数理解析研究所講究録
836巻 1993年 12-19

Inverse Problems - Gaussian Beams and Computations.

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I am a representative of an inverse problem group of the St.Petersburg Branch Mathematical Institute (POMI). The group consists of three members - M. Belishev, A. Katchalov, Ya. Kurylev. At the beginning of my talk I want to discuss some results obtained by our group during several years of investigations.

1. The problem under consideration.

Let's consider the hyperbolic equation in the domain Ω

$$u_{tt}(x,t) - L\,u(x,t) = 0, \quad (x,t)\in \Omega\times\mathbb{R}, \qquad (1)$$

where L is one of the following elliptical operators (see also the paper of Belishev & Kurylev and preprint of Belishev & Katchalov)

i) $Lu = \rho^{-1}\Delta u$, $\rho(x)$ is a density,

ii) $Lu = \mathrm{div}(\mu\,\mathrm{grad} u)$, $\mu(x)$ is a tension,

iii) $Lu = \Delta u - qu$, $q(x)$ is a potential,

iv) $Lu = \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{g(x)}\circ g^{ik}(x)\frac{\partial u}{\partial x^k}\right)$ is the Beltrami-Laplace operator in local coordinates x^i, g_{ik} is a metric tensor of the manifold Ω, $g^{ik}\circ g_{kl} = \delta^i_l$, $g = \det\{g_{ik}\}^n_{i,k=1}$.

The problem is to recover material parameters of media (density ρ, tension μ, potential q or metric tensor g_{ik}) in the domain Ω via inverse data.

2. Inverse data.

To solve these problems we use two type of data - spectral data and dynamic (nonstationary) inverse data (a nonstationary response operator).

i) Spectral inverse data.

To describe spectral inverse data let's consider a bounded domain Ω and the elliptical differential operator L with the Dirichlet or Neumann boundary condition on $\Gamma = \partial\Omega$. For the Neumann boundary condition spectral inverse data consist of the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ and traces on the boundary Γ of normalized eigenfunctions $\{\Phi_k(\gamma)\}_{k=1}^{\infty}$, $\gamma\in\Gamma$. For the Dirichlet boundary condition they consist of the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ and traces of normal derivatives of normalized eigenfunctions $\left\{\frac{\partial\Phi}{\partial n}k(\gamma)\right\}_{k=1}^{\infty}$.

ii) The nonstationar response operator.

The other possible variant of inverse data is a nonstationar response operator. To describe it let's consider the following well-posed initial-boundary value problem in the domain Ω

$$\begin{cases} u_{tt} - Lu = 0 \ \text{in} \ \Omega\times(0,T) \\ u|_{t<0} \equiv 0 \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma\times[0,T]} = f \end{cases} \tag{2}$$

(for the case ii we should replace $\frac{\partial u}{\partial n}$ on $\mu\frac{\partial u}{\partial n}$ in the boundary condition). The boundary function $f(\gamma,t)\in \mathbb{F}^T = L_2(\Gamma\times(0,T))$ or control generate a solution $u^f(x,t)$ of the problem (2) or wave field in Ω, $(x,t)\in\Omega\times(0,T)$. Let's consider a continuous linear operator R^T in $\mathbb{F}^T$, $R^T: f(\gamma,t) \longrightarrow g(\gamma,t) = u^f|_{\Gamma\times[0,T]}$. The operator R^T is known as a nonstationar response operator. It has an integral representation

$$(R^T f)(\gamma,t) = \int_{\Gamma\times[0,T]} d\Gamma_{\gamma'} dt' \Gamma(\gamma,\gamma',t-t')\circ f(\gamma',t'). \tag{3}$$

We consider this operator (or its kernel) as data to solve our inverse problems.

