数理解析研究所講究録
837巻 1993年 25-34

On the order of growth of the Kloosterman zeta function

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§1. Statement of the result.

Let Γ be a congruence subgroup and let Γ_∞ be the stabilizer of a cusp ∞. Let q be the smallest positive integer such that the transformation $\begin{pmatrix} 1 & q \\ & 1 \end{pmatrix}$ belongs to Γ_∞. For two non-zero integers m and n, we shall define the Kloosterman sum associated with the group Γ by

$$(1) \qquad S(m,n,c,\Gamma) := \sum_{\substack{d \,(\mathrm{mod}\, qc) \\ d\bar{d} \equiv 1 \,(\mathrm{mod}\, qc)}} e^{2\pi i \frac{md+n\bar{d}}{qc}} .$$

Accordingly, the Selberg-Kloosterman zeta function is defined by

$$(2) \qquad Z_{m,n}(s,\Gamma) := \sum_{c>0} \frac{S(m,n,c,\Gamma)}{c^{2s}} .$$

The above series converges absolutely for $Re(s) > 3/4$ by using the Weil estimate for the Kloosterman sum. Our purpose is to evaluate the zeta function $Z_{m,n}$ in $Re(s) > 1/2$. Before stating the result, we shall prepare some notations.

Let T be the set of exceptional eigenvalues s_j which satisfy $1/2 < s_j \leq 3/4$. For any positive δ chosen so that $(1/2, 1/2+2\delta) \cap T = \phi$, we shall define the domain U_δ to be $\{\sigma \mid 1/2 < \sigma < 1/2+\delta\} \times \{\tau \mid |\tau| \leq 1\}$ where $s = \sigma + i\tau$.

Our main result is the following

<u>Theorem</u>. The Kloosterman zeta function $Z_{m,n}(s,\Gamma)$ defined by (2) can be continued meromorphically to $\mathrm{Re}(s) > 1/2$ with at most a finite number of simple poles at $s = s_j$ (exceptional eigenvalue), and satisfies the following estimate:

$$(3) \qquad Z_{m,n}(s,\Gamma) = O\left(q^{2\sigma} \frac{|mn|^{1/2} |\tau|^{1/2}}{(\sigma - \frac{1}{2})^3} \right)$$

for $s = \sigma + i\tau$, $1/2 < \sigma < M$ as $|\tau| \geq 1$, where the implied constant depends solely on M, and

$$(4) \qquad Z_{m,n}(s,\Gamma) = O\left(q^{2\sigma} \frac{|mn|^{1/2}}{(\sigma - \frac{1}{2})^3 \sqrt{\tau^2 + (\sigma - \frac{1}{2})^2}} \right)$$

for $s \in U_\delta$ with an absolute constant in the O-symbol.

Remark. 1. Goldfeld - Sarnak [4 : Theorem 1] obtained $O\left(\frac{|mn|}{q^2} \frac{|\tau|^{1/2}}{\sigma - \frac{1}{2}} \mathrm{vol}(\mathcal{D}_\Gamma)\right)$ where $\mathcal{D}_\Gamma$ denotes the fundamental domain of Γ. Hejhal [6 : pp. 709] derived $O\left(\frac{1}{q^2} \frac{\min\{|m||n|^{1/2}, |m|^{1/2}|n|\}}{(\sigma - \frac{1}{2})^2} \frac{|\tau|^{1/2}}{\mathrm{vol}(\mathcal{D}_\Gamma)} \right)$. Thus, our result is a slight improvement of them with respect to the growth of m and n.

§ 2. Inner product formula.

The estimation of the Kloosterman zeta function $Z_{m,n}$ is based on the inner product formula for the non-holomorphic Poincaré series. Such formula has already been calculated by several authors. Roughly speaking, there are two-types of representations. The one is by Goldfeld-Sarnak and Hejhal, the other is by Kuznetsov [10: Lemma in section 4] and Deshouillers-Iwaniec [1: Lemma 4.1 and 4.3]. A few years ago, I could obtain a somewhat convenient representation for Fourier coefficients of the non-holomorphic Poincaré series ([18]). By using this, we can derive the new type of representation for the inner product of the non-holomorphic Poincaré series. This formula gives us the result stated in the theorem.

Let m be a non-zero integer. The non-holomorphic Poincaré series is defined by

$$(5)\qquad P_m(z, s, \Gamma) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i \frac{m}{\delta} x(\gamma z)} e^{-2\pi \frac{|m|}{\delta} y(\gamma z)} y(\gamma z)^s$$

where $\gamma z = x(\gamma z) + i y(\gamma z)$ for any $z \in H$ (complex upper half plane).

