Construction of Morse flows to a variational functional of harmonic map type

NORIO KIKUCHI
Department of Mathematics,
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi Kohoku-ku Yokohama-shi
Kanagawa-ken Japan, 223

In this paper we shall construct solutions of the parabolic differential equations associated to a simple variational functional, the Euler-Lagrange equations of which are linear equations.

Let $\Omega$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, with $C^2$-boundary $\partial \Omega$. In the following, a map $u$ means the one from $\Omega$ to $\mathbb{R}^M$, $M \geq 1$. For a map $u$ belonging to Sobolev space $H^1(\Omega)$, we consider the functional

$$F(u) = \int_{\Omega} A_{ij}^{\alpha \beta}(x) D_{\alpha}u^i(x) D_{\beta}u^j(x) dx,$$ (1)

where $u = (u^i), D_{\alpha}u^i = \partial u^i / \partial x^\alpha, 1 \leq i \leq M, 1 \leq \alpha \leq m$. The summation convention is used. The coefficients $A_{ij}^{\alpha \beta}(x), A_{ij}^{\alpha \beta} = A_{ji}^{\beta \alpha}$, are assumed to be bounded measurable in $\Omega$ and to satisfy the elliptic condition: There exists a positive $\lambda$ such that

$$A_{ij}^{\alpha \beta}(x) \xi_{\alpha}^i \xi_{\beta}^j \geq \lambda |\xi|^2 \text{ for } \xi = (\xi_{\alpha}^i) \in \mathbb{R}^{mM} \text{ and } x \in \Omega.$$ 

Hereafter, we use the notation

$$A(x)(Du, Du) = A_{ij}^{\alpha \beta}(x) D_{\alpha}u^i D_{\beta}u^j.$$ 

'Morse flows' of variational functional $F$ are defined as solutions of parabolic partial differential equations

$$\frac{\partial u^i}{\partial t} = D_{\beta}(A_{ji}^{\alpha \beta}(x) D_{\alpha}u^j) \quad (1 \leq i \leq M).$$ (2)
Let $u_0$ be a given map belonging to $H^1(\Omega)$ and $T$ a positive number. We take a positive integer $N$ and put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \cdots, N).$$

(3)

In the following, we use a function space

$$H^1_{u_0}(\Omega) = \{u \in H^1(\Omega); u - u_0 \in H^1_0(\Omega)\},$$

$H^1_0(\Omega)$ being the space obtained by taking the closure of $C_0^\infty(\Omega)$ in the space $H^1(\Omega)$. Beginning with $u_0$, we inductively construct two sequences of maps $u_n$ and functionals $F_n$, $1 \leq n \leq N$, as follows: For each $n$, $1 \leq n \leq N$, we introduce the functional

$$F_n(u) = \int_{\Omega} \left( A(x)(Du, Du) + \frac{1}{h}|u - u_{n-1}|^2 \right) dx$$

(4)

and define $u_n$ as a minimizer of $F_n$ in $H^1_{u_0}(\Omega)$, the existence of which is assured by the lower semi-continuity of $F_n$ with respect to weak convergence of $H^1(\Omega)$. We here remark the Euler-Lagrange equations of $F_n$ in $H^1_{u_0}(\Omega)$ are of the form: For $n$, $1 \leq n \leq N$,

$$\frac{u_n^i - u_{n-1}^i}{h} = D\beta(A_{ji}^\alpha\partial A_{\alpha}u_n^j) \quad (1 \leq i \leq M),$$

(5)

which are Rothe's approximate equations of (2). Upon comparing $u_{n-1}$ with a minimizer $u_n$ of $F_n$, we infer

$$\int_{\Omega} A(x)(Du_n, Du_n) dx + \int_{\Omega} \frac{1}{h}|u_n - u_{n-1}|^2 dx \leq \int_{\Omega} A(x)(Du_{n-1}, Du_{n-1}) dx$$

and hence have the following result.

**Theorem 0 ([4]).** For \{\{u_n\}(1 \leq n \leq N) constructed as above, there hold the estimates

$$\int_{\Omega} A(x)(Du_n, Du_n) dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx$$

(6)

for any $n$ $(1 \leq n \leq N)$ and

$$h \sum_{n=1}^{N} \int_{\Omega} \left( \frac{u_n - u_{n-1}}{h} \right)^2 dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx.$$

(7)
We define a map $u(t) \in H_{u_{O}}^{1}(\Omega), -h \leq t \leq T$, by means of the identities:

\[ u(t) = u_{n} \quad \text{for} \quad t_{n-1} < t \leq t_{n} \quad (1 \leq n \leq N) \]

and

\[ u(t) = u_{0} \quad \text{for} \quad -h \leq t \leq 0. \]  

(8)

We put

\[ \partial_{t}u(t) = \frac{1}{h}(u_{n} - u_{n-1}) \quad \text{for} \quad t_{n-1} < t \leq t_{n} \quad (1 \leq n \leq N) \]

and

\[ \tilde{u}(t) = u(t - h) \quad \text{for} \quad 0 \leq t \leq T. \]  

(9)

For the gradient of $u$ constructed as above, we have the estimate of higher integrability. To state the result, we shall prepare the notations as follows. We set

\[ Q = (0, T) \times \Omega. \]

For $z_{0} = (t_{n_{0}}, x_{0}) \in Q, 1 \leq n_{0} \leq N$ and positive $s$, we put

\[ Q_{s}(z_{0}) = \{t \in (0, T); t_{n_{0}} - s^{2} < t < t_{n_{0}}\} \times B_{s}(x_{0}), \]

where $B_{s}(x_{0}) = \{x \in \Omega; |x - x_{0}| < s\}$.