Remark 1. Spectral variant of the problem can be reduced to a nonstationary one. For example, for the Neumann boundary condition we have the following equality

$$(R^T f)(\gamma,t) = \int_{\Gamma\times[0,T]} d\Gamma_{\gamma'} dt' \{ \sum_{k=1}^{\infty} \frac{\sin\sqrt{\lambda_k}(t-t')}{\sqrt{\lambda_k}} \Phi_k(\gamma)\circ\Phi_k(\gamma')\}\circ f(\gamma',t'). \tag{4}$$

This equality expresses the nonstationar response operator via spectral inverse data. For the Dirichlet boundary conditions there is analogous formula. So further we may consider only dynamic inverse data to solve inverse problems.

3. Formulation the problem. BCT approach.

Where we can recover material parameters of the medium for the given response operator R^T? To answer this question we should define a subdomain Ω^T. Let $\tau(x) = \inf\{\,\mathrm{dist}(x,y);\ y\in\Gamma\}$, $x\in\Omega$ be a distance between the point of the domain Ω and the boundary Γ.

Distance (travel time) is measured in a metric generated by the elliptic operator L. Let the subdomain $\Omega^T = \{ x \in \Omega, \ \tau(x) < T\}$.

In a vicinity of Γ (more precisely outside the cut locus of geodesic normals to the boundary Γ) we have a semigeodesic (ray) coordinate system (γ, τ), $\gamma \in \Gamma$. **We can recover material parameters in Ω^T via given R^{2T}.** Doubling time is connected with very simple physical fact. To obtain information about a point x, $\tau(x) = T$ the wave should go from the boundary to the point x and then return to the boundary. Method of solving inverse problems bases upon the Boundary Control Theory (BCT) approach. The approach was put forward by M. Belishev. Further we describe very shortly main steps of the approach.

4. Recovering C^T operator.

Let's consider the functional space $\mathbb{H}$ in the domain Ω (inner space) with the inner product generated by the operator L. For the problem i) for example it is $L_{2,\rho}(\Omega)$. Using the solution of the problem (1) we can define a control operator $W^T: \mathbb{F}^T \to \mathbb{H}$, $W^T f = u^f(\cdot, T)$. It is clear that $W^T \mathbb{F}^T \subseteq \mathbb{H}^T$, where $\mathbb{H}^T = \{u \in \mathbb{H}, \ \operatorname{supp} u \in \Omega^T\}$. The main suggestion of BCT approach is

$$\operatorname{Cl} W^T \mathbb{F}^T = \mathbb{H}^T. \tag{5}$$

A system consisted of the domain Ω and material parameters (ρ, μ, q, g_{ik}) in it we call the controllable one if the property (5) is valid. It is equivalent to the Holmgren - John uniqueness theorem. If it is true we can use the inner product in the space $\mathbb{H}$ to define some new inner product in the space $\mathbb{F}^T$

$$\langle f, g\rangle = (W^T f, W^T g)_{\mathbb{H}} = (u^f(\cdot, T), u^g(\cdot, T))_{\mathbb{H}}. \tag{6}$$

So we have some operator C^T defined by the formula

$$(C^T f, g)_{\mathbb{F}} = \langle f, g\rangle. \tag{7}$$

It is possible to prove that C^T is a selfadjoint operator, C^T being positive for $T < T_*$. The critical time T_* is that of filling the domain Ω, $\Omega^{T_*} = \Omega$. Let Φ^T be a completion of $\mathbb{F}^T$ according C^T norm ($|f|_{\Phi^T} = \langle f, f\rangle$). The space Φ^T is called the outer space.

The C^T operator is the main instrument for recovering parameters of media. It can be obtained via the response operator R^{2T} by means of the Blagovestchensky formula

$$C^T := -\frac{1}{2} \circ [S^T]^* \circ Y^{2T} \circ R^{2T} \circ S^T, \tag{8}$$

where S^T is an operator of odd continuation through the time T

$$S^T f(\gamma,t) = \begin{cases} f(\gamma,t) & t\in (0,T) \\ -f(\gamma,2T-t) & t\in (T,2T) \end{cases}, \tag{9}$$

and Y^{2T} is an integrating operator.

