Gale's Theorem on an Infinite Network

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1. Introduction with problem setting

Let $X$ be a countable set of nodes, $Y$ be a countable set of arcs and $K$ be the node-arc incidence function. We assume that the graph $G = \{X, Y, K\}$ is connected, and has no self-loops as in [4]. Notice that $G$ is not necessarily locally finite. Let $R$ be the set of all real numbers and denote by $L(X; Z)$ the set of all functions from $X$ to a set $Z$. In particular, we set $L(X) = L(X; R)$ and $L(Y) = L(Y; R)$.

For each $y \in Y$, the nodes $x^+(y)$ and $x^-(y)$ are determined uniquely by the relation:

$$K(x^+(y), y) = 1 \text{ and } K(x^-(y), y) = -1.$$ 

Intuitively, $x^-(y)$ (resp. $x^+(y)$) is the initial (resp. terminal) node of $y$. For a nonempty subset $A$ of $X$, we put for simplicity

$$Q_-(A) = \{y \in Y; x^-(y) \in A \text{ and } x^+(y) \in X - A\}$$
$$Q_+(A) = \{y \in Y; x^-(y) \in X - A \text{ and } x^+(y) \in A\}.$$ 

Notice that $Q_-(A) \cup Q_+(A)$ is a cut $A \Theta(X - A)$ in [4].

In this paper we always assume that the functions $V, W \in L(Y)$, $\lambda \in L(X; R \cup \{-\infty\})$ and $\mu \in L(X; R \cup \{\infty\})$ satisfy the following conditions:

$$V(y) \leq W(y) \text{ on } Y; \quad (1)$$
$$\sum_{y \in Y} |V(y)| < \infty, \sum_{y \in Y} |W(y)| < \infty; \quad (2)$$
$$\lambda(x) \leq \mu(x) \text{ on } X, \sum_{x \in \Lambda} \lambda(x) < \infty, -\infty < \sum_{x \in \Gamma} \mu(x), \quad (3)$$
where $\Lambda = \{x \in X; \lambda(x) > 0\}, \Gamma = \{x \in X; \mu(x) < 0\}$.

The feasibility problem of Gale is to find $w \in L(Y)$ which has the following properties:

(G.1) \hspace{1cm} V(y) \leq w(y) \leq W(y) \text{ on } Y;

(G.2) \hspace{1cm} \lambda(x) \leq \sum_{y \in Y} K(x, y)w(y) \leq \mu(x) \text{ on } X.

The algebraic operations and order relation of $R$ are extended to $R \cup \{-\infty\}$ or $R \cup \{\infty\}$ in the usual way, i.e.,

$0 \cdot \infty = 0 \cdot (-\infty) = 0$;
$t + \infty = \infty, -\infty + t = -\infty$ for all $t \in R$;
$t \cdot \infty = \infty, t \cdot (-\infty) = -\infty$ for all $t > 0$.

To state our main theorem, we introduce a notation. For a subset $A$ of $X$ and a function $f \in L(X; R \cup \{-\infty\}) \cup L(X; R \cup \{\infty\})$, we put

$f(A) = \sum_{x \in A} f(x)$

if the sum is well-defined and $f(\emptyset) = 0$ for the empty set $\emptyset$. The quantity $w(Q)$ for a subset $Q$ of $Y$ and $w \in L(Y)$ is defined similarly.

Our aim of this paper is to prove the following theorem:

**Theorem 1.1** The feasibility problem of Gale has a solution if and only if the given functions $V, W, \lambda$ and $\mu$ satisfy the relation:

(H.1) \hspace{1cm} \lambda(A), -\mu(X - A) \leq W(Q_+(A)) - V(Q_-(A))

for every nonempty subset $A$ of $X$.

