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RATIONAL COHOMOLOGY OF WITT GROUPS

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Let k be an algebraically closed field of characteristic p and for each $n > 0$ let $W(n)$ denote the group of Witt vectors of length n. $W(n)$ is a commutative algebraic group. For reference, see Jacobson [2], Serre [6]. One of the important properties of the Witt groups is the following: Every commutative algebraic k-group whose underlying variety is an affine space is a homomorphic image of products of $W(n)$. We compute the rational cohomology of $W(n)$ for $n \geq 2$.

$$\mathrm{H}^*(W(n),k) = S((V^{n-1*})^{-1}\beta L^{\#}) \otimes E(R^{n-1*}L^{\#}),$$

where β is the Bockstein, V, the shift and R the restriction homomorphism and where $L^{\#}$ is the graded dual of the restricted Lie algebra $\mathrm{End}(G_a)$ identified with the first cohomology group $\mathrm{H}^1(G_a;k) \cong \oplus kx^{[p^i]}$. We also show the existence of the higher Bockstein for 1-dimensional cohomology classes of algebraic groups. As an application, we compute the rational cohomology of a family of commutative unipotent groups $V(n)$ and discuss the connection of these cohomology rings with that of the Steenrod algebra.

1 The ring of Witt vectors

Let $W = \mathbb{Q}(x_i, y_j, z_k)$, $0 \leq i,j,k < m$ be a polynomial $\mathbb{Q}$ algebra and let $W_n = W \times \cdots \times W$ be an n-fold product of W with componentwise addition " $+$ " and multiplication " $\cdot$ ". Define a new addition " $\oplus$ " and multiplication " $\odot$ " on W_n as follows:

$$\begin{aligned} a \oplus b &= \phi^{-1}(\phi a + \phi b) \\ a \odot b &= \phi^{-1}(\phi a \cdot \phi b), \end{aligned} \tag{1}$$

where, for $a = (a_0, \ldots, a_{m-1})$, $\phi a = (\phi a_0, \ldots, \phi a_{m-1})$ with $\phi a_r = \sum_0^r p^i a_i^{p^{r-i}}$. It's inverse ϕ^{-1} is defined inductively as: $\phi^{-1} a_0 = a_0$ and $\phi^{-1} a_r = \frac{1}{p^r}(a_r - \sum_{i=0}^{r-1} p^i (\phi^{-1} a_i)_i^{p^{r-i}})$. The triple $(W_n, \oplus, \odot)$ is a commutative ring over $\mathbb{Q}$ with $1 = (1, 0, \ldots, 0)$ as identity and $(0, \ldots, 0)$ as zero element. The map $\phi : W_n \to W_n$ is a ring isomorphism from $(W_n, \oplus, \odot)$ onto $(W_n, +, \cdot)$.

Consider generic vectors $x = (x_0, \ldots, x_{m-1})$ and $y = (y_0, \ldots, y_{m-1})$, with x_i and y_i indeterminate as above, then each component of $x \oplus y$ and $x \odot y$ are in fact a polynomial with integral coefficient. $(x \oplus y)_r$, $(x \odot y)_r \in \mathbb{Z}[x_0, y_0, \ldots, y_r]$ for $0 \leq r < m$. For example:

$$
\begin{aligned}
(x \oplus y)_0 &= x_0 + y_0 \\
(x \oplus y)_1 &= x_1 + y_1 - \frac{1}{p}\sum_{1}^{p-1} \binom{p}{i} x_0^i y_0^{p-i} \\
(x \odot y)_0 &= x_0 y_0 \\
(x \odot y)_1 &= x_0^p y_1 + x_1 y_0^p + p x_1 y_1.
\end{aligned} \tag{2}
$$

For an arbitrary commutative ring A of characteristic p, let $W_m(A)$ be the set of m-tuples $(a_0, \ldots, a_{m-1})$ with $a_i \in A$ and with addition and multiplication defined via the polynomials $(x \oplus y)_r$ and $(x \odot y)_r$ as follows: For any three elements a, b, $c \in W_m(A)$, let $s : \mathbb{Z}[x_i, y_i, z_i] \to W_m(A)$ be the map sending x_i, y_i, z_i to a_i, b_i, c_i respectively. Then $W_m(A)$ becomes an associative commutative ring of characteristic p with $(a \oplus b)_r = s(x \oplus y)_r$ and $(a \odot b)_r = s(x \odot y)_r)$, called *the ring of Witt vectors of length m.* In fact, W_m is a functor from commutative rings of characteristic p to commutative rings. The prime ring of $W_m(A)$ is isomorphic to $\mathbb{F}_{p^m}$. It consists of Witt vectors with coefficients in $\mathbb{F}_p$, the prime ring of A.

