem [2, 10]. They developed an interesting method for proving the unique normal form property for some non-Church-Rosser, non-left-linear term rewriting system $R$. The method is based on the fact that the unique normal form property of the original non-left-linear term rewriting system $R$ follows the Church-Rosser property of an associated left-linear conditional term rewriting system $R^L$ which is obtained form $R$ by linearizing the non-left-linear rules. In Klop and Bergstra [1] it is proven that non-overlapping left-linear conditional term rewriting systems are Church-Rosser. Hence, combining these two results, Klop and De Vrijer [4, 7, 6] showed that the term rewriting system $R$ has the unique normal form property if $R^L$ is non-overlapping. However, as their conditional linearization technique is based on the Church-Rosser property for the traditional conditional term rewriting system $R^L$, its application is restricted in non-overlapping $R^L$ (though this limitation may be slightly relaxed with $R^L$ containing only trivial critical pairs).

In this paper, we introduce a new conditional linearization based on a left-right separated conditional term rewriting system $R_L$. The point of our linearization is that by replacing a traditional conditional system $R^L$ with a left-right separated conditional system $R_L$ we can

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easily relax the non-overlapping limitation of conditional systems originated from Klop and Bergstra [1].

By developing a new concept of weighted reduction systems we present a sufficient condition for the Church-Rosser property of a left-right separated conditional term rewriting system $R_L$ which may have overlapping rewrite rules. Applying this result to our conditional linearization, we show a sufficient condition for the unique normal form property of a non-duplicating non-left-linear overlapping term rewriting system $R$.

Moreover, our result can be naturally applied to proving the Church-Rosser property of some non-duplicating non-left-linear overlapping term rewriting systems such as right-ground term rewriting systems. Oyamaguch and Ota [8] proved that non-E-overlapping right-ground term rewriting systems are Church-Rosser by using the joinability of E-graphs, and Oyamaguch extended this result into some overlapping systems [9]. The results by conditional linearization in this paper strengthen some part of Oyamaguchi’s results by E-graphs [8, 9], and vice versa. Hence, we believe that both approach should be working together for developing the potential of non-left-linear term rewriting system theory.

2 Reduction Systems

Assuming that the reader is familiar with the basic concepts and notations concerning reduction systems in [3, 5, 6], we briefly explain notations and definitions.

A reduction system (or an abstract reduction system) is a structure $A = \langle D, \rightarrow \rangle$ consisting of some set $D$ and some binary relation $\rightarrow$ on $D$ (i.e., $\rightarrow \subseteq D \times D$), called a reduction relation. A reduction (starting with $x_0$) in $A$ is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The identity of elements $x, y$ of $D$ is denoted by $x \equiv y$. $\equiv$ is the reflexive closure of $\rightarrow$, $\leftrightarrow$ is the symmetric closure of $\rightarrow$, $\Rightarrow$ is the transitive reflexive closure of $\rightarrow$, and $\Leftrightarrow$ is the equivalence relation generated by $\rightarrow$ (i.e., the transitive reflexive symmetric closure of $\rightarrow$).

If $x \in D$ is minimal with respect to $\rightarrow$, i.e., $\neg \exists y \in D[x \rightarrow y]$, then we say that $x$ is a normal form; let $NF$ be the set of normal forms. If $x \Rightarrow y$ and $y \in NF$ then we say $x$ has a normal form $y$ and $y$ is a normal form of $x$.

**Definition 2.1** $A = \langle D, \rightarrow \rangle$ is Church-Rosser (or confluent) iff

\[ \forall x, y, z \in D[x \Rightarrow y \land x \Rightarrow z \Rightarrow \exists w \in A, y \Rightarrow w \land z \Rightarrow w]. \]

**Definition 2.2** $A = \langle D, \rightarrow \rangle$ has unique normal forms iff

\[ \forall x, y \in NF[x \Rightarrow y \Rightarrow x \equiv y]. \]

The following fact observed by Klop and De Vrijer [7] plays an essential role in our linearization too.

**Proposition 2.3** [Klop and De Vrijer] Let $A_0 = \langle D, \rightarrow_0 \rangle$ and $A_1 = \langle D, \rightarrow_1 \rangle$ be two reduction systems with the sets of normal forms $NF_0$ and $NF_1$ respectively. Then $A_0$ has unique normal forms if each of the following conditions holds:
(i) $\rightarrow_1$ extends $\rightarrow_0$.
(ii) $A_1$ is Church-Rosser,
(iii) $NF_1$ contains $NF_0$.

3 Weight Decreasing Joinability

This section introduces the new concept of weight decreasing joinability. In the later sections this concept is used for analyzing the Church-Rosser property of conditional term rewriting systems with extra variables occurring in conditional parts of rewriting rules.

Let $N^+$ be the set of positive integers. $A = \langle D, \rightarrow \rangle$ is a weighted reduction system if $\rightarrow = \bigcup_{w \in N^+} \rightarrow_w$, that is, positive integers (weights $w$) are assigned to each reduction to represent costs.

A proof of $x \leadsto y$ is a sequence $P: x_0 \leftarrow_{w_1} x_1 \leftarrow_{w_2} \cdots \leftarrow_{w_n} x_n$ such that $x \equiv x_0$ and $y \equiv x_n$. The weight $w(P)$ of the proof $P$ is $\sum_{i=1}^{n} w_i$. We usually abbreviate a proof $P$ of $x \leadsto y$ by $P: x \leadsto y$. The form of a proof may be indicated by writing, for example, $P: x \leadsto \cdots \leadsto y$, $P': x \leftarrow \cdots \leftarrow y$, etc. We use the symbols $P, Q, \cdots$ for proofs.