5. Recovering the set of orthoprojectors.

Let's consider the set of orthoprojectors P^ξ, $\xi<T$ on the subspace $\mathbb{H}^\xi$: P^ξ: $\mathbb{H}^T \longrightarrow \mathbb{H}^\xi$ and corresponding set of orthoprojectors in the C^T norm on the space Φ^T, Q^ξ: $\Phi^T \longrightarrow \Phi^T$, $W^T Q^\xi = P^\xi W^T$. It is important that we can reconstruct the set of orthoprojectors Q^ξ (but not P^ξ) via C^T operator. We can do that using the following procedure.

i) Let's take a basis of controls $\{f_k\}_{k=1}^\infty$ in Φ^T.

ii) By means of the Shmidt orthogonalizing procedure we can reconstruct the orthonormal basis of controls $\{h_k^\xi\}_{k=1}^\infty$ in C^T norm $(C^T h_k^\xi, h_l^\xi) = (W^T h_k^\xi, W^T h_l^\xi) = \delta_{kl}$.

iii) The orthoprojector Q^ξ can be expressed in a term of the basis $\{h_k^\xi\}$, $Q^\xi = \sum \langle \circ, h_k^\xi\rangle\, h_k^\xi$.

6. Wave fields reconstruction.

The third step of the approach consist in the reconstruction of weighted wave fields $\tilde{u}^f(\gamma,\tau;T) = \beta_o(\gamma,\tau)\circ u^f(\gamma,\tau;T)$ corresponding to a control f. It is expressed in the semigeodesic coordinates (γ,τ). The weighted wave field $\tilde{u}^f$ is a product of wave field u^f on some factor $\beta_o(\gamma,\tau)$. The field $\tilde{u}^f(\gamma,\tau;T)$ can be expressed through C^T operator, projectors Q^ξ and control f

$$\tilde{u}^f(\gamma,\tau;T) = -\left\{ \frac{\partial}{\partial t} C^T Q^\xi_\perp f \right\} (\gamma, T-\xi-0). \tag{10}$$

This formula can be obtained using the singularities propagation theory.

These three steps of BCT approach are general for any type of equations described early. The last step of the approach consists in recovering unknown function β_o and the coordinate transformation $x^i = x^i(\gamma,\tau)$. It is individual for any type of operator L. Let's consider the last step of the procedure for the case i). The basic equality for recovering β_o and functions $x^i = x^i(\gamma,\tau)$ expressing cartesian coordinates x^i via semigeodesic coordinates γ,τ is the following

$$C^T q^T_\alpha = \int_{\Gamma\times(0,T)} [f(\gamma,t)a(\gamma) - R^T f(\gamma,t)\frac{\partial a}{\partial n}(\gamma)](T-t)d\Gamma_\gamma dt. \qquad (11)$$

In the formula (11) $a(x)$ is an analytic function $\Delta a(x) = 0$, q^T_α is a control corresponding the restriction of $a(x)$ on the Ω^T, $W^T q^T_\alpha = a|_{\Omega^T}$. If we apply the formula (11) to the constant function $a(x) = 1$, or coordinate functions $a(x) = x^i$ we can find corresponding controls q^T and P^T_i via C^T operator. Using these controls in the formula (10) we obtain β_0 and $x^i(\gamma,\tau)$. So for any control f we can recover the wave field $u^f(x,t)$ in cartesian coordinates x^i. Putting it in an equation (1) it is simple to obtain the density ρ in the domain Ω^T.

7. The Riemannian manifold reconstruction.

The described variant of the last step of the procedure can't be applied to the case iv) of recovering the Riemannian manifold. The method of solution in that case bases upon the existence of some special type of solutions for hyperbolic equations. These solutions are known as Gaussian beams or quasiphotons. They have the following properties:

i) The quasiphoton is a ray type asymptotic solution with a complex phase and without singularities.

ii) The quasiphoton is concentrated near the bicharacteristic of the wave equation. Physically speaking it is a wave field concentrated near a point moving along a geodesic with the velocity equal 1.