2 Witt groups

The underlying abelian group of W_n, denoted by $W(n)$ is a commutative algebraic group. It is commonly known as the Witt group of dimension, or length, n. There are natural homomorphisms among $W(n)$ for various $n \geq 1$:

(1)	The Frobenius homomorphism:	$F : W_m \to W_m :$	$F(a) = (a_0^p, \ldots, a_{n-1}^p)$,
(2)	The restriction homomorphism:	$R : W_m \to W_{m-1} :$	$R(a) = (a_0, \ldots, a_{n-2})$,
(3)	The shift homomorphism	$V : W_m \to W_{m+1} :$	$V(a) = (0, a_0, \ldots, a_{n-1})$.

R, F and V commute with each other and their product RFV is multiplication by p.

Similar to the ring W_n, the Hopf algebra associated to $W(n)$ is constructed first in characteristic 0, then followed by reduction mod p. Over the field of rational numbers $\mathbb{Q}$, consider the associated algebra $\mathbb{Q}[y_0, \cdots, y_{n-1}]$ of the additive $\mathbb{Q}$-vector group G_a^n. For $0 \leq j < n$ let $x_j = \psi(y_i) = p^j y_j + p^{j-1} y_{j-1}^p + \cdots + y_0^{p^{j-1}}$. The $\mathbb{Z}$ lattice $\mathbb{Z}[x_0, \ldots, x_{n-1}]$ of $\mathbb{Q}[V]$ generated by the x_i's is closed under comultiplication, counit and antipode. That is ψ is an automorphism on $\mathbb{Q}[y_0, \ldots, y_{n-1}]$ whose restriction to the $\mathbb{Z}$-lattice $\mathbb{Z}[y_0, \ldots, y_{n-1}]$ induces a Hopf algebra structure on its image $\mathbb{Z}[x_0, \ldots, x_{n-1}]$. For any field k of characteristic $p > 0$, $W(n)$ is defined to be the algebraic group associated to $\mathbb{Z}[x_0, \ldots, x_{n-1}] \otimes k = k[x_0, \ldots, x_{n-1}]$. The generator x_i is the function $x_i(a) = a_i$ for $a \in W(n)(A)$. The first few examples are

$\triangle x_0 = x_0 \otimes 1 + 1 \otimes x_0$ and from (2)

$$\triangle x_1 = x_1 \otimes 1 + 1 \otimes x_1 - \frac{1}{p}\sum \binom{p}{i} x_0^i \otimes x_0^{p-i}. \tag{3}$$

3 Cohomology of $W(n)$

Let G be an algebraic group defined over a field k and $k[G]$ be its coordinate algebra. For a G module M, the rational cohomology $\mathrm{H}^*(G; M)$ is the homology of the cobar complex.

$$C^n(G, M) = M \otimes I^n; \quad I \text{ is the augmentation ideal of } k[G],$$

with the coboundary $\partial^i : C^i(G, M) \to C^{i+1}(G, M)$

$$\partial^i(f_0\otimes\ldots\otimes f_i) = \sum_{j=0}^{i}(-1)^j f_0\otimes\ldots\otimes(\triangle(f_j) - f_j\otimes 1 - 1\otimes f_j)\otimes\ldots\otimes f_i \; + \; f_0\otimes\ldots\otimes f_i\otimes 1, \tag{4}$$

Let $k[G_a] = k[x]$ be the associted algebra of the additive algebraic group G_a. The rational cohomology of G_a is given (see Cline, Parshall, Scott and van der Kallen, 4.1 in [1]),

$$\mathrm{H}^*(G_a; k) \cong \begin{cases} S(\beta L^\#) \otimes E(L^\#) & \text{for } p \geq 3 \\ S(L^\#) & \text{for } p = 2, \end{cases}$$

where L is the restricted Lie algebra $\mathrm{End}(G_a)$, which can be identified with the infinite sum $\oplus_{i=0}^{\infty} k x^{p^i}$. Let $x(i)$ denotes the dual basis to x^{p^i} and identify it with the first cohomology class of $1 \otimes x^{p^i} \in C^1(G_a, k)$. $S(-)$ and $E(-)$ are the symmetric and exterior algebra and β denotes the (algebraic) Bockstein induced from the map $\bar{\beta} : C^1(G_a, k) \to C^2(G_a, k)$. For any monomial x^i

$$\bar{\beta} x^i = -\frac{1}{p}\sum_{j=1}^{p-1} \binom{p}{j} x^{ij} \otimes x^{i(p-j)} \tag{5}$$

Remark 3.1 For $p = 2$ we have $\beta x(i) = x(i)^2$. However for $p \geq 3$ β is not the usual Bockstein $\tilde{\beta}$ in the ordinary cohomology, which is induced from the long exact sequence from the extension

$$0 \to k \to W(2)(k) \to k \to 0,$$

but it is $\tilde{\beta} P^0$ (for detail see the appendix A1.5.2 in Ravenel [5]). Indeed, for $\mathrm{H}^*(G_a; k)$, P^0 is the Frobenius homomorphism in L, $P^0 x(i) = x(i+1)$ and $\beta x(i) = \tilde{\beta} x(i+1)$.