**Definition 3.1** A weighted reduction system $A = \langle D, \rightarrow \rangle$ is weight decreasing joinable iff
\[ \forall x, y \in D \{ \text{for any proof } P: x \leadsto y \text{ there exists some proof } P': x \leadsto \cdots \leadsto y \text{ such that } w(P) \geq w(P') \} . \]

It is clear that if a weighted reduction system $A$ is weight decreasing joinable then $A$ is Church-Rosser. We will now show a sufficient condition for the weight decreasing joinability.

**Lemma 3.2** Let $A$ be a weighted reduction system. Then $A$ is weight decreasing joinable if the following condition holds:

for any $x, y \in D \{ \text{for any proof } P: x \leftarrow \cdots \rightarrow y \text{ there exists a proof } P': x \leftarrow \cdots \leftarrow y \text{ such that (i) } w(P) > w(P'), \text{ or (ii) } w(P) \geq w(P') \text{ and } P': x \equiv \cdots \equiv y. \}$

**Proof.** The lemma can be easily proven by induction on the weight of a proof of $x \leadsto y$. $\square$

The following lemma is used to show the Church-Rosser property of non-duplicating systems.

**Lemma 3.3** Let $A_0 = \langle D, \rightarrow_0 \rangle$ and $A_1 = \langle D, \rightarrow_1 \rangle$. Let $P_i: x_i \leadsto y (i = 1, \cdots, n)$ and let $w = \sum_{i=1}^{n} w(P_i)$. Assume that for any $a, b \in D$ and any proof $P: a \rightarrow_1 b$ such that $w(P) \leq w$ there exists proofs $P': a \rightarrow_0 c \rightarrow_1 b$ with $w(P') \leq w(P)$ and $a \rightarrow_0 c \rightarrow_1 b$ for some $c \in D$. Then, there exist proofs $P_i': x_i \rightarrow_0 z (i = 1, \cdots, n)$ and $Q: y \rightarrow_0 z$ with $w(Q) \leq w$ for some $z$. 
Proof. By induction on \( w \). Base step \( w = 0 \) is trivial. Induction step: From I.H., we have proofs \( \hat{P}_i : x_i \rightarrow_0 z' \) (\( i = 1, \ldots, n - 1 \)) and \( \hat{Q} : y \rightarrow z' \) for some \( z' \) such that \( \sum_{i=1}^{n-1} w(P_i) \geq w(\hat{Q}) \). By connecting the proofs \( \hat{Q} \) and \( P_n \) we have a proof \( \check{P} : z' \rightarrow y \rightarrow x_n \). Since \( \sum_{i=1}^{n-1} w(P_i) \geq w(\hat{Q}) \) and \( w(\check{P}) = w(\hat{Q}) + w(P_n) \), it follows that \( w \geq w(\check{P}) \). By the assumption, we have proofs \( \check{P} : z' \rightarrow z \rightarrow x_n \) with \( w \geq w(\check{P}) \geq w(\check{P}) \) and \( z' \rightarrow z \rightarrow x_n \) for some \( z \). Thus we obtain proofs \( P'_i : x_i \rightarrow z \) (\( i = 1, \ldots, n \)).

By combining subproofs of \( \hat{P} : z' \rightarrow y \rightarrow x_n \) and \( \check{P} : z' \rightarrow z \rightarrow x_n \), we can make \( Q' : y \rightarrow z \rightarrow z \) and \( Q'' : y \rightarrow x_n \rightarrow z \). Note that \( w + w \geq w(\hat{P}) + w(\check{P}) = w(Q') + w(Q'') \). Thus \( w \geq w(Q') \) or \( w \geq w(Q'') \). Take \( Q' \) as \( Q \) if \( w \geq w(Q') \); otherwise, take \( Q'' \) as \( Q \). \( \square \)

## 4 Term Rewriting Systems

In the following sections, we briefly explain the basic notions and definitions concerning term rewriting systems [3, 5, 6].

Let \( \mathcal{F} \) be an enumerable set of function symbols denoted by \( f, g, h, \cdots \), and let \( \mathcal{V} \) be an enumerable set of variable symbols denoted by \( x, y, z, \cdots \) where \( \mathcal{F} \cap \mathcal{V} = \emptyset \). By \( T(\mathcal{F}, \mathcal{V}) \), we denote the set of terms constructed from \( \mathcal{F} \) and \( \mathcal{V} \). The term set \( T(\mathcal{F}, \mathcal{V}) \) is sometimes denoted by \( T \).

A substitution \( \theta \) is a mapping from a term set \( T(\mathcal{F}, \mathcal{V}) \) to \( T(\mathcal{F}, \mathcal{V}) \) such that for a term \( t \), \( \theta(t) \) is completely determined by its values on the variable symbols occurring in \( t \). Following common usage, we write this as \( t\theta \) instead of \( \theta(t) \).