Gale [2] proved this theorem in the case where $G$ is a finite graph without multiple arcs, i.e., for every two nodes, there exists at most one arc. An abstract Flow Theorem in B.Fuchssteiner and Lusky [1] and the theorem of Gale for infinite networks in M.M.Neumann [3] may be regarded as a generalization of the feasibility theorem of gale. In their problem settings, the set of nodes of the network is a nonempty set $S$ endowed some algebra $\Sigma$ of subsets and a flow is a biadditive set functions from $\Sigma \times \Sigma$ to an ordered real vector space which is Dedekind complete. Note that even if $S = X$, their infinite network is assumed not to have multiple arcs. Notice that the feasible solution in [1] and [2] does not give an answer to our flow even if $G$ has no multiple arcs.
2. Reduction of Theorem 1.1

First we prove the only if part of Theorem 1.1. Let $w$ be a feasible solution of $(G.1)$ and $(G.2)$ and $A$ be a nonempty subset of $X$. Then

$$
\lambda(A) \leq \sum_{x \in A} \sum_{y \in Y} K(x, y)w(y) \quad \text{by (G.2) and (2)}
$$

$$
= \sum_{y \in Y} w(y) \sum_{x \in A} K(x, y)
$$

$$
= \sum_{y \in Q_+(A)} w(y) - \sum_{y \in Q_-(A)} w(y)
$$

$$
\leq W(Q_+(A)) - V(Q_-(A)). \quad \text{by (G.1)}
$$

The inequality for $\mu(A)$ can be proved similarly.

To prove the "if" part, we may assume that $V = 0$. In fact, let $\tilde{V} = 0, \tilde{W} = W - V,$

$$
\tilde{\lambda}(x) = \lambda(x) + \sum_{y \in Y} K(x, y)V(y), \\
\tilde{\mu}(x) = \mu(x) + \sum_{y \in Y} K(x, y)V(y).
$$

If there exists $\tilde{w} \in L(Y)$ which satisfies the relation:

$$
0 \leq \tilde{w}(y) \leq \tilde{W}(y) \text{ on } Y, \\
\tilde{\lambda}(x) \leq \sum_{y \in Y} K(x, y)\tilde{w}(y) \leq \tilde{\mu}(x) \text{ on } X,
$$

then $w(y) = \tilde{w}(y) + V(y)$ satisfies $(G.1)$ and $(G.2)$.

3. Preliminaries

A function $f \in L(X)$ is called simple if its range is a finite set. Denote by $L_S(X)$ the set of real valued simple functions on $X$. Hereafter we put

$$
E = L_S(X) \text{ and } F = L_S(Y).
$$

For a subset $A$ of $X$ and a subset $Q$ of $Y$, denote by $\epsilon_A$ and $\varphi_Q$ their characteristic functions respectively. Denote by $L_S(Y;E)$ the set of $E$-valued functions on $Y$, i.e., $\psi \in L_S(Y;E)$ can be written in the form $\psi = \sum_{i=1}^{n} f_i \varphi_{Q_i}$, where $f_1, \cdots, f_n \in E$ and $Q_1, \cdots, Q_n$ are mutually disjoint subsets of $Y$.

For each $f \in L(X)$, let us define $\theta(f) \in L(Y)$ by
\[ \theta(f)(y) = \max\{0, \sum_{x \in X} K(x, y)f(x)\} \]
as in [1] and [2]. The following properties are easily seen:

1. \[ \theta(\epsilon_A) = \varphi_Q + (A) \]
2. \[ \theta(-\epsilon_A) = \varphi_Q - (A) \]
3. \[ \theta(f - g) = \theta(f) + \theta(-g) \]
for \( f, g \in L^+(X) \cap E \) such that \( f(x)g(x) = 0 \) on \( X \);
4. \[ \theta\left(\sum_{i=1}^{n} t_i\epsilon_{A_i}\right) = \sum_{i=1}^{n} t_i\theta(\epsilon_{A_i}) \]
for all \( t_1, \ldots, t_n \geq 0 \) and all \( A_i \) such that \( A_1 \supset \cdots \supset A_n \).

We prepare

**Lemma 3.1** For each \( \psi \in L_S(Y; E) \), the function \( \hat{\psi} \) defined by
\[ \hat{\psi}(y) := \theta(\psi(y))(y) \]
belongs to \( F \).

