TITLE:

The Complexity of Selecting Maximal Solutions

AUTHOR(S):

Chen, Zhi-Zhong; Toda, Seinosuke

CITATION:

Chen, Zhi-Zhong ...[et al]. The Complexity of Selecting Maximal Solutions. 数理解析研究所講究録 1993, 833: 11-21

ISSUE DATE:

1993-04

URL:

<http://hdl.handle.net/2433/83421>

RIGHT:

The Complexity of Selecting Maximal Solutions

Zhi-Zhong Chen

Department of Information Engineering, Mie University, Tsu-shi, Mie 514, Japan.

Seinosuke Toda

Department of Computer Science, University of Electro-Communications,
Chofu-shi, Tokyo 182, Japan.

1 Introduction

Intuitively, a maximization problem is to select a maximal solution for a given input according to some selection criterion. The maximal independent set problem (MIS) [5] and the minimal unsatisfiability problem (MinUnsat) [11] are two standard examples of such problems. Much work has been devoted to the study of maximization problems [1,2,3,4,5,7,9,11,12]. Most of the previous work has involved studying *specific* maximization problems and either finding an efficient algorithm (e.g., [5]) or proving the problem is hard to solve (e.g., [11]). An attractive alternative approach is to study maximization problems in a general framework and to prove general results.

In this paper, we formalize a maximization problem (MAXP) Q as a pair (D, R) , where D is the set of *instances* and $R : D \times \{0, 1\}^* \rightarrow \{true, false\}$ is the *instance-solution relation*. The objective in solving Q is to select, given an instance $x \in D$, a *maximal solution*, i.e., a binary string y such that $R(x, y)$ is true but changing one or more arbitrary 0-bits of y to 1-bits will change the value of $R(x, y)$ to false. As an example, consider MIS in our framework. For it, D is the set of all undirected graphs, and $R(G, b_1 b_2 \cdots b_n)$ is true if and only if G has n vertices (say, $1, 2, \dots, n$) and $\{i : b_i = 1\}$ is an independent set in G . Our goal is to demonstrate what factors make Q easy or hard to solve and how the factors influence the complexity of solving Q . We are able to find two such factors. One obvious factor is the complexity of R . This can be seen by comparing MIS with MinUnsat. The instance-solution relation of MIS is decidable in NC while that of MinUnsat is coNP-complete. Because of this gap, solving MinUnsat is much harder than solving MIS. In fact, MIS is solvable in NC [5,7] while solving MinUnsat is D^P -hard [11]. The other factor is whether R is *hereditary* or not, where R is said to be hereditary if and only if for every x and w , whenever $R(x, w)$ is true, $R(x, w)$ remains true even one or more arbitrary 1-bits of w are changed to 0-bits. The instance-solution relation of MIS (also MinUnsat) is hereditary. In [9], Papadimitriou considered the following problem (MinModel): Given a CNF boolean formula ϕ , find a satisfying truth assignment $\vec{a}$ to ϕ such that changing one or more arbitrary 1-bits of $\vec{a}$ to 0-bits will make $\vec{a}$ no longer satisfy ϕ . The instance-solution relation of MinModel is not hereditary but is decidable in NC. Unlike MIS, solving MinModel is obviously NP-hard.

In this paper, we restrict to consider only those MAXP's whose instance-solution relation is decidable in NP or coNP. We first consider upper bounds on the complexity of solving such MAXP's. Let $Q = (D, R)$ be a MAXP. The following give trivial upper bounds: (i) Q is solvable in FP if R is decidable in P and hereditary; (ii) Q is solvable in NPMV//OptP[$O(\log n)$] if R is decidable in NP; (iii) Q is solvable in FP^{NP} if R is decidable in coNP and hereditary; (iv) Q is solvable in $FP^{\Sigma_2^P}$ if R is decidable in coNP.

Our main results concerning upper bounds are the following:

(v) Suppose Q is a MAXP whose instance-solution relation is NP decidable. Let e be an arbitrary polynomial. Then, there exist a function $F \in \text{FP}_{\parallel}^{\text{NP}}$ and a polynomial p such that for every x , $\Pr[F(x, w) \text{ is a maximal solution of } x \text{ in } Q] \geq 1 - 2^{-e(|x|)}$, where $w \in \{0, 1\}^{p(|x|)}$ is randomly chosen under uniform distribution.

(vi) Suppose Q is a MAXP whose instance-solution relation is coNP decidable. Let e be an arbitrary polynomial. Then, there exist a function $F \in \text{FP}_{\parallel}^{\Sigma_2^{\text{P}}}$ and a polynomial p such that for every x , $\Pr[F(x, w) \text{ is a maximal solution of } x \text{ in } Q] \geq 1 - 2^{-e(|x|)}$, where $w \in \{0, 1\}^{p(|x|)}$ is randomly chosen under uniform distribution.

(v) and (vi) are shown by extending the technique used in [3].