Consider an extra constant \( \square \) called a hole and the set \( T(\mathcal{F} \cup \{ \square \}, \mathcal{V}) \). Then \( C \in T(\mathcal{F} \cup \{ \square \}, \mathcal{V}) \) is called a context on \( \mathcal{F} \). We use the notation \( C[\ldots, \square, \ldots] \) for the context containing \( n \) holes (\( n \geq 0 \)), and if \( t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{V}) \), then \( C[t_1, \ldots, t_n] \) denotes the result of placing \( t_1, \ldots, t_n \) in the holes of \( C[\ldots, \square, \ldots] \) from left to right. In particular, \( C[\square] \) denotes a context containing precisely one hole. \( s \) is called a subterm of \( t \equiv C[s] \). If \( s \) is a subterm occurrence of \( t \), then we write \( s \subseteq t \). If a term \( t \) has an occurrence of some (function or variable) symbol \( e \), we write \( e \in t \). The variable occurrences \( z_1, \ldots, z_n \) of \( C[z_1, \ldots, z_n] \) are fresh if \( z_1, \ldots, z_n \notin C[\square, \ldots, \square] \) and \( z_i \neq z_j \) (\( i \neq j \)).

A rewriting rule is a pair \( (l, r) \) of terms such that \( l \not\in \mathcal{V} \) and any variable in \( r \) also occurs in \( l \). We write \( l \rightarrow r \) for \( (l, r) \). A redex is a term \( l\theta \), where \( l \rightarrow r \). In this case \( r\theta \) is called a contractum of \( l\theta \). The set of rewriting rules defines a reduction relation \( \rightarrow \) on \( T(\mathcal{F}, \mathcal{V}) \) as follows:

\[
    t \rightarrow s \text{ iff } t \equiv C[l\theta], \ s \equiv C[r\theta] \\
    \text{for some rule } l \rightarrow r, \text{ and some } C[\square], \theta.
\]

When we want to specify the redex occurrence \( \Delta \equiv l\theta \) of \( t \) in this reduction, we write \( t \Delta \rightarrow s \).

**Definition 4.1** A term rewriting system \( R \) is a reduction system \( R = (T(\mathcal{F}, \mathcal{V}), \rightarrow) \) such that the reduction relation \( \rightarrow \) on \( T(\mathcal{F}, \mathcal{V}) \) is defined by a set of rewriting rules. If \( R \) has \( l \rightarrow r \) as a
rewriting rule, we write \( l \to r \in R \).

We say that \( R \) is left-linear if for any \( l \to r \in R \), \( l \) is linear (i.e., every variable in \( l \) occurs only once). If \( R \) has critical pair then we say that \( R \) is overlapping; otherwise non-overlapping [5, 6].

A rewriting rule \( l \to r \) is duplicating if \( r \) contains more occurrences of some variable than \( l \); otherwise, \( l \to r \) is non-duplicating. We say that \( R \) is non-duplicating if every \( l \to r \in R \) is non-duplicating.

## 5 Left-Right Separated Conditional Systems

In this section we introduce a new conditional term rewriting system \( R \) in which \( l \) and \( r \) of any rewrite rule \( l \to r \) do not share the same variable; every variable in \( r \) is connected to some variable in \( l \) thorough an equational condition. A decidable sufficient condition for the Church-Rosser property of \( R \) is presented.

\( V(t) \) denotes the set of variables occurring in a term \( t \).

**Definition 5.1** A left-right separated conditional term rewriting system is a conditional term rewriting system with extra variables in which every conditional rewrite rule has the form:

\[
l \to r \iff x_1 = y_1, \cdots, x_n = y_n
\]

with \( l, r \in T(\mathcal{F}, \mathcal{V}) \), \( V(l) = \{x_1, \cdots, x_n\} \) and \( V(r) \subseteq \{y_1, \cdots, y_n\} \) such that (i) \( l \) is left-linear, (ii) \( \{x_1, \cdots, x_n\} \cap \{y_1, \cdots, y_n\} = \phi \), (iii) \( x_i \neq x_j \) if \( i \neq j \), (iv) \( r \) does not contain more occurrences of some variables than the conditional part \( x_1 = y_1, \cdots, x_n = y_n \).

**Definition 5.2** Let \( R \) be a left-right separated conditional term rewriting system. We inductively define term rewriting systems \( R_i \) for \( i \geq 1 \) as follows:

\[
R_1 = \{l \to r \theta \mid l \to r \iff x_1 = y_1, \cdots, x_n = y_n \in R
\]

\[
\text{and } x_j \theta \equiv y_j \theta \ (j = 1, \cdots, n)\},
\]

\[
R_{i+1} = \{l \to r \theta \mid l \to r \iff x_1 = y_1, \cdots, x_n = y_n \in R
\]

\[
\text{and } x_j \theta \overset{R_i}{\longrightarrow} y_j \theta \ (j = 1, \cdots, n)\}.
\]

In \( R_{i+1} \), proofs of \( x_j \theta \overset{R_i}{\longrightarrow} y_j \theta \ (j = 1, \cdots, n) \) are called subproofs associating with one step reduction by \( l \to r \theta \). Note that \( R_i \subseteq R_{i+1} \) for all \( i \geq 1 \). We have \( s \overset{R}{\longrightarrow} t \) if and only if \( s \overset{R_i}{\longrightarrow} t \) for some \( i \).