**Proof.** By definition,
\[ \psi(y) = \sum_{i=1}^{n} f_i\varphi_{Q_i}(y) \]
with \( f_i \in L_S(X) \) and mutually disjoint subsets \( Q_i \) of \( Y \). In case \( y \) does not belong any one of \( Q_i \), \( \psi(y) = 0 \in L(X) \) and \( \theta(\psi(y))(y) = 0 \). If \( y \in Q_i \), then \( \psi(y) = f_i \) and \( \hat{\psi}(y) = \theta(f_i)(y) \). Thus
\[ \hat{\psi}(y) = \sum_{i=1}^{n} [\theta(f_i)(y)]\varphi_{Q_i}(y) \]
and \( \hat{\psi} \in F \).

**Lemma 3.2** Let \( f \in L^+_S(X) := L_S(X) \cap L^+(X) \) and assume that the number of elements in the range of \( f \) is equal to \( n \). Then there exist non-negative numbers \( t_1, \ldots, t_n \) and subsets \( A_1, \ldots, A_n \) of \( X \) such that \( A_1 \supset \cdots \supset A_n \) and
\[ f(x) = \sum_{i=1}^{n} t_i\epsilon_{A_i}(x) \]

**Proof.** There existe a class \( \{B_i\} \) of mutually disjoint subsets of \( X \) such that \( f(x) = \alpha_i \) on \( B_i(i = 1, \ldots, n) \) and \( \alpha_i \neq \alpha_j \) if \( i \neq j \). Clearly we have
\[ f(x) = \sum_{i=1}^{n} \alpha_i\epsilon_{B_i}(x) \]
Without any loss of generality, we may assume that \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \). Define \( A_i \) and \( t_i \) as follows:
\[ A_i = \bigcup_{j=i}^{n}B_j (1 \leq i \leq n); \]
\[ t_1 = \alpha_1 \text{ and } t_i = \alpha_i - \alpha_{i-1} (2 \leq i \leq n). \]

Then our assertion is easily seen.

4. Proof of Theorem 1.1

To prove the reduced "if" part of Theorem 1.1, we assume that condition (H.1) holds and \( V = 0 \).

Let us put

\[ G = L_S(Y; E). \]

Then \( G \) is a linear space. We shall identify each \( f \in E \) with \( \psi_f = f \varphi_Y \in G \).

In this sense, \( E \subset G \).

Let us introduce the convex cones \( K_{\lambda} \) and \( K_{\mu} \) in \( E \):

\[ K_{\lambda} = \{ f \in L_S^+(X); \sum_{x \in X} f(x)|\lambda(x)| < \infty \}; \]
\[ K_{\mu} = \{ g \in L_S^+(X); \sum_{x \in X} g(x)|\mu(x)| < \infty \}. \]

Now assume that \( W \in L^+(Y) \) and \( W(Y) = \sum_{y \in Y} W(y) < \infty \). We shall consider a functional \( \rho \) on \( G \) as in [1] and [3]:

\[ \rho(\psi) := \sum_{y \in Y} \hat{\psi}(y)W(y). \]

To verify \( \rho(\psi) \) is finite, let \( \psi = \sum_{i=1}^{n} f_i \varphi_Q, \) as in Lemma 3.1. Let \( m_i = \min\{f_i(x); x \in X\} \) and \( M_i = \max\{f_i(x); x \in X\} \). Then

\[ \theta(f_i)(y) \leq M_i - m_i, \]
\[ 0 \leq \hat{\psi}(y) \leq \max\{\theta(f_i)(y); i = 1, \cdots, n\} \leq c(\hat{\psi}) \text{ on } Y, \]
where \( c(\hat{\psi}) = \max\{M_i - m_i; i = 1, \cdots, n\} \). Therefore

\[ 0 \leq \rho(\psi) \leq c(\hat{\psi})W(Y) < \infty. \]