We then show that $\text{NPMV//OptP}[O(\log n)]$ is also a lower bound for solving those MAXP's whose instance-solution relation is decidable in NP or is decidable in P but not hereditary, and that $\text{FP}_{\parallel}^{\Sigma_2^{\text{P}}}$ is also a lower bound for solving those MAXP's whose instance-solution relation is decidable in coNP but not hereditary. Combining the upper and lower bounds, we obtain characterizations of $\text{NPMV//OptP}[O(\log n)]$ and $\text{FP}_{\parallel}^{\Sigma_2^{\text{P}}}$ via MAXP's. As an important consequence of the characterization of $\text{NPMV//OptP}[O(\log n)]$, we obtain the first natural complete problem for $\text{NPMV//OptP}[O(\log n)]$. The problem (called X -MinModel) is defined as follows: Given a CNF boolean formula ϕ and a subset X of the set of variables in ϕ , find a satisfying truth assignment $\vec{a}$ to ϕ such that changing one or more arbitrary 1-bits of $\vec{a}$ corresponding to variables in X to 0-bits will make $\vec{a}$ no longer satisfy ϕ . X -MinModel was first considered by Papadimitriou in [9], and was claimed *without a precise proof* to be Δ_2^{P} -complete there. However, Papadimitriou later withdrew his claim and thus left the complexity of X -MinModel open [10]. In [3], we proved that the complexity of X -MinModel is roughly captured by $\text{FP}_{\parallel}^{\text{NP}}$. Now, the results in this paper give, for the first time, the exact complexity of solving X -MinModel.

We also characterize complexity classes of sets via MAXP's. The following are shown:

(a) coNP is the class of all sets L that can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is P-decidable.

(b) D^{P} is the class of all sets L that can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is NP-decidable.

(c) D^{P} is the class of all sets L that can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is coNP-decidable and hereditary.

(d) Π_2^{P} is the class of all sets L that can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is coNP-decidable.

As consequences, we obtain several new natural problems that are $\leq_m^{\text{P}}$ -complete for coNP or D^{P} .

2 Preliminaries

We use $\Sigma = \{0, 1\}$ as our alphabet. By a *set*, we mean a subset of Σ^* . Similarly, by a *string*, we mean a string in Σ^* . We denote by $|x|$ the length of a finite string x . The bits

of a finite string with length n are indexed from left to right as the 1st, 2nd, $\dots$, n th bits, respectively. For a finite string x , we usually identify x with the set of all indices i such that the i th bit of x is 1. Thus we will often use some set-theoretical notations for finite strings. A finite string x is *smaller* than another finite string y if either $|x| < |y|$ or $|x| = |y|$ and $x \subset y$. A *maximal string* in a set S of finite strings is a string in S that is not smaller than any other string in S .

We assume a one-to-one pairing function from $\Sigma^* \times \Sigma^*$ to Σ^* that is polynomial-time computable and polynomial-time invertible. For strings x and y , we denote the output of the pairing function by $\langle x, y \rangle$; this notation is extended to denote any k -tuples for $k > 2$ in a usual manner. W.l.o.g., we assume that $|\langle x, y \rangle|$ depends only on $|x|$ and $|y|$.

For any finite set A , $||A||$ denotes the number of strings in A . For a set L , $\bar{L}$ denotes its complement (i.e., $\Sigma^* - L$), and χ_L denotes the characteristic function of L . For a class $\mathbf{C}$ of sets, $\text{co}\mathbf{C}$ denotes the class of all sets whose complement is in $\mathbf{C}$. Let Σ^n denote the set of all strings with length n . For two sets L_1 and L_2 , $L_1 \oplus L_2$ denotes the set $\{0x : x \in L_1\} \cup \{1y : y \in L_2\}$.

All functions considered here are ones from Σ^* to $\Sigma^* \cup \{\#\}$. The symbol $\#$ is assumed to be not in Σ^* . We consider both single-valued functions and multi-valued functions, but by a *function* we mean a (partial) single-valued function. For a multi-valued function G , $G(x)$ denotes the set of all possible values of G at x . Thus, when $G(x) = \emptyset$, the multi-valued function G is *undefined* at the argument x .

We assume that the reader is familiar with the basic concepts from the theory of computational complexity. Our computational models are variations of standard Turing machines. A machine is either an acceptor or a transducer, and may be deterministic or nondeterministic. An acceptor is denoted by M or M_i while a transducer is denoted by T or T_i . A deterministic (resp., nondeterministic) Turing machine is abbreviated as DTM (resp., NTM). On a given input, a branch of a (nondeterministic) machine may halt by entering either a rejecting state or an accepting state. For simplicity, we say that a branch of a machine *halts* if the branch halts by entering an accepting state. Let $L(M)$ denote the set of all strings accepted by M . A transducer T *computes a string y on input x* if some branch of T on input x halts with y on the output tape. $T(x)$ denotes the set of all strings computed by T on input x . A DTM T *computes a function f* if for all $x \in \Sigma^*$, $T(x) = \emptyset$ if $f(x)$ is undefined, and the unique element of $T(x)$ is $f(x)$ otherwise.