The weight \( w(s \overset{R}{\longrightarrow} t) \) of one step reduction \( s \overset{R}{\longrightarrow} t \) is inductively defined as follows:

(i) \( w(s \overset{R}{\longrightarrow} t) = 1 \) if \( s \overset{R}{\longrightarrow} t \),

(ii) \( w(s \overset{R}{\longrightarrow} t) = 1 + w(\mathcal{P}_1) + \cdots + w(\mathcal{P}_n) \) if \( s \overset{R_i}{\longrightarrow} t \ (i \geq 1) \), where \( \mathcal{P}_1, \cdots, \mathcal{P}_n \ (m \geq 0) \) are subproofs associating with one step reduction \( s \overset{R_{i+1}}{\longrightarrow} t \).
Let \( l \rightarrow r \leftarrow x_1 = y_1, \cdots, x_m = y_m \) and \( l' \rightarrow r' \leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n \) be two rules in a left-right separated conditional term rewriting system \( R \). Assume that we have renamed the variables appropriately, so that two rules share no variables. Assume that \( s \notin V \) is a subterm occurrence in \( l \), i.e., \( t \equiv C[s] \), such that \( s \) and \( l' \) are unifiable, i.e., \( s\theta \equiv l'\theta \), with a minimal unifier \( \theta \). Note that \( r\theta \equiv r', r'\theta \equiv r' \), \( y_i\theta \equiv y_i \) \( (i = 1, \cdots, m) \) and \( y'_j\theta \equiv y'_j \) \( (j = 1, \cdots, n) \) as \( \{x_1, \cdots, x_m\} \cap \{y_1, \cdots, y_m\} = \phi \) and \( \{x'_1, \cdots, x'_n\} \cap \{y'_1, \cdots, y'_n\} = \phi \). Thus, from \( l\theta \equiv C[s]\theta \equiv C\theta[l'\theta] \), two reductions starting with \( l\theta \), i.e., \( l\theta \rightarrow C\theta[l'\theta] \) and \( l\theta \rightarrow r \), can be obtained by using \( l \rightarrow r \leftarrow x_1 = y_1, \cdots, x_m = y_m \) and \( l' \rightarrow r' \leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n \) if we have subproofs of \( x_1\theta \not\rightarrow y_1, \cdots, x_m\theta \not\rightarrow y_m \) and \( x'_1\theta \not\rightarrow y'_1, \cdots, x'_n\theta \not\rightarrow y'_n \). Then we say that \( l \rightarrow r \leftarrow x_1 = y_1, \cdots, x_m = y_m \) and \( l' \rightarrow r' \leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n \) are overlapping, and
\[
E \vdash (C\theta[l'\theta], r)
\]
is a conditional critical pair associated with the multiset of equations \( E = \{x_1\theta = y_1, \cdots, x_m\theta = y_m, x'_1\theta = y'_1, \cdots, x'_n\theta = y'_n\} \) in \( R \). We may choose \( l \rightarrow r \leftarrow x_1 = y_1, \cdots, x_m = y_m \) and \( l' \rightarrow r' \leftarrow x'_1 = y'_1, \cdots, x'_n = y'_n \) to be the same rule, but in this case we shall not consider the case \( s \equiv l \). If \( R \) has no critical pair, then we say that \( R \) is non-overlapping.

\( E \cup E' \) denotes the union of multisets \( E \) and \( E' \). We write \( E \sqsubseteq E' \) if no elements in \( E \) occur more than \( E' \).

**Definition 5.3** Let \( E \) be a multiset of equations \( t' = s' \) and a fresh constant \( \bullet \). Then relations \( t \sim \_E t \) and \( t \sim \_E s \) on terms are inductively defined as follows:

(i) \( t \sim \_E t \),

(ii) \( t \sim \_E s \),

(iii) If \( t \sim \_E s \), then \( s \sim \_E t \),

(iv) If \( t \sim \_E r \) and \( r \sim \_E s \), then \( t \sim \_E s \),

(v) If \( t \sim \_E s \), then \( C[t] \sim \_E C[s] \),

(vi) If \( l \rightarrow r \leftarrow x_1 = y_1, \cdots, x_n = y_n \in R \) and \( x_i\theta \sim y_i\theta \) \( (i = 1, \cdots, n) \), then \( C[\theta] \sim \_E C[\theta] \)

where \( E = E_1 \sqcup \cdots \sqcup E_n \),

(vii) If \( t \sim \_E r \), then \( t \sim \_E s \).

**Lemma 5.4** Let \( E = \{p_1 = q_1, \cdots, p_m = q_m, \bullet, \cdots, \bullet\} \) be a multiset in which \( \bullet \) occurs \( n \) times \( (n \geq 0) \), and let \( \mathcal{P}_i : p_i\theta \not\rightarrow q_i\theta \) \( (i = 1, \cdots, m) \).

1. \( t \sim \_E s \) then there exists a proof \( \mathcal{Q} : t\theta \not\rightarrow s\theta \) with \( w(\mathcal{Q}) \leq \sum_{i=1}^m w(\mathcal{P}_i) + n \).

2. \( t \sim \_E s \) then there exists a proof \( \mathcal{Q}' : t\theta \not\rightarrow s\theta \) with \( w(\mathcal{Q}') \leq \sum_{i=1}^m w(\mathcal{P}_i) + n + 1 \).
Proof. By induction on the construction of $t \sim s$ in Definition 5.3, we prove (1) and (2) simultaneously.

Base Step: Trivial as (i) $t \sim s \equiv t$ or (ii) $t \sim s$ of Definition 5.3.

