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A BRIEF REPORT ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES

BY

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§ 1. INTRODUCTION. This is a brief expository article on the zeros of a class of generalised Dirichlet series. The theory is due to R. Balasubramanian and myself developed individually and jointly in several papers. The reason for developing the theory is as follows. In [13] G.H. Hardy proved that (we will always write $s = \sigma + it$),

$$\zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{du}{u^s} \right) + \frac{1}{s-1} \quad (\sigma > 0) \tag{1}$$

has infinity of zeros with $\sigma = \frac{1}{2}$. Hence the same is true trivially of the series

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} \quad (\sigma > 0). \tag{2}$$

This cannot be generalised in a neat way. For example (using the functional equation of $\zeta(s)$) it is easy to see that there are lots of real constants a for which $\zeta(s) - a$ has no zeros at all with $\sigma = \frac{1}{2}$. Also the analogue of Hardy's result to zeta-functions of algebraic number fields is not known. (However we know the analogue for the zeta-functions of ideal classes, and also for their real linear combinations, of any quadratic field. See K. Chandrasekharan and Raghavan Narasimhan [11] for these results and the earlier results of E. Hecke). It is even more difficult to prove the analogue of the result (due to A. Selberg [23], N. Levinson [14] and J.B. Conrey [12]) that the number of zeros of $\zeta(s)$ in $(\sigma = \frac{1}{2}, 0 \leq t \leq T)$ is $\gg T \log T$. In fact it is not even known whether the zeta-function of an algebraic number field has infinity of zeros in $| \sigma - \frac{1}{2} | < \delta$ for every fixed $\delta > 0$.

§ 2. SOME METHODS. So we considered the zeros of the function $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ (where $1 = \lambda_1 < \lambda_2 < \cdots, \frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C, C > 1$ being a positive constant and $\{a_n\}$ is any sequence of complex numbers. We assume further that $F(s)$ has abscissa of absolute convergence $\sigma = 1$ and that it has an analytic continuation in $(\sigma \geq \frac{1}{2} - \delta_0, T \leq t \leq 2T)$ and there maximum of $| F(s) |$ does not exceed T^A, A and δ_0 being any fixed positive constants. Under these conditions we proved that (for $T = T_\nu \to \infty) F(s)$ has $\gg_\varepsilon T^{1-\varepsilon}$ zeros, $\varepsilon > 0$ being any arbitrary constant. The method may be called *the mean square lower bound method.* We would like to call it method number 1. For this method see the zeros I[15] and II[16]. (In the text we abbreviate "on the zeros of a class of generalised Dirichlet series" to "zeros" while referring to papers I to XIV. Also for some convenience we refer to the papers "on the zeros of $\zeta'(s) - a$" and "on the zeros of $\zeta(s) - a$" respectively as "zeros XII" and "zeros XIII"). The method 1 depends upon Borel-Caratheodory theorem and Hadamard's three circles theorem and in a sense it is originally due to J.E. Littlewood. *Method number 2* (developed in zeros II[16]) *is the density argument method.* This gives the lower bound $> T\ Exp(-(log\ T)^\varepsilon)$ for all $T \geq T_0(\varepsilon, \delta_0, A)$, under the restrictions $x^{-1} \sum_{x \leq n \leq 2x} | a_n |^2 > x^{-\varepsilon}$ for all $x \geq x_0(\varepsilon), a_n = O(n^\varepsilon)$ and further $\lambda_n = n$. *Method number* 3 (developed in zeros III[1] and IV[2]) can be called *the mean square and the mean fourth power method.* This can be described as follows. Let δ be a constant satisfying $0 < \delta < \delta_0$ and let

$$\frac{1}{T} \int_T^{2T} | F(\frac{1}{2} - \delta + it) |^2 dt \gg \psi^2 \text{ and } \frac{1}{T} \int_T^{2T} | F(\frac{1}{2} - \delta + it) |^4 dt \ll \psi^4 \quad (3)$$

where ψ exceeds a positive constant power of T. Then it is easy to see that on a "well-spaced" set S of points in $[T, 2T]$ with real part $\frac{1}{2} - \delta$ (the cardinality of S being $\gg T$) we have $| F(\frac{1}{2} - \delta + it) | \gg \psi$. (This gives $\gg T$ zeros in $(\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T)$ by Borel-Caratheodory theorem and Hadamard's three circles theorem). But by combining this fact with an extension (see zeros III[1]) of a lemma of E.C. Titchmarsh (see Theorem 9.14 of [21]) to $F(s)$ it is possible to obtain the lower bound $\gg T \; log \; T$, which is trivially optimal. Moreover if

$$\frac{1}{T}\int_T^{2T} \mid F(\frac{1}{2}+it) \mid^2 dt \ll_\varepsilon T^\varepsilon \quad (\forall \varepsilon > 0) \tag{4}$$