Classes in the first three levels of the polynomial-time hierarchy are denoted in the usual way: $\mathbf{P}$, $\mathbf{NP}$, coNP , $\Sigma_2^{\mathbf{P}}$, $\Pi_2^{\mathbf{P}} = \text{co}\Sigma_2^{\mathbf{P}}$. Let $\mathbf{D}^{\mathbf{P}} = \{L_1 \cap L_2 : L_1 \in \mathbf{NP} \text{ and } L_2 \in \text{coNP}\}$.

$\mathbf{FP}$ denotes the class of all functions computed by polynomial-time bounded DTM's. Let A be a set. $\mathbf{FP}^A$ denotes the class of all functions computed by polynomial-time bounded deterministic oracle Turing machines (DOTM) with oracle A . $\mathbf{FP}_{\parallel}^A$ denotes the class of all functions F for which there exists a polynomial-time bounded DOTM T such that T , while computing $F(x)$ for a given x , prepares all its query strings before asking them to the oracle A . More precisely, a function F is in $\mathbf{FP}_{\parallel}^A$ if there exist two functions f and g in $\mathbf{FP}$ such that for all strings x , $F(x) = g(x, \chi_A(y_1) \cdots \chi_A(y_m))$, where $f(x) = \langle y_1, \dots, y_m \rangle$. For a class $\mathbf{C}$ of sets, $\mathbf{FP}^{\mathbf{C}} = \cup_{A \in \mathbf{C}} \mathbf{FP}^A$ and $\mathbf{FP}_{\parallel}^{\mathbf{C}} = \cup_{A \in \mathbf{C}} \mathbf{FP}_{\parallel}^A$.

An *NP metric Turing machine* is a polynomial-time bounded NTM T such that on every input, every branch of T outputs a binary number and halts [6]. $\text{OptP}[O(\log n)]$ denotes the class of all (total) integer-valued functions H for which there exist a polynomial p and an NP metric Turing machine T such that for every x , $H(x) \leq p(|x|)$ and

$H(x)$ equals to the maximum number in $T(x)$. $\text{NPMV//OptP}[O(\log n)]$ denotes the class of all (partial) multi-valued functions G for which there exist an NTM T and a function $H \in \text{OptP}[O(\log n)]$ such that for every x , $G(x) = T(\langle x, H(x) \rangle)$.

A *maximization problem* (MAXP) Q is a pair (D, R) , where (i) D is the set of *instances* and (ii) $R : D \times \Sigma^* \rightarrow \{\text{true}, \text{false}\}$ is the *instance-solution relation*.

R is said to be *hereditary* if for every $x \in D$ and every $w \in \Sigma^*$, whenever $R(x, w)$ is true, $R(x, w')$ is also true for every w' with $|w'| = |w|$ and $w' \subset w$. Let $x \in D$. A string w is called a *solution* of x if $R(x, w)$ is true. A *maximal solution* of x is a maximal string in the set of all solutions of x . The objective in solving Q is to compute, given an instance $x \in D$, a maximal solution of x .

Each MAXP $Q = (D, R)$ considered in this paper is required to satisfy the following: (1) D is P-decidable (i.e., decidable in polynomial time), (2) there is a polynomial p such that for every $x \in D$ and every string w , whenever $R(x, w)$ is true, $|w| \leq p(|x|)$, and (3) R is NP-decidable or coNP-decidable.

Definition 2.1 A function F *solves* Q if for every $x \in D$, (a) $F(x)$ is undefined if x has no solution in Q and (b) $F(x)$ is a maximal solution of x in Q otherwise. A multi-valued function G *solves* Q if for every $x \in D$, (a) $G(x) = \emptyset$ if x has no solution in Q and (b) $G(x)$ is nonempty and each element of $G(x)$ is a maximal solution of x in Q otherwise. Q is *solvable* in a class $\mathbf{H}$ of (single-valued or multi-valued) functions if some $H \in \mathbf{H}$ solves Q .

Definition 2.2 Let F be a function, and let G be a multi-valued function. Then, F (resp., G) is *reducible* to Q if there exist two functions f, g in FP such that for every x , $f(x) \in D$ and $g(x, w) = F(x)$ (resp., $g(x, w) \in G(x)$) for every maximal solution w of $f(x)$ in Q . Q is *hard* for a class $\mathbf{H}$ of (single-valued or multi-valued) functions if every $H \in \mathbf{H}$ is reducible to Q . Q is *complete* for a class $\mathbf{H}$ of (single-valued or multi-valued) functions if Q is solvable in and hard for $\mathbf{H}$. Q is *hard* for a class $\mathbf{C}$ of sets if Q is hard for the class $\{\chi_L : L \in \mathbf{C}\}$.

Definition 2.3 The set $L_Q = \{\langle x, w \rangle : w \text{ is a maximal solution of } x \text{ in } Q\}$ is called the *decision problem associated with* Q .

3 Upper bounds

In this section, we show upper bounds on the complexity of solving MAXP's. The following proposition shows trivial upper bounds.

Proposition 3.1 Let $Q = (D, R)$ be a MAXP.

- (1) If R is hereditary and P-decidable, then Q is solvable in FP.
- (2) If R is NP-decidable, then Q is solvable in $\text{NPMV//OptP}[O(\log n)]$.
- (3) If R is hereditary and coNP-decidable, then Q is solvable in FP^{NP} .
- (4) If R is coNP-decidable, then Q is solvable in $\text{FP}^{\Sigma_2^P}$.