The solution depends upon a parameter ε characterizing the diameter of the quasiphoton and some other parameters (initial coordinates and impulses of the quasiphoton etc.).

It is convenient to describe the solution in a vicinity of geodesic $x = X(t)$ in a coordinate system (η,t), $\eta = x - X(t)$. The quasiphoton can be expressed as follows

$$U^\varepsilon(x,t) = \exp\{-i\varepsilon^{-1}\theta(\eta,t)\}\cdot\sum_{l=0}^{\infty} U_{(l)}(\eta,t)(i\varepsilon)^l, \qquad (12)$$

where

$$\theta(\eta,t) = \theta_i(t)\eta^i + \frac{1}{2}\Gamma_{ij}(t)\eta^i\eta^j + \dots, \qquad (13)$$

$$U_{(l)}(\eta,t) = U_{(l)0}(t) + \dots \qquad (14)$$

are complex functions.

Let's substitute the series (12) in the wave equation (1) and equalize expressions at different powers of the parameter ε. We obtain the Hamilton - Jacobi equation for the phase function θ and transport equations for amplitude functions $U_{(l)}$. Let's substitute the Taylor series (13) in the Hamilton - Jacobi equation end equalize at different powers of coordinates η. The first and second terms of the decomposition leads to the Hamilton system of ordinary differential equation for coordinates $X^i(t)$ and impulses $\theta_i(t)$ (both functions are real ones) with the hamiltonian $H = \sqrt{g^{ik}(X)\theta_i\theta_j}$. The behavior of the solution depends upon the initial coordinates X^i_o and impulse θ_{io}.

For a complex matrix Γ we obtain the matrix Riccati equation

$$\frac{d\Gamma}{dt} = A + B\circ\Gamma + \Gamma\circ B^T + \Gamma\circ R\circ\Gamma, \tag{14}$$

where coefficients of matrices A, B and R are expressed through the second derivatives of the hamiltonian H. It is possible to prove (see preprint of Belishev and Katchalov) that the matrix Γ is a symmetric matrix with a positive defined imaginary part for any time t if it is true for the time t=0. It can be expressed in a form $\Gamma = Z\circ Y^{-1}$ with complex matrices Z, and Y, $\det(Y) \neq 0$ $\forall$ $t\geq 0$. Matrices Z, and Y, are solutions of a linear system of ordinary differential equations along the geodesic $x= X(t)$. For other terms of the decomposition (13) we obtain the linear systems of ordinary differential equations. The Taylors decomposition of transport equations leads to linear systems of ordinary differential equations on time t. For the main term of the solution we can obtain the following expression

$$U_{(0)0}(t) = \frac{C_{(0)0}}{g^{1/4}(X(t))\circ\det^{1/2}(Y(t))}. \tag{16}$$

It is clear that this solution is not zero for any t. So we have a solution with desirable properties.

Now let's use controls $\{f^{\varepsilon}_k\}$, which generate quasiphotons moving along different geodesics originated on the boundary Γ in the step three of our scheme. By means of these controls we can evaluate corresponding weighted wave fields $\tilde{u}^f(\gamma,\tau;T)$ in a semigeodesic coordinates. So we can see the small domain (where the wave field is not small) near the point moving along a geodesic with the velocity equal 1. Changing parameters of quasiphotons we can recover any geodesic originated on the boundary Γ in semigeodesic coordinates. This information is enough to recover the topology as well as the Riemannian structure of the manifold. The problem is pure geometric one. In more detail the procedure of recovering the Riemannian manifold is given in the preprint of Belishev and Katchalov.

8. Results of computations.

At the end of my paper I want to show some results of computation for one dimensional case. It is a case of recovering the density of a string via spectral data (see the paper of Belishev and Katchalov). All computations were carried out on the base of BCT approach. Using the above described procedure we reduced the problem of recovering the density of the string to that of solution a great number of linear system of algebraic equations with Toeplitz matrices. An effective method for solving these systems of equations was suggested in the above mentioned paper.