In terms of $x(i)$ and $\beta x(i) := y(i+1)$ we write

$$\mathrm{H}^*(G_a;k) \cong \begin{cases} \bigotimes_{i=0}^{\infty} k[y(i+1)] \otimes E(x(i)) & \text{for } p \geq 3 \\ \bigotimes_{i=0}^{\infty} k[x(i)] & \text{for } p = 2. \end{cases}$$

Now we consider the cohomology of $W(n)$. For each pair of positive integers n, m, the homomorphisms R and V induce an extension

$$0 \to W(m) \to W(n+m) \to W(n) \to 0.$$

In particular, for $n-1$ and 1 we have the extension

$$0 \to G_a \to W(n) \to W(n-1) \to 0 \tag{6}$$

which corresponds to the coextension of Hopf algebras:

$$k[x_{n-1}] \leftarrow k[x_0, \ldots, x_{n-1}] \leftarrow k[x_0, \ldots, x_{n-2}],$$

To compute $\mathrm{H}^*(W(n);k)$ for $n \geq 2$ we apply the Hochschild-Serre's spectral sequence

$$E_2^{*,*}(n) = \mathrm{H}^*(W(n-1); \mathrm{H}^*(G_a;k)) \Longrightarrow \mathrm{H}^*(W(n);k).$$

For $n = 2$ and $p \geq 3$

$$E_2^{*,*}(2) \cong \bigotimes_{i=0}^{\infty} k[y_0(i+1), y_1(i+1)] \otimes E(x_0(i), x_1(i))$$

The differential in $C^*(W(n), k)$ is given by (3), (4) and (5)

$$\partial_1 x_1^{p^i} = \Delta x_1^{p^i} - (x_1^{p^i} \otimes 1 - 1 \otimes x_1^{p^i}) = \beta x_0^{p^i}.$$

So the induced differential in the spectral sequence is $d_2 x_1(i) = y_0(i+1)$. Hence

$$E_3^{*,*}(2) \cong \bigotimes_{i=0}^{\infty} k[y_1(i+1)] \otimes E(x_0(i)).$$

By Cartan-Serre's transgression theorem (see the appendix A.1.5.2 in [5])

$$d_3 y_1(i+1) = d_3(\tilde{\beta} P^0 x_1(i)) = \tilde{\beta} P^0 d_2 x_1(i) = \tilde{\beta} P^0 y_0(i+1) = \tilde{\beta} y_0(i+2) = 0.$$

Therefore $E_3^{*,*}(2) \cong E_\infty^{*,*}(2)$ and we have just proved the following theorem for $n = 2$.

Theorem 3.2 *(Compare VII, 9, Lemma 4 in [6]). For any integer $n \geq 1$,*

$$\mathrm{H}^*(W(n);k) \cong \bigotimes_{i=0}^{\infty} k[y_{n-1}(i+1)] \otimes E(x_0(i)) \quad \textit{for } p \geq 3 \tag{7}$$

$$\cong \bigotimes_{i=0}^{\infty} k[x_{n-1}^2(i)] \otimes E(x_0(i)) \quad \textit{for } p = 2. \tag{8}$$

Proof: The map of extensions

$$\begin{array}{ccccccccc} 0 & \to & G_a & \xrightarrow{V} & W(2) & \longrightarrow & G_a & \to & 0 \\ & & \| & & \downarrow V^{n-2} & & \downarrow V^{n-1} & & \\ 0 & \to & G_a & \xrightarrow{V^{n-1}} & W(n) & \longrightarrow & W(n-1) & \to & 0 \end{array}$$

induces a map of spectral sequences

$$\begin{array}{ccccc} E_2^{*,*}(n) & \cong & \mathrm{H}^*(W(n-1);\mathrm{H}^*(G_a;k)) & \Longrightarrow & \mathrm{H}^*(W(n);k) \\ & & \downarrow V^{n-2*} & & \downarrow V^{n-2*} \\ E_2^{*,*}(2) & \cong & \mathrm{H}^*(\ G_a\ ;\mathrm{H}^*(G_a;k)) & \Longrightarrow & \mathrm{H}^*(W(2);k). \end{array}$$