We next proceed to show two other non-trivial upper bounds. To do this, we need several definitions and a known result.

Definition 3.1 Let $\mathbf{F}$ be a class of functions. Then we define a class $\text{RP} \cdot \mathbf{F}$ of

multi-valued functions as follows: A multi-valued function G is in $\text{RP} \cdot \mathbf{F}$ if for every polynomial e , there exist a function $F \in \mathbf{F}$ and a polynomial p such that for every string x , (a) $F(x, w)$ is undefined for all $w \in \{0, 1\}^{p(|x|)}$ if $G(x)$ is undefined and (b) $\Pr[F(x, w) \in G(x) \cup \{\#\}] = 1$ and $\Pr[F(x, w) \in G(x)] \geq 1 - 2^{-e(|x|)}$ otherwise, where w is a random string chosen from $\{0, 1\}^{p(|x|)}$. Intuitively speaking, G is in $\text{RP} \cdot \mathbf{F}$ if for every string x , we can randomly pick up an element of $G(x)$ using a function in $\mathbf{F}$.

Notation: For $k \geq 1$, $[1, k]$ denotes the set of all integers i with $1 \leq i \leq k$.

Definition 3.2 Let S be a finite set and let k be a positive integer. A *weight function* over S is a function from the elements of S to positive integers. A *k -weight function* over S is a weight function f over S such that for each $s \in S$, $f(s)$ is in $[1, k]$. A *random k -weight function* over S is a k -weight function f over S such that for each $s \in S$, $f(s)$ is chosen uniformly and independently from $[1, k]$. The *weight* of a subset S' of S under a weight function f is $\sum_{s \in S'} f(s)$. Note that for every k -weight function over S , the weight of each subset of S under f is no more than $k||S||$ and that the empty set $\emptyset$ is the unique subset of S with weight 0.

Lemma 3.1 [8]. Let $\mathbf{S}$ be a nonempty family of subsets of a finite set S . Then, for any random k -weight function f over S with $k \geq 2||S||$, $\Pr[\text{There is a unique maximum weight set in } \mathbf{S} \text{ under } f] \geq \frac{1}{2}$.

Now we are ready to show the two non-trivial upper bounds. The idea used in the proof is a generalization of the one used in [3].

Theorem 3.1 Let $Q = (D, R)$ be a MAXP.

- (1) If R is NP-decidable, then Q is solvable in $\text{RP} \cdot \text{FP}_{\parallel}^{\text{NP}}$.
- (2) If R is coNP-decidable, then Q is solvable in $\text{RP} \cdot \text{FP}_{\parallel}^{\Sigma_2^P}$.

Proof. We only show a proof for (2). (1) can be shown in a similar manner.

(2) We first explain the idea behind the proof. Let p_Q be a polynomial such that for all $x \in D$, the length of each solution of x is no more than $p_Q(|x|)$. Let x be an instance of Q . Then, we consider $\mathbf{S}$, the family of all solutions of x with maximum length. To find a maximal solution for x , we first get a random $2p_Q(|x|)$ -weight function f over $[1, p_Q(|x|)]$. Then, by Lemma 3.1, with probability at least $\frac{1}{2}$, there is a unique solution in $\mathbf{S}$ of maximum weight. To find this unique solution of maximum weight, it suffices to ask only one round of parallel queries to a Σ_2^P oracle set. Since the weight assigned to each element of $[1, p_Q(|x|)]$ is positive, all maximum weight solutions are maximal solutions (but not necessarily solutions of maximum 1-bits). In order to get the high probability of success, we may perform several copies of this computation in parallel.

We now proceed to give the precise proof. Let p_Q be a polynomial that bounds the lengths of solutions of x from above. For convenience, let $n_x = p_Q(|x|)$ for all $x \in D$. Then we define five sets as follows:

- $$\begin{aligned} L_R &= \{x : x \text{ has a solution}\}, \\ B_1 &= \{\langle x, i \rangle : 0 \leq i \leq n_x \text{ and } x \text{ has a solution of length } i\}, \\ B_2 &= \{\langle x, i, f, j \rangle : x \in D, 0 \leq i \leq n_x, f \text{ is a } 2^{1+\lceil \log_2 n_x \rceil}\text{-weight function over } [1, n_x], \\ &\quad 0 \leq j \leq i2^{1+\lceil \log_2 n_x \rceil}, \text{ and } x \text{ has a solution } u \text{ such that } |u| = i \text{ and } j \text{ is the weight of } u \\ &\quad \text{under } f\}, \\ B_3 &= \{\langle x, i, f, j \rangle : x \in D, 0 \leq i \leq n_x, f \text{ is a } 2^{1+\lceil \log_2 n_x \rceil}\text{-weight function over } [1, n_x], \end{aligned}$$

$0 \leq j \leq i2^{1+\lceil \log_2 n_x \rceil}$, and x has two or more solutions u such that $|u| = i$ and j is the weight of u under f }, and

$B_4 = \{\langle x, i, f, j, k \rangle : x \in D, 0 \leq i \leq n_x, f \text{ is a } 2^{1+\lceil \log_2 n_x \rceil}\text{-weight function over } [1, n_x], 0 \leq j \leq i2^{1+\lceil \log_2 n_x \rceil}, 1 \leq k \leq i, \text{ and } x \text{ has a solution } u \text{ such that } |u| = i, j \text{ is the weight of } u \text{ under } f, \text{ and the } k\text{th bit of } u \text{ is } 1\}$.