On the fig. 1-6 we can see the exact density of a string (solid line) and the computed density of the same string (dashed line). The deviation between both lines is small enough (less then 2-3%). The time of computation is approximately 10 sec. for IBM PC AT.

References

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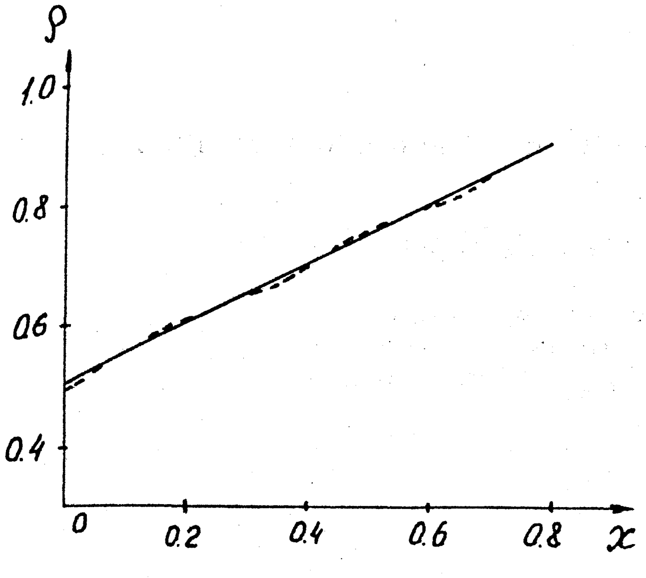