it follows (by an easy extension of a lemma of J.E. Littlewood) that the number of zeros of $F(s)$ in $(\sigma \geq \frac{1}{2}+\delta, T \leq t < 2T)$ is $O(T)$. Thus $F(s)$ has $\gg T \; log \; T$ zeros in $(\mid \sigma - \frac{1}{2} \mid \leq \delta, T \leq t \leq 2T)$. Example $F(s) = \sum_{n=1}^{\infty}((-1)^{n-1} exp(\sqrt{log \; n}) n^{-s})$. *Method number* 4 can be called *the mean first power and the mean second power method.* Here we prove

$$\frac{1}{T}\int_T^{2T} \mid F(\frac{1}{2}-\delta+it) \mid dt \gg \psi \text{ and } \frac{1}{T}\int_T^{2T} \mid F(\frac{1}{2}-\delta+it) \mid^2 dt \ll \psi^2, \tag{5}$$

ψ being the same as before. The method of deducing that $F(s)$ has $\gg T \; log \; T$ zeros in $(\mid \sigma - \frac{1}{2} \mid \leq \delta, T \leq t \leq 2T)$ is the same as before (we still require (4)). But the first inequality in (5) is possible in more general situations than those which prove (3) and (4). For example we can take $F(s) = \zeta(s) + \sum_{n=1}^{\infty} \chi(n) n^{-s}$, $\{\chi(n)\}$ being any sequence of complex numbers with $\sum_{n \leq x} \chi(n) = O(1)$. We can relax this last condition to $\sum_{n \leq x} \chi(n) = O(x^{\frac{1}{2}-\delta_0})$ and $\chi(n) = O(1)$. But in this case we can only prove that the number of zeros of $F(s)$ in $(\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T)$ is $\gg T \; log \; T (loglog \; T)^{-1}$. Thus in order to prove the same lower bound for the number of zeros of $F(s)$ in $(\mid \sigma - \frac{1}{2} \mid \leq 2\delta, T \leq t \leq 2T)$ we have to assume (4) (see zeros III[1], IV[2], V[17] and VI[3]). *Method number* 5 is quite general and can be called *the log* $F(s)$ *method.* In this method we first obtain (under the assumption that certain regions are zero-free) an upper bound for $\mid F(s) \mid$. Next under the conditions $\frac{1}{x}\sum_{n \leq x} \mid a_n \mid^2 \geq Exp\left(-\frac{C_1 \; log \; x}{loglog \; x}\right)$ where

$C_1 > 0$ is a constant and $x \geq x_0$, the method enables us to prove (by contradiction to the bound for $\mid F(s) \mid$) that in $\left(\sigma \geq \frac{1}{2} - \frac{D}{loglog\ T}, T \leq t \leq 2T\right)$, $F(s)$ has $\gg T^{1-\varepsilon}$ zeros provided $D = D(\varepsilon, C_1, A)$ (see zeros VII[18]). The conditional upper bound for $\mid F(s) \mid$ depends on an extension of some ideas of J.E. Littlewood and A. Selberg, due to myself and A. Sankaranarayanan (see [19] and [20]). *Method number* 6 can be called *the Euler product method.* This in brief depends on the assumptions $\lambda_n = n$ and that $F(s)$ has an Euler product and so very restrictive. (However it is applicable to zeta and L-functions of algebraic number fields). But it gives powerful results without the use of Functional equation for $F(s)$ (see zeros IX[5]. For some other results with $\delta = \delta(T) \rightarrow 0$ see zeros VIII[4] where the Euler product is not used). *Method number* 7 can be called *the kth power method* (see zeros X[6], XI[7] and XII[8]). The method consists of a clever (conditional) lower bound for

$$\frac{1}{T}\int_{T}^{T+H} \mid (F(s))^k \mid^2 dt, 0 < k < 1.$$

It is applicable to the study of zeros in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ when $\delta = \delta(T) \rightarrow 0$ more rapidly than $C_1(loglog\ T)^{-1}$ and also in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq 2T)$ where $\delta > 0$ is a constant. The results are sometimes very general. It gives for example that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that the number of zeros of $F(s) = \sum_{n=1}^{\infty}\left((-1)^{n-1} exp(\sqrt{log\ n}) n^{-s}\right)$ in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq T + T^{\varepsilon})$ is $\gg T^{\varepsilon}$ for $T \geq T_o(\varepsilon)$. *Method number* 8 can be called *the Titchmarsh phenomenon method.* This method depends upon Euler product and hence very special (but it is applicable to ζ and L-functions of algebraic number fields). It gives for example that given any $\varepsilon > 0$ and any $\delta > 0 \left(\varepsilon < \frac{1}{10}, \delta < \frac{1}{10}\right)$ there are $\gg T^{\varepsilon}$ zeros of $\zeta(s) - a$ (a, a non-zero complex constant) in $(\sigma \geq 1 - \delta, T \leq t \leq T + T^{\varepsilon})$ for $T \geq T_0(\varepsilon, \delta, a)$ (see zeros XIII[9]). For further details of these eight methods we refer the reader to the concerned papers. (Two general reference books are (i) the famous book [21] of E.C. Titchmarsh revised and edited by D.R. Heath-Brown and the book [22] by E.C. Titchmarsh which is also a famous classic). It requires a much longer article to give more details.