Induction Step: If we have $t \sim s$ by (iii) (iv) (v) and $t \sim s$ by (vi) of Definition 5.3, then from the induction hypothesis (1) and (2) clearly follow. Assume that $t \sim s$ by (v) of Definition 5.3. Then we have a rule $l \rightarrow r \equiv x_1 = y_1, \cdots, x_k = y_k$ such that $t \equiv C[\theta']$, $s \equiv C[r\theta']$, $x_i\theta' \equiv y_i\theta'$ (i = 1, \cdots, k) for some $\theta'$ and $E = E_1 \cup \cdots \cup E_k$. Then we have a proof $Q'$: $t\theta \rightarrow s\theta$ with $w(Q') \leq \sum_{i=1}^{m} w(P_i) + n$. Therefore we have a proof $Q$: $t\theta \rightarrow s\theta$ with $w(Q) \leq \sum_{i=1}^{m} w(P_i) + n + 1$. \hfill \Box

**Theorem 5.5** Let $R$ be a left-right separated conditional term rewriting system. Then $R$ is weight decreasing joinable if for any conditional critical pair $E \vdash (q, q')$ one of the following conditions holds:

(i) $q \sim_{E'} q'$ for some $E'$ such that $E' \subseteq E \cup \{\bullet\}$ or,

(ii) $q \sim_{E_1} \bullet \sim_{E_2} q'$ for some $E_1$ and $E_2$ such that $E_1 \cup E_2 \subseteq E$ or,

(iii) $q \sim_{E'} q'$ (or $q' \sim_{E'} q$) and $E' \subseteq E \cup \{\bullet\}$.

**Note.** The above conditions (i) (ii) (iii) are decidable if $R$ has finite rewrite rules. Thus, the theorem presents a decidable condition for guaranteeing the Church-Rosser property of $R$.

**Proof.** The theorem follows from Lemma 3.2 if for any $P$: $t \rightarrow p \rightarrow s$ (t \neq s) there exists some proof $Q$: $t \rightarrow s$ such that (i) $w(P) > w(Q)$, or (ii) $w(P) \geq w(Q)$ and $Q$: $t \equiv s$. Hence we will show a proof $Q$ satisfying (i) or (ii) for a given proof $P$: $t \rightarrow p \rightarrow s$.

Let $P$: $t \rightarrow P \rightarrow s$ where two redexes $\Delta \equiv \theta \theta'$ and $\Delta' \equiv \theta' \theta''$ are associated with two rules $r_1$: $l \rightarrow r \equiv x_1 = y_1, \cdots, x_m = y_m$ and $r_2$: $l' \rightarrow r' \equiv x'_1 = y'_1, \cdots, x'_n = y'_n$, respectively.

**Case 1.** $\Delta$ and $\Delta'$ are disjoint. Then $p \equiv C[\Delta, \Delta']$ for some context $C[\cdots]$ and $P$: $t \equiv C[t', \Delta'] \Delta' C[\Delta, s'] \equiv s$ for some $t'$ and $s'$. Thus, we can take $Q$: $t \equiv C[t', \Delta'] \Delta' C[t', s'] \Delta C[\Delta, s'] \equiv s$ with $w(Q) = w(P)$.

**Case 2.** $\Delta'$ occurs in $\theta$ of $\Delta \equiv \theta \theta'$ (i.e., $\Delta'$ occurs below the pattern $l$). Without loss of generality we may assume that $r_1$: $C_L[x_1, \cdots, x_m] \rightarrow C_R[y_1, \cdots, y_n] \equiv x_1 = y_1, \cdots, x_m = y_m$ (all the variable occurrences are displayed and $n \leq m$), $P'$: $p \equiv C[C_L[p_1, \cdots, p_m]] \Delta t \equiv C[C_R[t_1, \cdots, t_n]]$ with subproofs $P_i$: $p_i \triangleleft t_i$ (i = 1, \cdots, m), and $P''$: $p \equiv C[C_L[p_1, p_2, \cdots, p_m]] \Delta' s \equiv C[C_L[p'_1, p_2, \cdots, p_m]]$ by $p_1 \triangleleft p'_1$. Thus $w(P) = w(P') + w(P'')$ and $w(P') = 1 + \sum_{i=1}^{m} w(P_i)$. Since we have a proof $Q'$: $p'_1 \triangleleft t_1$ with $w(Q') = w(P'') + w(P_1)$, we can apply $r_1$ to $s \equiv C[C_L[p'_1, p_2, \cdots, p_m]]$ too. Then, we have a proof $Q$: $s \equiv C[C_L[p'_1, \cdots, p_m]] \rightarrow t \equiv C[C_R[t_1, \cdots, t_n]]$ with $w(Q) = 1 + w(Q') + \sum_{i=2}^{m} w(P_i) = w(P)$.

**Case 3.** $\Delta$ and $\Delta'$ coincide by the application of the same rule, i.e., $r = r_1 = r_2$. (Note. In a left-right separated conditional term rewriting system the application of the same rule at
the same position does not imply the same result as the variables occurring in the left-hand side of a rule does not cover that in the right-hand side. Thus this case is necessary even if the system is non-overlapping.) Let the rule applied to $\Delta$ and $\Delta'$ be: $C_L[x_1, \ldots, x_m] \rightarrow C_R[y_1, \ldots, y_n] \leftarrow x_1 = y_1, \ldots, x_m = y_m$ (all the variable occurrences are displayed and $n \leq m$), and let $\mathcal{P}': p \equiv C[C_L[p_1, \ldots, p_m]_{\Delta}] t \equiv C[C_R[t_1, \ldots, t_n]]$ with subproofs $\mathcal{P}'_i: p_i \xrightarrow{\Delta} t_i (i = 1, \ldots, m)$ and $\mathcal{P}''_i: p \equiv C[C_L[p_1, \ldots, p_m]_{\Delta} s \equiv C[C_R[s_1, \ldots, s_n]]$ with subproofs $\mathcal{P}''_i: p_i \xrightarrow{s} s_i (i = 1, \ldots, m)$. Here $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'') = 1 + \sum_{i=1}^{n} w(\mathcal{P}_i) + 1 + \sum_{i=1}^{m} w(\mathcal{P}'_i)$.