Fig. 1

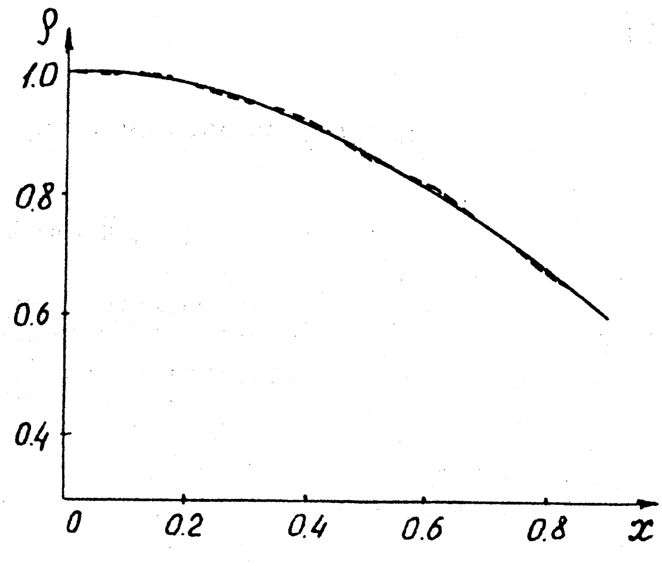

Fig. 2

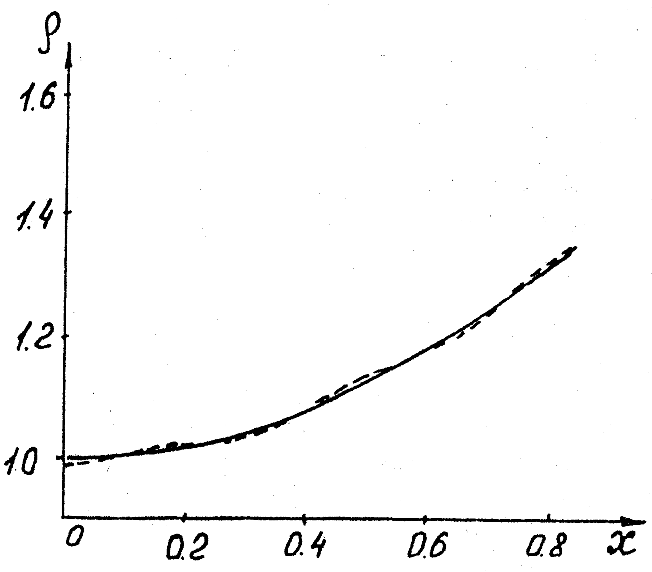

Fig. 3

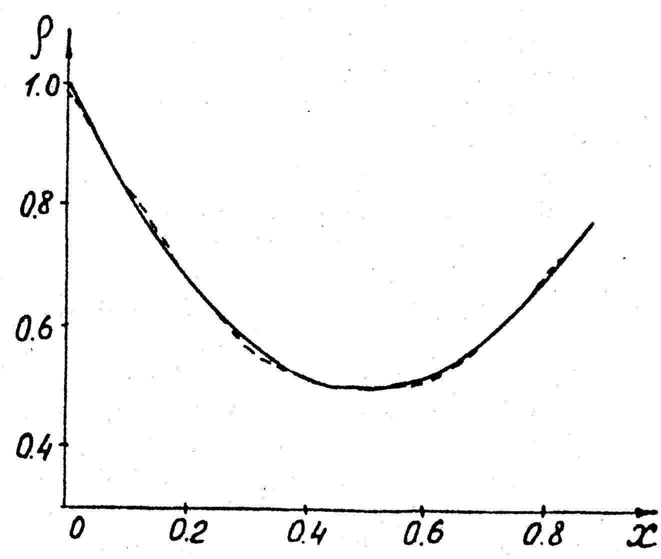

Fig. 4

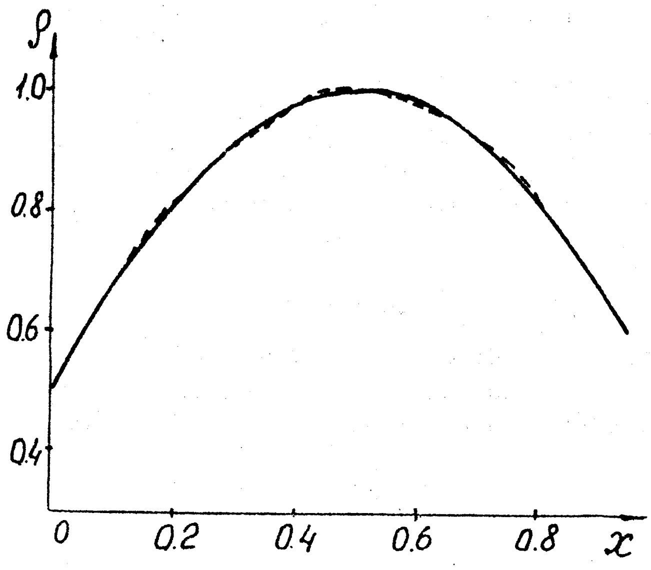

Fig. 5

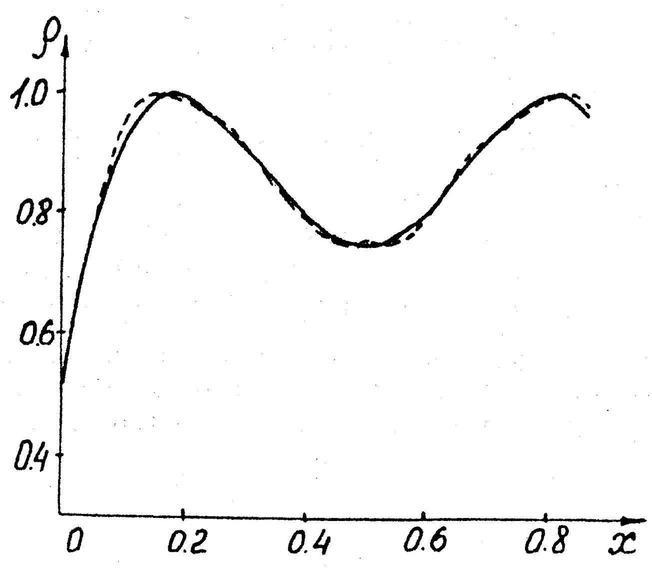

Fig. 6