Obviously, L_R, B_1, B_2, B_3 , and B_4 are in Σ_2^P . Let $B = (((L_R \oplus B_1) \oplus B_2) \oplus B_3) \oplus B_4$. Then, $B \in \Sigma_2^P$.

Let e be an arbitrary polynomial. We define a polynomial p as follows: $p(i) = e(i) \cdot (p_Q^2(i) + p_Q(i))$. Below, we define a DOTM T which uses B as an oracle set. Given an input $\langle x, w \rangle$ with $x \in D$ and $w \in \{0, 1\}^{p(|x|)}$, T operates as follows:

Step 1: T checks whether x has a solution by asking a query to L_R . If x has no solution, then T halts by entering a rejecting state.

Step 2: T finds n_1 , the length of the longest solutions of x . This is done by asking the queries $\langle x, 0 \rangle, \langle x, 1 \rangle, \dots, \langle x, n_x \rangle$ to the oracle set B_1 .

Step 3: T computes $n_2 = 2^{1+\lceil \log_2 n_x \rceil}$ and constructs, from w , n_2 -weight functions $f_1, f_2, \dots, f_{e(|x|)}$ over the set $[1, n_x]$ as follows:

Step 3.1: T first computes $e(|x|)$ strings $w_1, \dots, w_{e(|x|)}$ from w such that $|w_1| = \dots = |w_{e(|x|)}| = n_x \log_2 n_2$ and the string $w_1 w_2 \dots w_{e(|x|)}$ is a prefix of w (the remaining part of w is ignored), and then for each $1 \leq k \leq e(|x|)$, it partitions w_k into n_x substrings $w_{k,1}, \dots, w_{k,n_x}$ each of length $\log_2 n_2$. (Note: T can do this because $n_x^2 + n_x \geq n_x \log_2 n_2$.)

Step 3.2: For each $1 \leq k \leq e(|x|)$ and each l in $[1, n_x]$, T sets $f_k(l) = d_{k,l} + 1$, where $d_{k,l}$ is the integer whose binary representation is $w_{k,l}$.

Step 4: For each $1 \leq k \leq e(|x|)$, T computes the maximum number m_k with $\langle x, n_1, f_k, m_k \rangle \in B_2$. This is done by asking the queries $\langle x, i, f_k, j \rangle$ with $0 \leq i \leq n_x$, $1 \leq k \leq e(|x|)$, and $0 \leq j \leq in_2$ to the oracle set B_2 . (Note: In this step, T asks the queries of the form $\langle x, i, f_k, j \rangle$ for all possible values of i, k , and j because the machine needs to prepare all queries independently of each other.)

Step 5: For $1 \leq k \leq e(|x|)$ and $1 \leq l \leq n_1$, T computes $a_{k,l} = \chi_{B_4}(\langle x, n_1, f_k, m_k, l \rangle)$. This is done by asking the queries $\langle x, i, f_k, j, l \rangle$ with $0 \leq i \leq n_x$, $1 \leq k \leq e(|x|)$, $0 \leq j \leq in_2$, and $1 \leq l \leq i$ to the oracle set B_4 . (Note: In this step, T asks the queries of the form $\langle x, i, f_k, j, l \rangle$ for all possible values i, k, j , and l because the machine needs to prepare all queries independently of each other.)

Step 6: For each $1 \leq k \leq e(|x|)$, T checks whether $\langle x, n_1, f_k, m_k \rangle \in B_3$. This is done by asking the queries $\langle x, i, f_k, j \rangle$ with $0 \leq i \leq n_x$, $1 \leq k \leq e(|x|)$, and $0 \leq j \leq in_2$ to the oracle set B_3 . If for some k , $\langle x, n_1, f_k, m_k \rangle \notin B_3$, then T outputs $a_{k,1} a_{k,2} \dots a_{k,n_1}$ and halts; otherwise, T outputs the special symbol $\#$ and halts.

Let F denote the function computed by T with oracle B . We can easily see that T is polynomial-time bounded and all query strings are prepared independently of each other; this means that the query strings made by T on input $\langle x, w \rangle$ can be realized as parallel queries to the oracle set B . Thus, F is in $\text{FP}_{\parallel}^{\Sigma_2^P}$.

Let G be a multi-valued function defined by $G(x) = \{F(x, w) : w \in \{0, 1\}^{p(|x|)} \text{ and } F(x, w) \text{ is defined}\} - \{\#\}$. We show that G solves Q and is in $\text{RP} \cdot \text{FP}_{\parallel}^{\Sigma_2^P}$. To this end, we first prove two claims.

Claim 1 Suppose that for some k with $1 \leq k \leq e(|x|)$, x has a unique solution with

length n_1 and of weight m_k under f_k . Then, the string $a_{k,1}a_{k,2}\cdots a_{k,n_1}$ output by T is a maximal solution of x .