For two Γ-automorphic functions $f(z)$ and $g(z)$, the inner product $\langle f, g \rangle$ is defined by

$$\langle f, g \rangle := \int_{H/\Gamma} f(z) \overline{g(z)} \frac{dx\, dy}{y^2}.$$

Let $P_m(z, s, \Gamma)$ and $P_n(z, w, \Gamma)$ $(s, w \in C)$ be two non-holomorphic Poincaré series. It then follows from the Rankin-Selberg method that

$$\langle P_m(z,s,\Gamma),\ P_m(z,\bar{w},\Gamma)\rangle$$
$$= g\int_0^\infty a_m(y,s,n,\Gamma)\, y^{w-2} e^{-2\pi\frac{|m|}{g}y}\, dy$$

where $a_m(y,s,n,\Gamma)$ denotes the n^{th} Fourier coefficient of P_m. Substituting the formula stated in [18 : Theorem. B] for a_m, and calculating the right side in the above, we can obtain the following

<u>Proposition</u>. For $Re(s) > 3/4$. and $Re(w) > 3/4$, we have

$$\langle P_m(z,s,\Gamma),\ P_m(z,\bar{w},\Gamma)\rangle = \delta_{m,n}\, g\left(4\pi\frac{|m|}{g}\right)^{1-s-w}\Gamma(s+w-1)$$

(6) $$+\, 2 f_{m,n}(s,w)\sum_{c>0} S(m,n,c,\Gamma)\, c^{-(1+s)} K_{w-s}(\alpha)$$

$$-\, \varepsilon_{m,n} f_{m,m}(s,w)\sum_{c>0} S(m,n,c,\Gamma)\, c^{-(1+s)}\alpha^{w} R_{m,n}(s,w,c,\Gamma)$$

where $\delta_{m,n}$ is the Kronecker symbol, $\varepsilon_{m,n} = \begin{cases} 1 & mn>0 \\ 0 & mn<0 \end{cases}$, $\alpha = 4\pi\frac{|mn|^{1/2}}{gc}$

and
$$f_{m,n}(s,w) = 2^{4-s-3w}\pi^{2-w} g^{w-1}\frac{\Gamma(s+w-1)}{\Gamma(s)\Gamma(w)}|m|^{\frac{1-s}{2}}|n|^{\frac{s-2w+1}{2}},$$

moreover

(7) $$R_{m,n}(s,w,c,\Gamma) = \int_0^1 K_{w-s}(\alpha u^{1/2})\, u^{\frac{s+w}{2}}(1-u)^{-1/2} J_1(\alpha(1-u)^{1/2})\, du$$
$$+\int_0^1 K_{w-s}(\alpha u^{1/2})\, u^{\frac{s+w-2}{2}}(1-u)^{1/2} J_1(\alpha(1-u)^{1/2})\, du.$$

(K_ν and J_ν denote respectively the modified Bessel function and the Bessel function).

To derive the formula (6), we use the summation formula for the Bessel function:

$$(1+t^2)^{-s/2} J_s(\alpha(1+t^2)^{1/2}) = 2\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(s+2\ell)\Gamma(s+\ell)}{\ell!}\frac{J_{s+\ell}(\alpha t)}{t^{s}}\frac{J_{s+2\ell}(\alpha)}{\alpha^{s}}$$

[16 : pp. 366, (3)]

and basic properties concerning the Jacobi polynomial. For the detailed proof, see [19].

§ 3. Proof of the theorem.

In this section, we shall give an outline how to derive the assertion of the theorem from the formula (6). We first recall the decomposition for the modified Bessel function:

$$K_\nu(\alpha) = 2^{\nu-1}\alpha^{-\nu}\Gamma(\nu) + 2^{\nu-1}\pi^{-1/2}\alpha^{-\nu}\Gamma(\nu+\tfrac{1}{2})\,Q(\nu,\alpha)$$

where

$$Q(\nu,\alpha) = \int_{-\infty}^{\infty} \frac{(e^{-i\alpha t}-1)}{(1+t^2)^{\nu+(1/2)}}\,dt .$$

This decomposition is valid for $\mathrm{Re}(\nu) > 1/2$. We apply this decomposition to the second term in (6) and we put $W = s+1$. Then, the equality (6) may be rewritten as

$$2^{-2s} g\,\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}\,|m|^{-1} Z_{m,n}(s,\Gamma)$$

$$(8) \qquad = \langle P_m(z,s,\Gamma),\, P_n(z,\bar{s}+1,\Gamma)\rangle - \delta_{m,n}\, g\,(4\pi\tfrac{|m|}{b})^{-2s}\Gamma(2s)$$

$$- f_{m,n}(s,s+1)\sum_{c>0} S(m,n,c,\Gamma)\,c^{-(1+s)}\,\alpha^{s-1}\,Q(1,\alpha)$$

$$+ f_{m,n}(s,s+1)\sum_{c>0} S(m,n,c,\Gamma)\,c^{-(1+s)}\,\alpha^{s+1}\,R_{m,n}(s,s+1,c,\Gamma)$$