Thus we have a proof $\mathcal{Q}$: $t \equiv C[C_R[t_1, \ldots, t_n]] \xrightarrow{s} C[C_R[p_1, \ldots, p_n]] \xrightarrow{s} C[C_R[s_1, \ldots, s_n]] \equiv s$ with $w(\mathcal{Q}) = \sum_{i=1}^{n} w(\mathcal{P}_i) + \sum_{i=1}^{m} w(\mathcal{P}'_i) < w(\mathcal{P})$.

Case 4. $\Delta'$ occurs in $\Delta$ but neither Case 2 nor Case 3 (i.e., $\Delta'$ overlaps with the pattern $l$ of $\Delta \equiv l \theta$). Then, there exists a conditional critical pair $[p_1 = q_1, \ldots, p_m = q_m] \vdash \{q, q'\}$ between $r_1$ and $r_2$, and we can write $\mathcal{P}: t \equiv C[q\theta], p \equiv C[\Delta] s \equiv C[q'\theta]$ with subproofs $\mathcal{P}_i: p_i \xrightarrow{s} q_i \theta (i = 1, \ldots, m)$. Thus $w(\mathcal{P}) = \sum_{i=1}^{n} w(\mathcal{P}_i) + 2$. From the assumption about critical pairs the possible relations between $q$ and $q'$ are give in the following subcases.

Subcase 4.1. $q \sim q'$ for some $E'$ such that $E' \subseteq E \cup \{x\}$. By Lemma 5.4 and $E' \subseteq E \cup \{x\}$, we have a proof $\mathcal{Q}'$: $q\theta \xrightarrow{\Delta} q'\theta$ with $w(\mathcal{Q}') = \sum_{i=1}^{n} w(\mathcal{P}_i) + 1 < w(\mathcal{P})$. Hence it is obtained that $\mathcal{Q}$: $t \equiv C[q\theta] s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$.

Subcase 4.2. $q \sim q'$ for some $E_1$ and $E_2$ such that $E_1 \cup E_2 \subseteq E$. By Lemma 5.4 and $E_1 \cup E_2 \subseteq E$, we have a proof $\mathcal{Q}'$: $q\theta \rightarrow q\theta$ with $w(\mathcal{Q}') = \sum_{i=1}^{n} w(\mathcal{P}_i) + 2 < w(\mathcal{P})$. Hence we can take $\mathcal{Q}$: $t \equiv C[q\theta] s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$.

Subcase 4.3. $q \sim q'$ (or $q \sim q$) and $E' \subseteq E \cup \{x\}$. By Lemma 5.4 and $E' \subseteq E \cup \{x\}$, we have a proof $\mathcal{Q}'$: $q\theta \rightarrow q\theta$ with $w(\mathcal{Q}') = \sum_{i=1}^{n} w(\mathcal{P}_i) + 2 < w(\mathcal{P})$. Hence we obtain $\mathcal{Q}$: $t \equiv C[q\theta] s \equiv C[q'\theta]$ with $w(\mathcal{Q}) < w(\mathcal{P})$. For the case of $q' \sim q$ we can obtain $\mathcal{Q}$: $s \rightarrow t$ with $w(\mathcal{Q}) < w(\mathcal{P})$ similarly. □

**Corollary 5.6** Let $R$ be a left-right separated conditional term rewriting system. Then $R$ is weight decreasing joinable if $R$ is non-overlapping.

**Example 5.7** Let $R_L$ be the left-right separated conditional term rewriting system with the following rewriting rules:

\[
R_L \begin{cases}
  f(x', x'') \rightarrow h(x, f(x, b)) & x' = x, x'' = x \\
  f(g(y'), y'') \rightarrow (h(y, f(g(y), a)) & y' = y, y'' = y \\
  a \rightarrow b
\end{cases}
\]

Here, we have a conditional critical pair

\[
[g(y') = x, y' = x, y'' = y] \vdash \{h(x, f(x, b)), h(y, f(g(y), a))\}
\]

Since $h(x, f(x, b)) \sim h(y'', f(x, b))_{[g(y') = x]} \sim h(y'', f(g(y'), b))_{[g(y') = x]} \sim h(y, f(g(y), b))_{[g(y) = x]} \sim h(y, f(g(y), a))$, we have $h(x, f(x, b)) \sim h(y, f(g(y), a))$ where $E' = [g(y') = x, y'' = x, y'' = x]$. □
$y, y' = y, \bullet$. Thus, from Theorem 5.5 it follows that $R_L$ is weight decreasing joinable. □

In Theorem 5.5 we request that every conditional critical pair $E \vdash (q, q')$ satisfies (i), (ii) or (iii). However, it is clear that we can ignore the conditional critical pairs which cannot appear in the actual proofs of $R$. Thus, we can strengthen Theorem 5.5 as follows.