Notice that \( \theta \) is sublinear, i.e., \( \theta(\alpha f + \beta g)(y) \leq \alpha \theta(f)(y) + \beta \theta(g)(y) \) on \( Y \) for every \( f, g \in L_S(X) \) and \( \alpha, \beta \geq 0 \). Therefore for \( \psi = \psi_1 + \psi_2, (\psi_1, \psi_2 \in G) \), we have

\[ \hat{\psi}(y) \leq \hat{\psi}_1(y) + \hat{\psi}_2(y) \text{ on } Y. \]

Namely the mapping \( \psi \rightarrow \hat{\psi} \) is sublinear.
Lemma 4.1 Assume that condition (H.1) holds with $V = 0$. Then
\[
\sum_{x \in X} \lambda(x)f(x) - \sum_{x \in X} \mu(x)g(x) \leq \rho(f - g)
\]
holds for every $f \in K_{\lambda}$ and $g \in K_{\mu}$.

Proof. For $f \in K_{\lambda}$ and $g \in K_{\mu}$, we put $\tilde{f} = (f - g)^+$ and $\tilde{g} = (f - g)^-$. Then
\[
\tilde{f} \in K_{\lambda}, \tilde{g} \in K_{\mu}, \tilde{f}(x)\tilde{g}(x) = 0 \text{ on } X \text{ and } f - \tilde{f} = g - \tilde{g} \in K_{\lambda} \cap K_{\mu}.
\]
By Lemma 3.2, $\tilde{f}$ and $\tilde{g}$ can be expressed as follows:
\[
\tilde{f} = \sum_{i=1}^{m} \alpha_{i} \epsilon_{A_{i}} \text{ and } \tilde{g} = \sum_{j=1}^{n} \beta_{j} \epsilon_{B_{j}},
\]
where $\alpha_{i}, \beta_{j} \geq 0$, $A_{1} \supset \cdots \supset A_{m}$ and $B_{1} \supset \cdots \supset B_{n}$. By using the properties of $\theta$, we have
\[
\rho(f - g) = \rho(\tilde{f} - \tilde{g})
\]
\[
= \sum_{y \in Y} [\theta(\tilde{f} - \tilde{g})(y)\varphi_{Y}(y)]W(y) \quad \text{by (\theta.3)}
\]
\[
= \sum_{y \in Y} [\theta(\tilde{f})(y) + \theta(-\tilde{g})(y)]W(y)
\]
\[
= \sum_{y \in Y} [\theta(\sum_{i=1}^{m} \alpha_{i} \epsilon_{A_{i}})(y) + \theta(-\sum_{j=1}^{n} \beta_{j} \epsilon_{B_{j}})(y)]W(y) \quad \text{by (\theta.4)}
\]
\[
= \sum_{y \in Y} \sum_{i=1}^{m} \alpha_{i}[\theta(\epsilon_{A_{i}})(y)]W(y) + \sum_{j=1}^{n} \beta_{j}[\theta(-\epsilon_{B_{j}})(y)]W(y) \quad \text{by (\theta.1) and (\theta.2)}
\]
\[
= \sum_{j=1}^{m} \alpha_{i}W(Q_{+}(A_{i})) + \sum_{j=1}^{n} \beta_{j}W(Q_{-}(B_{j}))
\]
\[
\geq \sum_{i=1}^{m} \alpha_{i}\lambda(A_{i}) - \sum_{j=1}^{n} \beta_{j}\mu(B_{j})
\]
\[
= \sum_{x \in X} \lambda(x)f(x) - \sum_{x \in X} \mu(x)g(x) \quad \text{by (3)}
\]
\[
\geq \sum_{x \in X} \lambda(x)f(x) - \sum_{x \in X} \mu(x)g(x).
\]
For each \( h \in K := K_\lambda - K_\mu \), define \( \Phi(h) \) by

\[
\Phi(h) = \sup \{ \sum_{x \in X} \lambda(x)f(x) - \sum_{x \in X} \mu(x)g(x); h = f - g, f \in K_\lambda, g \in K_\mu \}.
\]

Then it is easily seen that \( \Phi \) is superlinear on \( K \), i.e.,

\[
\Phi(\alpha h_1 + \beta h_2) \geq \alpha \Phi(h_1) + \beta \Phi(h_2)
\]

for every \( h_1, h_2 \in K \) and \( \alpha, \beta \geq 0 \). Notice that by Lemma 4.1

(4.1) \( \Phi(h) \leq \rho(h) \) for all \( h \in K \).