By elementary evaluations, we can obtain that the third and the fourth terms are holomorphic in $Re(s) > 1/2$ and satisfy the following estimate:

$$(9) \qquad O\left(\left|\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1)}\right| \cdot \left|\frac{m}{n}\right|^{1/2} q \frac{1}{(\sigma-\frac{1}{2})^2}\right)$$

where the implied constant is an absolute constant. For the first term, we use the Cauchy-Schwarz inequality. Then we have

$$|\langle P_m, P_n \rangle| \leq 4\pi \frac{|m|}{q} |s| \, \|\mathcal{R}_\lambda\| \, \|P_m(z, s+1)\| \cdot \|P_n(z, s+1)\|$$

where $\mathcal{R}_\lambda$ $(\lambda = s(1-s))$ denotes the resolvent of the automorphic Laplacian. For the norm of the resolvent $\mathcal{R}_\lambda$, it is well known to hold that

$$(10) \qquad \|\mathcal{R}_\lambda\| \leq \begin{cases} \dfrac{1}{|\tau|(2\sigma-1)} & \text{for } \sigma > 1/2 \text{ and } |\tau| \geq 1 \\[2ex] \dfrac{1}{(\sigma-\frac{1}{2})\sqrt{\tau^2+(\sigma-\frac{1}{2})^2}} & \text{for } s \in U_S \end{cases}$$

As for the norm $\|P_m(z, s+1)\|$, we can obtain the following

<u>Lemma</u>. $\quad \|P_m(z, s+1)\| = O\left(q^{1+\sigma} |m|^{-1/2} \dfrac{1}{\sigma - \frac{1}{2}}\right)$

for $1/2 < \sigma < M$, where the constant in the O-symbol depends only on M.

Remark. 2. In order to derive the estimation of the lemma, we have to apply the formula (6) again. Indeed, $\|P_m(z, s+1)\|^2 = \langle P_m(z, s+1), P_m(z, s+1) \rangle$. (see [19]).

Combining (10) and the result in the lemma, we conclude

$$(11)\qquad |\langle P_m, P_m\rangle| = \begin{cases} O\left(q^{1+2\sigma}\left|\frac{m}{n}\right|^{1/2}|\tau|^{1/2}\frac{1}{(\sigma-\frac{1}{2})^3}\right) & \text{for } 1/2<\sigma<M \\ & \quad |\tau|\geq 1 \\ O\left(q^{1+2\sigma}\left|\frac{m}{n}\right|^{1/2}\frac{1}{(\sigma-\frac{1}{2})^3\sqrt{\tau^2+(\sigma-\frac{1}{2})^2}}\right) & \text{for } s\in U_\delta \end{cases}$$

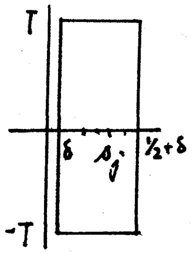
Applying (9) and (11) to the equation (8) and using Stirling's formula, we can obtain the assertion of the theorem.

§4. Application to sums of Kloosterman sums.

The estimation of the Kloosterman zeta function $Z_{m,n}(s)$ in $\mathrm{Re}(s) > 1/2$ has a very important application. In fact, combining a well-known method in analytic number theory, we can obtain a certain estimation for sums of Kloosterman sums. This is the reason why we try to refine the estimation of the Kloosterman zeta function.

By Cauchy's theorem, it follows that

$$\int_{\partial E} Z_{m,m}\left(\frac{1+s}{2}\right)\frac{x^s}{s}\,ds = \sum_{s_j} B_j \frac{x^{2s_j-1}}{2s_j-1}$$



where $E = [\delta, \frac{1}{2}+\delta] \times [-T, T]$ and the point s_j ranges over exceptional eigenvalues and the constantant B_j can be expressed precisely by using Fourier coefficients of Maass wave forms (see [4]).

From the Phragmén-Lindelöf principle and (3), we see

$$Z_{m,n}\left(\frac{1+s}{2}\right) = O\left(\frac{|mn|^{1/2}}{\delta^3}|r|^{\left(\frac{1}{2}+\delta-\sigma\right)}\right)$$

for $|r| \geq 2$ and $\delta < \sigma < \frac{1}{2}+\delta$. Thus

$$\int_{horiz}(\quad)\,ds = O\left\{\frac{|mn|^{1/2}}{T}x^{1/2}(\log x)^2\right\}.$$

Moreover from (3) and (4),

$$\int_{\delta-iT}^{\delta+iT}(\quad)\,ds = O\left\{|mn|^{1/2}\left(T^{1/2}+(\log x)\right)(\log x)^3\right\}.$$

It is well known to hold that

$$\int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT}(\quad)\,ds = \sum_{c<x}\frac{S(m,n,c,\Gamma)}{c} + O\left(\frac{|mn|^{1/2}}{T}x^{1/2}\log x\right).$$

Gathering together, we conclude that

$$(12) \qquad \sum_{c<x}\frac{S(m,n,c,\Gamma)}{c} = \sum_{s_j} B_j\frac{x^{2s_j-1}}{2s_j-1} + O\left(|mn|^{1/2}x^{1/6}(\log x)^2\right).$$

Non-existence of the point s_j is equivalent to Selberg's eigenvalue conjecture (see [8], [15]). In addition, if we succeed to derive the estimate:

$$Z_{m,n}(s,\Gamma) = O\left(\frac{|r|^{\varepsilon}}{(\sigma-\frac{1}{2})^a}\right) \qquad \text{for } \sigma > \tfrac{1}{2}, \text{ and any positive } a,$$

we can prove the Linnik-Selberg conjecture. This conjecture seems to be very difficult. Anyway our standpoint is far from this conjecture.

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