**Corollary 5.8** Let $R$ be a left-right separated conditional term rewriting system. Then $R$ is weight decreasing joinable if any conditional critical pair $E \vdash (q, q')$ such that $E$ is satisfiable in $R$ satisfies (i), (ii) or (iii) in Theorem 5.5.

**Note.** The satisfiability of $E$ is generally undecidable.

### 6 Conditional Linearization

The original idea of the conditional linearization of non-left-linear term rewriting systems was introduced by De Vrijer [4], Klop and De Vrijer [7] for giving a simpler proof of Chew's theorem [2, 10]. In this section, we introduce a new conditional linearization based on left-right separated conditional term rewriting systems. The point of our linearization is that by replacing traditional conditional systems with left-right separated conditional systems we can easily relax the non-overlapping limitation because of the results of the previous section.

Now we explain a new linearization of non-left-linear rules. For instance, let consider a non-duplicating non-left-linear rule $f(x, x, x, y, y, z) \rightarrow g(x, x, x, z)$. Then, by replacing all the variable occurrences $x, x, y, y, z$ from left to right in the left handside with distinct fresh variable occurrences $x', x'', x'''$, $y', y'', y'$ respectively and connecting every fresh variable to corresponding original one with equation, we can make a left-right separated conditional rule $f(x', x'', x''', y', y'', y') \rightarrow g(x, x, x, z) \Leftarrow x' = x, x'' = x, x''' = x, y' = y, y'' = y, z' = z$. More formally we have the following definition, the framework of which originates essentially from De Vrijer [4], Klop and De Vrijer [7].

**Definition 6.1** (i) If $r$ is a non-duplicating rewrite rule $l \rightarrow r$, then the (left-right separated) conditional linearization of $r$ is a left-right separated conditional rewrite rule $r_L$: $l' \rightarrow r \Leftarrow x_1 = y_1, \ldots, x_m = y_m$

such that $l' \theta \equiv l$ for the substitution $\theta = [x_1 := y_1, \ldots, x_m := y_m]$.

(ii) If $R$ is a non-duplicating term rewriting system, then $R_L$, the conditional linearization of $R$, is defined as the set of the rewrite rules $\{r_L | r \in R\}$.

**Note.** The non-duplicating limitation of $R$ in the above definition is necessary to guarantee that $R_L$ is a left-right separated conditional term rewriting system.

**Note.** The above conditional linearization is different form the original one by Klop and De Vrijer [4, 7] in which the left-linear version of a rewrite rule $r$ is a traditional conditional rewrite
rule without extra variables in the right handside and the conditional part. Hence, in the case \( r \) is already left-linear, Klop and De Vrijer [4, 7] can take \( r \) itself as its conditional linearization. On the other hand, in our definition we cannot take \( r \) itself as its conditional linearization because \( r \) must be translated into a left-right separated rewrite rule.

**Theorem 6.2** If a conditional linearization \( R_L \) of a non-duplicating term rewriting system \( R \) is Church-Rosser, then \( R \) has unique normal forms.

**Proof.** By Proposition 2.3, similar to Klop and De Vrijer [4, 7]. \( \square \)

**Example 6.3** Let \( R \) be the non-duplicating term rewriting system with the following rewriting rules:

\[
R \begin{cases} f(x,x) \rightarrow h(x,f(x,b)) \\
f(g(y),y) \rightarrow h(y,f(g(y),a)) \\
a \rightarrow b \end{cases}
\]

Note that \( R \) is non-left-linear and non-terminating. Then we have the following \( R_L \) as the linearization of \( R \):

\[
R_L \begin{cases} f(x',x'') \rightarrow h(x,f(x,b)) \iff x' = x, x'' = x \\
f(g(y'),y'') \rightarrow h(y,f(g(y),a)) \iff y' = y, y'' = y \\
a \rightarrow b \end{cases}
\]

In Example 5.7 the Church-Rosser property of \( R_L \) has already been shown. Thus, form Theorem 6.2 it follows that \( R \) has unique normal forms. \( \square \)

## 7 Church-Rosser Property of Non-Duplicating Systems

In the previous section we have shown a general method based on the conditional linearization technique to prove the unique normal form property for non-left-linear overlapping non-duplicating term rewriting systems. In this section we show that the same conditional linearization technique can be used as a general method for proving the Church-Rosser property of some class of non-duplicating term rewriting systems.

**Theorem 7.1** Let \( R \) be a right-ground (i.e., no variables occur in the right handside of rewrite rules) term rewriting system. If the conditional linearization \( R_L \) of \( R \) is weight decreasing joinable then \( R \) is Church-Rosser.