Claim 2 Suppose that x has solutions and w is randomly chosen from $\{0,1\}^{p(|x|)}$. Then, $\Pr[F(x, w) \text{ is a maximal solution of } x] \geq 1 - 2^{-e(|x|)}$.

Proof. Since w is randomly chosen from $\{0,1\}^{p(|x|)}$, the functions $f_1, f_2, \dots, f_{e(|x|)}$ constructed in Step 3 must be random n_2 -weight functions over $[1, n_x]$. Note that $n_2 = 2^{1+\lceil \log_2 n_x \rceil} \geq 2n_x$. Thus, from Claim 1 and Lemma 3.1, we have that

$$\begin{aligned} & \Pr[F(x, w) \text{ is a maximal solution of } x] \\ &= \Pr[(\exists k, 1 \leq k \leq e(|x|)) a_{k,1}a_{k,2}\cdots a_{k,n_1} \text{ is a maximal solution of } x] \\ &\geq \Pr[(\exists k, 1 \leq k \leq e(|x|)) x \text{ has a unique solution with maximum length and of maximum weight under } f_k] \\ &\geq 1 - \prod_{k=1}^{e(|x|)} \Pr[x \text{ has two or more solutions with maximum length and of maximum weight under } f_k] \\ &\geq 1 - \left(\frac{1}{2}\right)^{e(|x|)} = 1 - 2^{-e(|x|)}. \end{aligned}$$

In the case when x has no solution, $F(x, w)$ is undefined for all $w \in \{0,1\}^{p(|x|)}$ by Step 1 and hence $G(x) = \emptyset$. On the other hand, when x has solutions, $G(x) \neq \emptyset$ by Claim 2 and the definition of G , and each element of $G(x)$ is a maximal solution of x by Claim 1. Therefore, G solves Q .

If $G(x) = \emptyset$, we know that x has no solution by the discussions in the last paragraph and thus that $F(x, w)$ is undefined for all $w \in \{0,1\}^{p(|x|)}$ by Step 1. On the other hand, if $G(x) \neq \emptyset$, then x has solutions by the discussions in the last paragraph, $\Pr[F(x, w) \in G(x) \cup \{\#\}] = 1$ by Step 6 and the definition of G , and $\Pr[F(x, w) \in G(x)] \geq 1 - 2^{-e(|x|)}$ by Claim 2. Therefore, G is in $\text{RP}\cdot\text{FP}_{\parallel}^{\Sigma_2^P}$.

4 Hardness of solving MAXP's

In the light of Proposition 3.1(1), the following proposition shows that FP is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is P-decidable and hereditary.

Proposition 4.1 There is MAXP $Q = (D, R)$ such that R is P-decidable and hereditary and Q is hard for FP.

By Proposition 3.1(2), the following theorem shows that $\text{NPMV//OptP}[O(\log n)]$ is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is P-decidable but not hereditary.

Theorem 4.1 There is a MAXP $Q = (D, R)$ such that R is P-decidable (but not hereditary) and Q is hard for $\text{NPMV//OptP}[O(\log n)]$.

The following corollary is immediate from the proof of Theorem 4.1.

Corollary 4.1 The following problem (called X -MaxModel hereafter) is complete for $\text{NPMV//OptP}[O(\log n)]$:

Instance: A CNF boolean formula ϕ and a subset X of the set of variables in ϕ .

Output: A truth assignment $\vec{a}$ to the variables in X such that $\vec{a}$ can be extended to a satisfying truth assignment to ϕ but no $\vec{b}$ with $\vec{a} \subset \vec{b}$ and $|\vec{a}| = |\vec{b}|$ can be extended to

a satisfying truth assignment to ϕ .

X -MaxModel is essentially the same problem as considered by Papadimitriou in Section 3 of [9]. In [9], Papadimitriou claimed *without a precise proof* that the problem is complete for FP^{NP} . However, he later withdrew his claim and thus left the complexity of the problem open [10]. In [3], we proved that the complexity of the problem is roughly captured by $\text{FP}_{\parallel}^{\text{NP}}$. Now, Corollary 4.1 gives, for the first time, the exact complexity of the problem. Corollary 4.1 is also of special interest in the sense that no natural problem complete for $\text{NPMV//OptP}[O(\log n)]$ has been shown before.

By modifying the proof of Theorem 4.1, we can show that $\text{NPMV//OptP}[O(\log n)]$ is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is NP-decidable and hereditary.

Theorem 4.2 There is a MAXP $Q = (D, R)$ such that R is NP-decidable and hereditary and Q is hard for $\text{NPMV//OptP}[O(\log n)]$.

The instance-solution relation of X -MaxModel is NP-decidable but not hereditary. A natural question arises: Are there natural MAXP's Q such that Q is hard for $\text{NPMV//OptP}[O(\log n)]$ and the instance-solution relation of Q is either NP-decidable and hereditary or P-decidable (but not hereditary)? Unfortunately, we are unable to settle this question. However, we below show that the question will have a positive answer if $\text{NPMV//OptP}[O(\log n)]$ in it is replaced by $\text{FP}_{\parallel}^{\text{NP}}$.