**Theorem 1.** For the map $u$ defined as in (8), there exist positive $C$ and $\varepsilon$ not depending on $h$ such that

\[
\left( \frac{1}{Q_{r/2}(z_{0})} |Du|^{2+\varepsilon} dz \right)^{1/(2+\varepsilon)} \leq C \left( \frac{1}{Q_{r/2}(z_{0})} |Du|^{2} dz \right)^{1/2} + ch^{(\overline{p}-1)(m+2)/2m} \left( \frac{1}{Q_{r}(z_{0})} |\partial_{t}u|^{(1+\varepsilon/2)\overline{p}} |u - \tilde{u}|^{(1+\varepsilon/2)(2-\overline{p})} dz \right)^{1/(\overline{p}+2)} \]

holds for any $Q_{r}(z_{0}) \subset Q$ and any $\overline{p}, 1 < \overline{p} < 2.$

(10)

Noting the estimates in Theorem 0 and 1 are valid uniformly in $h$, there holds the existence theorem of a weak solution to (2) with the gradient of higher integrability.
By a weak solution to parabolic system (2), we mean a map $u \in L^\infty \cap H^1 \cap H^1 (0, T, L^2 (\Omega)$ such that

$$
\int_Q \frac{\partial u^i}{\partial t} \varphi^i \, dz + \int_Q A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} \varphi^j \, dz = 0
$$

for any $\varphi \in C_0^\infty (Q)$.

**Theorem 2.** There exists a weak solution $u$ to (2) satisfying the initial and boundary conditions:

$$u(t) \in H^1_u (\Omega) \quad \text{for almost every } t \in (0, T)$$

and

$$\lim_{t \downarrow 0} u(t) = u_0 \quad \text{in } L^2 (\Omega).$$

The solution $u$ satisfies the estimate:

$$
\left( \frac{\int_{Q_r(z_0)} |Du|^{2+\varepsilon} \, dz}{\int_{Q_{r/2}(z_0)} |Du|^2 \, dz} \right)^{1/(2+\varepsilon)} \leq C \left( \frac{\int_{Q_r(z_0)} |Du|^2 \, dz}{\int_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} \, dz} \right)^{1/2}
$$

for $Q_r(z_0) \subset Q$, where $C$ and $\varepsilon$ are positive numbers as in Theorem 1.

Furthermore, if $A_{ij}^{\alpha\beta}(x)$ are continuous in $\Omega$, $u$ is Hölder continuous in $Q$ with any component $\alpha, 0 < \alpha < 1$.

The existence proof of solutions follows from the estimates in Theorem 0. Noting the higher integrability (10) of $Du$ and the estimate (7) and paralleling the method developed in [2], it follows from Campanato’s fundamental result [1] that Hölder continuity of $u$ is derived. The estimate (10) in Theorem 1 is derived from the following estimate of Caccioppoli type, for the verification of which we have only to follow the method due to Giaquinta-Stuwe([3]).

For positive $s$ satisfying $B_s(x_0) \subset \Omega$ and $u \in L^1 (Q)$, we put ([6])

$$
u_s = u_s(t) = \int_{B_s(x_0)} \eta(x) u(t, x) \, dx \quad \text{for } 0 < t < T,
$$

for $0 < t < T$, (11)

where $\eta(x) = 1$ on $B_{s/2}(x_0)$ and $|D\eta| \leq 4/s$.  

Lemma (Caccioppoli type estimate). For the map $u$ defined as in (8), there exists a positive $C$ not depending on $h$ such that

$$
\int_{Q_{r}(z_{0})} |Du|^{2}dz \leq Cr^{-2} \int_{Q_{2r}(z_{0})} |u-u_{2r}|^{2}dz
$$

$$
+Ch^{\overline{p}-1} \int_{Q_{2r}(z_{0})} |\partial_{t}u|^{\overline{p}}|u-u|^{2-\overline{p}}dz
$$

holds for any $Q_{2r}(z_{0}) \subset Q, z_{0} = (t_{n_{0}}, x_{0}), 1 \leq n_{0} \leq N$ and for any $\overline{p}, 1 < \overline{p} < 2,$ where $|\partial_{t}u|^{\overline{p}}|u-u|^{2-\overline{p}}, 1 < \overline{p} < 2,$ belongs to $L^{p}(Q)$ with some $p, p > 1,$ satisfying $p \leq m/(m - 2 + \overline{p}).$

We shall only sketch our proof. Let $k$ and $l$ be positive numbers satisfying $r < k < l < 2r.$ As a comparison map in functional $F_{n}$, we adopt $v_{n}, 1 \leq n \leq N,$ defined by

$$
v_{n} = u_{n} - h\eta(u_{n} - u_{n,l}),
$$

where $u_{n}$ is a minimizer of $F_{n}$ in $H^{1}_{u_{0}}(\Omega)$ and $u_{n,l}$ is defined as in (11).

We make the classification between $(l - k)^{2}$ and $h$ ([5]):

$$
(l - k)^{2} \leq 4h, \quad \text{(12)}
$$

$$
(l - k)^{2} > 4h. \quad \text{(13)}
$$

We treat each case of (12) and (13) and follow the iteration procedure ([2]) to obtain each estimate. By adding both the estimates, we arrive at the estimate in Lemma, which is available under no restriction of (12) and (13).

The term $|\partial_{t}u|^{\overline{p}}|u-u|^{2-\overline{p}}$ is assured to belong to $L^{p}(Q)$ with some $p, p > 1,$ satisfying $p \leq m/(m - 2 + \overline{p}),$ which is verified to hold from the global estimates (6) and (7).

References