Clearly \( K \) is a convex subset of \( G \). For a sublinear functional \( \rho \) on \( G \) and a superlinear functional \( \Phi \) on \( K \) which satisfy (4.1), we can apply the Sandwich Theorem in [1]. Thus there exists a linear functional \( \xi \) on \( G \) such that

(4.2) \( \Phi(h) \leq \xi(h) \) for every \( h \in K \),

(4.3) \( \xi(\psi) \leq \rho(\psi) \) for every \( \psi \in G \).

For each \( y \in Y \), let us put

\[
\psi_y^+ := \epsilon_{\{x^+(y)\}} \varphi_{\{y\}} \quad \text{and} \quad \psi_y^- := \epsilon_{\{x^-(y)\}} \varphi_{\{y\}}.
\]

Then we have \( \psi_y^+ \), \( \psi_y^- \in G \) and \( \psi_y^+ + \psi_y^- = \epsilon_{e(y)} \varphi_{\{y\}} \) with \( e(y) = \{x^+(y), x^-(y)\} \), so that

\[
\rho(\psi_y^+) = W(y), \rho(-\psi_y^+) = 0 \quad \text{and} \quad \rho(\psi_y^+ + \psi_y^+) = \rho(-(-\psi_y^+ + \psi_y^-)) = 0
\]

Now we define \( w \in L(Y) \) by

(4.4) \( w(y) := \xi(\psi_y^+) = \xi(\epsilon_{\{x^+(y)\}} \varphi_{\{y\}}) \).

By (4.3) and the above observation, we obtain

\[
0 \leq w(y) \leq W(y) \quad \text{on} \quad Y \quad \text{and} \quad \xi(\psi_y^-) = -w(y).
\]

Our next goal is to prove that \( w \) satisfies (G.2) with \( V = 0 \). Let \( a \in X \) be any node such that \( \lambda(a) \in R \) and put
\[ Y' = \{ y \in Y; a \in e(y) \}. \]

Then, for every \( y \in Y' \)
\[ \theta(\epsilon_{\{a\}}(y)) = \theta(-\epsilon_{\{a\}}(y)) = 0, \]
so that
\[ \rho(\epsilon_{\{a\}}\varphi_{Y'}) = \rho(-\epsilon_{\{a\}}\varphi_{Y'}) = 0. \]

Therefore, by (4.4), \( \xi(\epsilon_{\{a\}}\varphi_{Y'}) = 0. \) For simplicity, put
\[ Y_+(a) = \{ y \in Y; K(a, y) = 1 \} \]
\[ Y_-(a) = \{ y \in Y; K(a, y) = -1 \}. \]

By (4.2), we have
\[
\lambda(a) = \sum_{x \in X} \lambda(x)\epsilon_{\{a\}}(x) \\
\leq \xi(\epsilon_{\{a\}}\varphi_Y) \\
= \sum_{y \in Y_+(a)} \xi(\psi^+_y) + \sum_{y \in Y_-(a)} \xi(\psi^-_y) + \xi(\epsilon_{\{a\}}\varphi_{Y'}) \\
= \sum_{y \in Y_+(a)} w(y) - \sum_{y \in Y_-(a)} w(y) \\
= \sum_{y \in Y} K(a, y)w(y).
\]

Similarly we have
\[
\sum_{y \in Y} K(a, y)w(y) \leq \mu(a)
\]
for every \( a \in X \) such that \( \mu(a) \in R \). The resulting estimates are obvious if \( \lambda(a) = -\infty(\mu(a) = \infty) \). This completes the proof.
Reference