Definition 4.1 A MAXP $Q = (D, R)$ is *paddable* if there are two functions f and g in FP such that for every list $I = \langle x_1, x_2, \dots, x_m \rangle$ of instances of Q , $f(I) \in D$ and for every maximal solution w of $f(I)$, $g(I, w)$ gives a maximal solution for each x_i .

Lemma 4.1 If a MAXP is paddable and hard for NP, then it is hard for $\text{FP}_{\parallel}^{\text{NP}}$.

Theorem 4.3 The following MAXP's are hard for $\text{FP}_{\parallel}^{\text{NP}}$.

(1) MAXIMAL MODEL (MaxModel)

Instance: A CNF boolean formula ϕ .

Output: A *maximal satisfying truth assignment* to ϕ , i.e., a satisfying truth assignment $\vec{a}$ to ϕ such that there is no other satisfying truth assignment $\vec{b}$ to ϕ with $\vec{a} \subset \vec{b}$.

(2) MAXIMAL CUBIC SUBGRAPH (MaxCubSubgraph)

Instance: An undirected graph G .

Output: A maximal subset F of $E(G)$ such that every vertex in the graph $(V(G), F)$ has either degree 3 or degree 0. (Note: $V(G)$ and $E(G)$ denote hereafter the sets of vertices and edges of G , respectively.)

(3) MAXIMAL SATISFIABILITY (MaxSat)

Instance: A CNF boolean formula $\phi = \{C_1, C_2, \dots, C_m\}$.

Output: A maximal subset ϕ' of ϕ that is satisfiable.

(4) MAXIMAL k -COLORABILITY ($k \geq 3$) (Max- k -Colorability)

Instance: An undirected graph G .

Output: A maximal subset of $V(G)$ whose induced subgraph is k -colorable.

(5) MAXIMAL HAMILTONIAN SUBGRAPH (MaxHamSubgraph)

Instance: A pair $\langle G, w \rangle$ of a connected undirected graph and a vertex in G .

Output: A maximal subset U of $V(G)$ such that $w \in U$ and the subgraph induced by U has a Hamiltonian circuit.

We here note that a different proof for the $\text{FP}_{\parallel}^{\text{NP}}$ -hardness of MaxModel has been given in [3]. Note that the instance-solution relations of the first two problems in Theorem 4.3 are P-decidable but not hereditary, while the instance-solution relations of the third and fourth problems in Theorem 4.3 are NP-decidable and hereditary. The last problem in Theorem 4.3 is a concrete MAXP whose instance-solution relation is NP-decidable but not hereditary.

For those MAXP's Q whose instance-solution relation is coNP-decidable and hereditary, we are only able to show a loose lower bound.

Proposition 4.2 There is a MAXP $Q = (D, R)$ such that R is a coNP-decidable hereditary relation and Q is hard for $\text{FP}_{\parallel}^{\text{NP}}$.

In the light of Theorem 3.1(2), the following theorem shows that $\text{FP}_{\parallel}^{\Sigma_2^{\text{P}}}$ is a nearly optimal lower bound on the complexity of solving MAXP's whose instance-solution relation is coNP-decidable but not hereditary.

Theorem 4.4 There is a MAXP $Q = (D, R)$ such that R is coNP-decidable and Q is hard for $\text{FP}_{\parallel}^{\Sigma_2^{\text{P}}}$.

5 Characterizations of coNP, D^{P} and Π_2^{P}

The following proposition can be easily proved.

Proposition 5.1 Let $Q = (D, R)$ be a MAXP.

- (1) If R is P-decidable and hereditary, then L_Q is in P.
- (2) If R is P-decidable, then L_Q is in coNP.
- (3) If R is NP-decidable, then L_Q is in D^{P} .
- (4) If R is coNP-decidable and hereditary, then L_Q is in D^{P} .
- (5) If R is coNP-decidable, then L_Q is in Π_2^{P} .

Similar to Proposition 4.1, we can simply show that P is a tight lower bound on the complexity of L_Q for MAXP's Q whose instance-solution relation is P-decidable and hereditary.

The following theorem gives us characterizations of coNP, D^{P} , and Π_2^{P} via MAXP's.

Theorem 5.1 The following hold:

- (1) A set L is in coNP if and only if it can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is P-decidable.
- (2) A set L is in D^{P} if and only if it can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is NP-decidable (and hereditary).
- (3) A set L is in D^{P} if and only if it can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is coNP-decidable and hereditary.

(4) A set L is in Π_2^P if and only if it can be expressed as $L = \{x : f(x) \text{ is a maximal solution of } x \text{ in } Q\}$ for some $f \in \text{FP}$ and some MAXP Q whose instance-solution relation is coNP-decidable.

From the proof of Theorem 5.1(1), we easily see that there is a MAXP whose instance-solution relation is P-decidable (but not hereditary) and whose associated decision problem is $\leq_m^P$ -complete for coNP. However, the following proposition gives us two concrete such MAXP's.