**Proof.** Let \( R \) and \( R_L \) have reduction relations \( \rightarrow \) and \( \rightarrow_L \) respectively. Since \( \rightarrow \) extends \( \rightarrow_L \) and \( R_L \) is weight decreasing joinable, the theorem clearly holds if we show the claim: for any \( t, s \) and \( P \): \( t \overset{P}{\rightarrow} s \) there exist proofs \( Q \): \( t \overset{Q}{\rightarrow}_L r \overset{Q}{\rightarrow}_L s \) with \( w(P) \geq w(Q) \) and \( t \overset{P}{\rightarrow}_L r \overset{Q}{\rightarrow}_L s \).
for some term $r$. We will prove this claim by induction on $w(\mathcal{P})$. **Base Step** $w(\mathcal{P}) = 0$ is trivial. **Induction Step** $w(\mathcal{P}) = w$ ($w > 0$): Form the weight decreasing joinability of $R_L$, we have a proof $\mathcal{P}'$: $t \rightarrow^{\cdot \cdot \cdot} s$ with $w \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \rightarrow^{\cdot \cdot \cdot} s \rightarrow^{\cdot \cdot \cdot}$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m] \rightarrow C_R \leftarrow x_1 = x, \cdots, x_m = x$ (all the variable occurrences are displayed) is a linearization of $C_L[x, \cdots, x] \rightarrow C_R$ and $\mathcal{P}''$: $t \equiv C[C_L[t_1, \cdots, t_m]] \rightarrow s' \equiv C[C_R]$ with subproofs $\mathcal{P}_i$: $t_i \rightarrow^{\cdot \cdot \cdot} t'_i$ ($i = 1, \cdots, m$) for some $t'_i$. Then, from Lemma 3.3 and the induction hypothesis we have proofs $t_i \rightarrow^{\cdot \cdot \cdot} t''_i$ ($i = 1, \cdots, m$). Hence we can take the reduction $t \equiv C[C_L[t_1, \cdots, t_m]] \rightarrow s' \equiv C[C_R]$. Let $\mathcal{P}_i$: $s''_i \rightarrow^{\cdot \cdot \cdot} s''_i$. From $w > w(\mathcal{P})$ and I.H., we have $\mathcal{Q}$: $s''_i \rightarrow^{\cdot \cdot \cdot} s''_i$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $s''_i \rightarrow^{\cdot \cdot \cdot} s''_i$ for some $r$. Thus, the theorem follows. □

The following corollary is originally proven by Oyamaguchi [8].

**Corollary 7.2** [Oyamaguchi] Let $R$ be a right-ground term rewriting system having a non-overlapping conditional linearization $R_L$. Then $R$ is Church-Rosser.

Next we relax the right-ground limitation of $R$ in Theorem 7.1.

**Theorem 7.3** Let $R$ be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear and no non-linear variables in $l$ occur in $r$. If the conditional linearization $R_L$ of $R$ is weight decreasing joinable then $R$ is Church-Rosser.

**Proof.** The proof is similar to that of Theorem 7.1. Let $R$ and $R_L$ have reduction relations $\rightarrow$ and $\rightarrow_L$ respectively. Since $\rightarrow$ extends $\rightarrow$ and $R_L$ is weight decreasing joinable, the theorem clearly holds if we show the claim: for any $t$, $s$ and $\mathcal{P}$: $t \rightarrow_L s$ there exist proofs $\mathcal{Q}$: $t \rightarrow_L r \rightarrow_L s$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $t \rightarrow_L r \rightarrow_L s$ for some term $r$. We will prove this claim by induction on $w(\mathcal{P})$. **Base Step** $w(\mathcal{P}) = 0$ is trivial. **Induction Step** $w(\mathcal{P}) = w$ ($w > 0$): Form the weight decreasing joinability of $R_L$, we have a proof $\mathcal{P}'$: $t \rightarrow_L s$ with $w \geq w(\mathcal{P}')$. Let $\mathcal{P}'$ have the form $t \rightarrow_L s \rightarrow_L s$. Without loss of generality we may assume that $C_L[x_1, \cdots, x_m, y_1] \rightarrow C_R[y] \leftarrow x_1 = x, \cdots, x_m = x, y_1 = y$ (all the variable occurrences are displayed) is the linearization of $C_L[x, \cdots, y] \rightarrow C_R[y]$ and $t \equiv C[C_L[t_1, \cdots, t_m, p_1]] \rightarrow s' \equiv C[C_R[p]]$ with subproofs $\mathcal{P}_i$: $t_i \rightarrow_L t'_i$ ($i = 1, \cdots, m$) for some $t'$ and $p_1 \rightarrow_L p$. Then, we can take $t \equiv C[C_L[t_1, \cdots, t_m, p_1]] \rightarrow s' \equiv C[C_R[p]] \rightarrow \hat{s} \equiv C[C_R[p]] \rightarrow s$ with the weight $w(\mathcal{P}')$. Let $\mathcal{P}''$: $t \equiv C[C_L[t_1, \cdots, t_m, p_1]] \rightarrow s' \equiv C[C_R[p_1]]$. Then, from Lemma 3.3 and the induction hypothesis we have proofs $t_i \rightarrow_L t''_i$ ($i = 1, \cdots, m$). Hence we can take the reduction $t \equiv C[C_L[t_1, \cdots, t_m, p_1]] \rightarrow C[C_L[t'', \cdots, t''_m, p_1]] \rightarrow s' \equiv C[C_R[p_1]]$. Let $\mathcal{P}_i$: $s''_i \rightarrow_L s''_i$. From $w > w(\mathcal{P})$ and I.H., we have $\mathcal{Q}$: $s''_i \rightarrow_L s''_i$ with $w(\mathcal{P}) \geq w(\mathcal{Q})$ and $s''_i \rightarrow_L s''_i$ for some $r$. Thus, the theorem follows. □
Corollary 7.4 Let $R$ be a term rewriting system in which every rewrite rule $l \rightarrow r$ is right-linear and no non-linear variables in $l$ occur in $r$. If the conditional linearization $R_L$ of $R$ is non-overlapping then $R$ is Church-Rosser.

References


