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Construction of Morse flows to a variational functional of harmonic map type

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In this paper we shall construct solutions of the parabolic differential equations associated to a simple variational functional, the Euler-Lagrange equations of which are linear equations.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^m, m \geq 2 \), with \( C^2 \)-boundary \( \partial \Omega \). In the following, a map \( u \) means the one from \( \Omega \) to \( \mathbb{R}^M, M \geq 1 \). For a map \( u \) belonging to Sobolev space \( H^1(\Omega) \), we consider the functional

\[
F(u) = \int_{\Omega} A_{ij}^{\alpha\beta}(x) D_{\alpha}u^i(x) D_{\beta}u^j(x) dx,
\]

where \( u = (u^i), D_{\alpha}u^i = \partial u^i / \partial x^\alpha, 1 \leq i \leq M, 1 \leq \alpha \leq m \). The summation convention is used. The coefficients \( A_{ij}^{\alpha\beta}(x), A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} \), are assumed to be bounded measurable in \( \Omega \) and to satisfy the elliptic condition: There exists a positive \( \lambda \) such that

\[
A_{ij}^{\alpha\beta}(x) \xi^i_{\alpha} \xi^j_{\beta} \geq \lambda |\xi|^2 \quad \text{for} \quad \xi = (\xi^i_{\alpha}) \in \mathbb{R}^{mM} \quad \text{and} \quad x \in \Omega.
\]

Hereafter, we use the notation

\[
A(x)(Du, Du) = A_{ij}^{\alpha\beta}(x) D_{\alpha}u^i D_{\beta}u^j.
\]

'Morse flows' of variational functional \( F \) are defined as solutions of parabolic partial differential equations

\[
\frac{\partial u^i}{\partial t} = D_{\beta}(A_{ij}^{\alpha\beta}(x) D_{\alpha}u^j) \quad (1 \leq i \leq M).
\]
Let $u_0$ be a given map belonging to $H^1(\Omega)$ and $T$ a positive number. We take a positive integer $N$ and put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \cdots, N). \quad (3)$$

In the following, we use a function space

$$H_{u_0}^1(\Omega) = \{u \in H^1(\Omega); u - u_0 \in H^1_0(\Omega)\},$$

$H^1_0(\Omega)$ being the space obtained by taking the closure of $C_0^\infty(\Omega)$ in the space $H^1(\Omega)$. Beginning with $u_0$, we inductively construct two sequences of maps $u_n$ and functionals $F_n$, $1 \leq n \leq N$, as follows: For each $n$, $1 \leq n \leq N$, we introduce the functional

$$F_n(u) = \int_{\Omega} \left( A(x)(Du, Du) + \frac{1}{h} |u - u_{n-1}|^2 \right) dx \quad (4)$$

and define $u_n$ as a minimizer of $F_n$ in $H^1_{u_0}(\Omega)$, the existence of which is assured by the lower semi-continuity of $F_n$ with respect to weak convergence of $H^1(\Omega)$. We here remark the Euler-Lagrange equations of $F_n$ in $H^1_{u_0}(\Omega)$ are of the form: For $n$, $1 \leq n \leq N$,

$$\frac{u_n^i - u_{n-1}^i}{h} = D_\beta (A_{ji}^{\alpha\beta} D_\alpha u_n^j) \quad (1 \leq i \leq M), \quad (5)$$

which are Rothe’s approximate equations of (2). Upon comparing $u_{n-1}$ with a minimizer $u_n$ of $F_n$, we infer

$$\int_{\Omega} A(x)(Du_n, Du_n) dx + \int_{\Omega} \frac{1}{h} |u_n - u_{n-1}|^2 dx \leq \int_{\Omega} A(x)(Du_{n-1}, Du_{n-1}) dx$$

and hence have the following result.

**Theorem 0 ([4]):** For $\{u_n\} (1 \leq n \leq N)$ constructed as above, there hold the estimates

$$\int_{\Omega} A(x)(Du_n, Du_n) dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx \quad \text{for any } n \ (1 \leq n \leq N) \quad (6)$$

and

$$h \sum_{n=1}^{N} \int_{\Omega} \left| \frac{u_n - u_{n-1}}{h} \right|^2 dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx. \quad (7)$$
We define a map $u(t) \in H_{u_0}^{1}(\Omega), -h \leq t \leq T$, by means of the identities:

$$u(t) = u_n \quad \text{for} \quad t_{n-1} < t \leq t_n \quad (1 \leq n \leq N)$$

and

$$u(t) = u_0 \quad \text{for} \quad -h \leq t \leq 0. \tag{8}$$

We put

$$\partial_t u(t) = \frac{1}{h} (u_n - u_{n-1}) \quad \text{for} \quad t_{n-1} < t \leq t_n \quad (1 \leq n \leq N)$$

and

$$\tilde{u}(t) = u(t - h) \quad \text{for} \quad 0 \leq t \leq T. \tag{9}$$

For the gradient of $u$ constructed as above, we have the estimate of higher integrability. To state the result, we shall prepare the notations as follows. We set

$$Q = (0, T) \times \Omega.$$ 

For $z_0 = (t_{n_0}, x_0) \in Q, 1 \leq n_0 \leq N$ and positive $s$, we put

$$Q_s(z_0) = \{t \in (0, T); t_{n_0} - s^2 < t < t_{n_0}\} \times B_s(x_0),$$

where $B_s(x_0) = \{x \in \Omega; |x - x_0| < s\}$.