Proposition 5.2 The decision problems associated with MaxModel and MaxCubSubgraph are $\leq_m^P$ -complete for coNP:

The following corollary follows immediately from the proof of Theorem 5.1(2) and Cook's theorem.

Corollary 5.1 The decision problem associated with X -MaxModel is $\leq_m^P$ -complete for D^P .

We next show three natural MAXP's whose instance-solution relations are in NP and whose associated decision problems are $\leq_m^P$ -complete for D^P .

Proposition 5.3 The decision problems associated with MaxSat, Max- k -Colorability and MaxHamSubgraph are $\leq_m^P$ -complete for D^P :

We next show a natural MAXP whose instance-solution relation is coNP-decidable and hereditary and whose associated decision problem is $\leq_m^P$ -complete for D^P . Other such natural MAXP's may be found in [2,11,12].

Proposition 5.4 The following problem is $\leq_m^P$ -complete for D^P :

Instance: A triple $\langle \phi, X, \vec{a} \rangle$, where ϕ is a CNF boolean formula, X is a set of variables appearing only positively in ϕ , and $\vec{a}$ is a truth assignment to the variables in X .

Question: Is it the case that $\vec{a}$ has no extension satisfying ϕ but each $\vec{b} \in \Sigma^{\|X\|}$ with $\vec{a} \subset \vec{b}$ has an extension satisfying ϕ ?

6 Conclusion

In this paper, we have suggested a general framework for studying the complexity of solving maximization problems. Our results are summarized in Table 1 and Table 2. The results give, systematically, characterizations of several important complexity classes via MAXP's. An important consequence of the results is that the complexity of the problem X -MinModel is exactly captured by $\text{NPMV} // \text{OptP}[O(\log n)]$, giving an answer to an open question of Papadimitriou [9].

As seen from Table 1, the complexity of solving those MAXP's whose instance-solution relation is coNP-decidable and hereditary is unclear. Two obvious open questions are to ask whether the trivial upper bound FP^{NP} can be lowered and to ask whether the trivial lower bound $\text{FP}_{\parallel}^{\text{NP}}$ can be raised. As a step toward the investigation of the two questions, we may first consider what is the complexity of solving MinUnsat (or other natural such problems). Although $\text{FP}_{\parallel}^{\text{NP}}$ is a loose lower bound, proving the $\text{FP}_{\parallel}^{\text{NP}}$ -hardness of solving MinUnsat seems to be a hard task in the sense that at least the ideas used in proving the

D^P -hardness of the decision problem associated with MinUnsat do not work [11]. Also, showing that MinUnsat is solvable in a class below FP^{NP} needs new ideas; at least, our ideas used in the proof of Theorem 3.1 do not seem to be applicable.

It would be also interesting to consider the complexity of MAXP's whose instance-solution relation is C -decidable and hereditary for some complexity class C below P . These MAXP's are obviously solvable in FP . Are they solvable in a class below FP or is there such a MAXP Q that solving Q is complete for FP (say, under $\leq_{1-T}^{NC}$ reductions)? The two questions are important in parallel computation in the case when $C \subseteq NC$.

References

- [1] R. Anderson, A Parallel Algorithm for the Maximal Path Problem, *Proc. 17th ACM Symposium on Theory of Computing* (1985) 33-37.
- [2] J.-Y. Cai and G.E. Meyer, Graph Minimal Uncolorability is D^P -Complete, *SIAM J. Comput.*, **16** (1987), pp. 259-277.
- [3] Z.-Z. Chen and S. Toda, The Complexity of Finding a Minimal Satisfying Truth Assignment to a Boolean Formula, In submission (1992).
- [4] Z.-Z. Chen, A Randomized NC Algorithm for the Maximal Tree Cover Problem, *Inform. Process. Lett.* **40** (1991) 241-246.
- [5] R.M. Karp and A. Wigderson, A Fast Parallel Algorithm for the Maximal Independent Set Problem, *Proc. 16th ACM Symposium on Theory of Computing* (1984) 266-272.
- [6] M.W. Krentel, The Complexity of Optimization Problems, *J. Comput. System Sci.*, **36** (1988), pp. 490-509.
- [7] M. Luby, A Simple Parallel Algorithm for the Maximal Independent Set Problem, *Proc. 17th ACM Symposium on Theory of Computing* (1985) 1-10.
- [8] K. Mulmuley, U.V. Vazirani and V.V. Vazirani, Matching is as Easy as Matrix Inversion, *Proc. 19th ACM Symposium on Theory of Computing* (1987), pp. 345-354.
- [9] C.H. Papadimitriou, On Selecting a Satisfying Truth Assignment, *Proc. 32nd IEEE Symposium on the Foundations of Computer Science* (1991), pp. 163-169.
- [10] C.H. Papadimitriou, Personal Communication (1992).
- [11] C.H. Papadimitriou and D. Wolfe, The Complexity of Facets Resolved,, *Proc. 26th IEEE Symposium on the Foundations of Computer Science* (1985), pp. 74-78.
- [12] C.H. Papadimitriou and M. Yannakakis, The Complexity of Facets (and Some Facets of Complexity), *J. Comput. System Sci.*, **28** (1982), pp. 244-259.