§ 3. **A RECENT RESULT.** Recently R. Balasubramanian and myself have proved (see zeros XIV[10]) some very nice results on the zeros of $G(s) = \sum_{n=1}^{\infty}(a_n b_n \lambda_n^{-s})$ which are further developments of the results proved in (zeros III[1], IV[2], V[17] and IV[3]). Here a_n should not be confused with the sequence occuring in the beginning of § 2. From now on $\{a_n\}$ is any *bounded sequence of complex numbers* satisfying some further conditions. We begin by mentioning some special cases. Let $\{\alpha_n\}$ be any sequence of real numbers with $| \alpha_n | \leq 10^{-5}$ (we have not tried to get optimal constants in place of 10^{-5}). Then for all δ with $0 < \delta < \frac{1}{10}$ we have the following results.

$$\#\{\text{zeros of } \sum_{n=1}^{\infty}(n+\alpha_n)^{-s} \text{ in } (| \sigma - \frac{1}{2} | \leq \delta, T \leq t \leq 2T)\} \gg_\delta T \log T$$

and

$$\#\{\text{zeros of } \sum_{n=1}^{\infty}(-1)^{n-1}e^{\sqrt{\log n}}(n+\alpha_n)^{-s} \text{ in } (| \alpha - \frac{1}{2} | \leq \delta, T \leq t \leq 2T)\} \gg_\delta T \log T.$$

Here in the first result by the infinite sum we mean the analytic continuation in $\sigma > 0$.

The more general result deals with the case $\lambda_n = f(n) + \alpha_n$ for $n \geq n_0, f(x)$ being twice continuously differentiable and is further subject to

(i) $f(x) \sim x$ as $x \to \infty$

(ii) $f'(x)$ lies between two positive constants for $x \geq x_0$, and

(iii) $(f'(x))^2 - f(x)f''(x)$ lies between two positive constants for $x \geq x_0$.

The sequence $\{b_n\}$ is subject to $| b_n | \asymp g(n)$ where

(i) $g(n)n^\eta$ is monotonic increasing for $\forall\ \eta > 0$ and $n \geq n_0(\eta)$,

(ii) $g(n)n^{-\eta}$ is monotonic decreasing for $\forall\ \eta > 0$ and $n \geq n_0(\eta)$, and

(iii) for all $X \geq X_0$ we have $\sum_{X \leq n \leq 2X} \mid b_{n+1} - b_n \mid \ll g(X)$.

The sequence $\{a_n\}$ is again somewhat general. (It is subject to the analytic continuation of $G(s)$ in $(\sigma \geq \frac{1}{2} - \delta_0, T \leq t \leq 2T)$ and $\mid G(s) \mid < T^A$ and some further conditions). We can take for example $\sum_{n \leq x} a_n = x + O(1)$ and $f(n) = n$ or $a_n = (-1)^{n-1}$ and $f(n)$ subject to the general condition. In these two cases the conditions mentioned in the parenthesis are already satisfied. We can prove that the numbber of zeros in $(\mid \sigma - \frac{1}{2} \mid \leq \delta, T \leq t \leq 2T)$ is $\gg_\delta T\ log\ T$. We can also manage with some other conditions which are less restrictive. For example $\sum_{n \leq x} a_n = x + O(x^\theta)$ with a constant $\theta < \frac{1}{2}$ and $f(n) = n$. In this case we can prove that the number of zeros of $G(s)$ in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ is $\gg_\delta T\ log\ T(loglog\ T)^{-1}$. But only when $\theta = \varepsilon$, an arbitrary positive constant we can prove that the number of zeros of $G(s)$ in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq 2T)$ is $O(T)$. In all these cases the bound for $\mid \alpha_n \mid$ will depend upon constants other than δ. There are yet another set of results (on the zeros of a class of generalised Dirichlet series - XV, to appear) for example those which deal with zeros of $\sum_{n=1}^{\infty} d(n)(n + \alpha_n)^{-s}$ and those of $\sum_{n=1}^{\infty} d_3(n)(n + \alpha_n)^{-s}$, which are ~~optional and close to~~ optimal ~~respectively~~. results .

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