**Theorem 1.** For the map $u$ defined as in (8), there exist positive $C$ and $\epsilon$ not depending on $h$ such that

$$\left( \int_{Q_r/2(z_0)} |Du|^{2+\epsilon} \, dz \right)^{1/(2+\epsilon)} \leq C \left( \int_{Q_r/2(z_0)} |Du|^2 \, dz \right)^{1/2}$$

$$+ ch^{(\bar{p}-1)(m+2)/2m} \left( \int_{Q_r(z_0)} |\partial_t u|^{(1+\epsilon/2)\bar{p}} |u - \tilde{u}|^{(1+\epsilon/2)(2-\bar{p})} \, dz \right)^{1/(2+\epsilon)} \tag{10}$$

holds for any $Q_r(z_0) \subset Q$ and any $\bar{p}, 1 < \bar{p} < 2$.

Noting the estimates in Theorem 0 and 1 are valid uniformly in $h$, there holds the existence theorem of a weak solution to (2) with the gradient of higher integrability.
By a weak solution to parabolic system (2), we mean a map $u \in L^\infty \ (0, \infty), H^1 (\Omega) \cap H^1 \ (0, T), L^2 (\Omega)$ such that

$$\int_{Q} \frac{\partial u^i}{\partial t} \varphi^i dz + \int_{Q} A_{ij}^{\alpha\beta} (x) D_{\alpha} u^i D_{\beta} \dot{\psi} dz = 0$$

for any $\varphi \in C_0^\infty (Q)$.

**Theorem 2.** There exists a weak solution $u$ to (2) satisfying the initial and boundary conditions:

$$u(t) \in H^1_{u_0} (\Omega) \quad \text{for almost every } t \in (0, T)$$

and

$$\lim_{t \downarrow 0} u(t) = u_0 \quad \text{in } L^2 (\Omega).$$

The solution $u$ satisfies the estimate:

$$\left( \frac{\int_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} dz}{\int_{Q_r(z_0)} |Du|^2 dz} \right)^{1/(2+\varepsilon)} \leq C \left( \frac{\int_{Q_{r}(z_0)} |Du|^2 dz}{\int_{Q_{r}(z_0)} |Du|^2 dz} \right)^{1/2}$$

for $Q_r(z_0) \subset Q$, where $C$ and $\varepsilon$ are positive numbers as in Theorem 1. Furthermore, if $A_{ij}^{\alpha\beta} (x)$ are continuous in $\Omega$, $u$ is Hölder continuous in $Q$ with any component $\alpha, 0 < \alpha < 1$.

The existence proof of solutions follows from the estimates in Theorem 0. Noting the higher integrability (10) of $Du$ and the estimate (7) and paralleling the method developed in [2], it follows from Campanato's fundamental result [1] that Hölder continuity of $u$ is derived. The estimate (10) in Theorem 1 is derived from the following estimate of Caccioppoli type, for the verification of which we have only to follow the method due to Giaquinta-Stuwe([3]).

For positive $s$ satisfying $B_s(x_0) \subset \Omega$ and $u \in L^1 (Q)$, we put ([6])

$$u_s = u_s (t) = \int_{B_s(x_0)} \eta(x) u(t, x) dx$$

for $0 < t < T$, (11)

where $\eta(x) = 1$ on $B_{s/2}(x_0)$ and $|D \eta| \leq 4/s$. 
Lemma (Caccioppoli type estimate). For the map \( u \) defined as in (8), there exists a positive \( C \) not depending on \( h \) such that
\[
\int_{Q_r(z_0)} |Du|^2 \, dz \leq Cr^{-2} \int_{Q_{2r}(z_0)} |u - u_{2r}|^2 \, dz
+ Ch^{\overline{p}-1} \int_{Q_{2r}(z_0)} |\partial_t u|^\overline{p} |u - \tilde{u}|^{2-\overline{p}} \, dz
\]
holds for any \( Q_{2r}(z_0) \subset Q, z_0 = (t_n, x_0), 1 \leq n_0 \leq N \) and for any \( \overline{p}, 1 < \overline{p} < 2 \), where \( |\partial_t u|^\overline{p} |u - \tilde{u}|^{2-\overline{p}}, 1 < \overline{p} < 2 \), belongs to \( L^p(Q) \) with some \( p, p > 1 \), satisfying \( p \leq m/(m-2+\overline{p}) \).

We shall only sketch our proof. Let \( k \) and \( l \) be positive numbers satisfying \( r < k < l < 2r \). As a comparison map in functional \( F_n \), we adopt \( v_n, 1 \leq n \leq N \), defined by
\[
v_n = u_n - h\eta(u_n - u_{n,l}),
\]
where \( u_n \) is a minimizer of \( F_n \) in \( H^1_0(\Omega) \) and \( u_{n,l} \) is defined as in (11).

We make the classification between \((l - k)^2 \) and \( h \) ([5]):
\[
(l - k)^2 \leq 4h, \tag{12}
\]
\[
(l - k)^2 > 4h. \tag{13}
\]
We treat each case of (12) and (13) and follow the iteration procedure ([2]) to obtain each estimate. By adding both the estimates, we arrive at the estimate in Lemma, which is available under no restriction of (12) and (13).

The term \( |\partial_t u|^\overline{p} |u - \tilde{u}|^{2-\overline{p}} \) is assured to belong to \( L^p(Q) \) with some \( p, p > 1 \), satisfying \( p \leq m/(m-2+\overline{p}) \), which is verified to hold from the global estimates (6) and (7).

References