By induction, we assume

$$\mathrm{H}^*(W(n-1);k) \cong \bigotimes_{i=0}^{\infty} k[y_{n-2}(i+1)] \otimes E(x_0(i)).$$

Since $V^{n-2*}y_j(i+1) = y_{j-n+2}(i+1)$, and $V^{n-2*}x_j(i) = x_{j-n+2}(i)$, where $y_j(i+1) = x_j(i) = 0$ for $j < 0$, we get

$$d_2 x_{n-1}(i) = y_{n-2}(i+1) \text{ modulo the ideal } (x_0(i)),$$

from the naturality and from the result for $n = 2$. Hence $E_3^{*,*}(n)$ is isomorphic to the formular in the theorem, and we see that $E_3^{*,*}(n) \cong E_\infty^{*,*}(n)$ by the same reason as in the case $n = 2$.

The proof for the case $p = 2$ is by similar arguments exchanging $y_j(i+1)$ with $x_j(i)^2$. □

Corollary 3.3 *The map F^* on $\mathrm{H}^*(W(n);k)$ induced from the Frobenius map is injective.*

Proof: This follows from the Theorem since $F^*x_j(i) = x_j(i+1)$ and $F^*y_j(i+1) = y_j(i+2)$. □

4 Higher Bockstein operations

Recall that $\mathrm{H}^*(W(n);k)$ is generated by $y_{n-1}(i+1)$ and $x_0(i)$. We may and will hereafter assume that $y_{n-1}(i+1) \in \mathrm{H}^2(W(n);k)$ has a representative in $C^2(W(n+1),k)$ of the form

$$Y = \partial^1 x_n^{p^i} = \Delta x_n^{p^i} - (x_n^{p^i} \otimes 1 + 1 \otimes x_n^{p^i})$$

since $V^{n-1*}(\Delta x_n) = \Delta x_1$ and $\partial^2\partial^1(x_n^{p^i}) = 0$ in $C^3(W(n+1),k)$, so Y is a cocycle. For $n=1$ we have the Bockstein $\beta x_0(i) = y_0(i+1)$. For $n \geq 1$ we define the higher Bockstein β_n for $W(n)$ by: $\beta_n x_0(i) = y_{n-1}(i+1)$, setting $\beta = \beta_1$. In general

Definition 4.1 *Let G be an algebraic group defined over k. For an element $x \in \mathrm{H}^1(G;k)$ and an integer $n \geq 1$ we define the higher Bockstein of x to be an element $\beta_n x = y$ in $\mathrm{H}^2(G;k)$ if there is a map $q : G \to W(n)$ of algebraic k-groups such that the induced map $q^* : \mathrm{H}^*(W(n);k) \to \mathrm{H}^*(G;k)$ satisfies $q^*x_0(0) = x$ and $q^*y_{n-1}(1) = y$.*

Theorem 4.2 *Let G be an algebraic k-group. For each element $x \in \mathrm{H}^1(G;k)$ such that $\beta_1(x) = \cdots = \beta_n(x) = 0$, then $\beta_{n+1}(x)$ is defined.*

Proof: For an element $x \in \mathrm{H}^1(G;k)$, let $\tilde{x} \in C^1(G,k)$ be a representative of x. Then $\partial^1\tilde{x} = 0$ implies that $\tilde{x}$ is primitive and we get a Hopf algebra homomorphism:

$$k[G_a] \cong k[\tilde{x}] \hookrightarrow k[G]$$

which induces a homomorphism of algebraic groups $q : G \to G_a$ such that $q^*x(0) = x$. Hence the theorem is true for $n = 1$.

Now suppose $\beta_1 x = \cdots = \beta_n x = 0$. The last equality implies there is an algebraic group homomorphism $q : G \to W(n)$ with $q^*x_0(0) = x$ and $q^*y_{n-1}(1) = 0$ in $\mathrm{H}^*(G;k)$. Let $\tilde{x} \in C^1(G;k)$ be such that $\partial^1\tilde{x}$ represents $q^*y_{n-1}(1)$ in $C^2(G,k)$. Define a map

$$\phi : k[W(n+1)] \to k[G]$$

as follows: $\phi|_{k[x_0,\ldots,x_{n-1}]} = q$ and $\phi(x_n) = \tilde{x}$. The map ϕ is a map of Hopf algebra such that $\phi^*x_0(0) = x$ and $\phi^*y_n(1) := \beta_{n+1}x$. This finishes the proof of the theorem. □

As a consequence of this Theorem, we can explicitly write down β_n in the cobar complex. For any sequence $I = (i_0, \ldots)$, with $i_s \geq 0$, for all $s \geq 0$, let a^I denote $a_0^{i_0}a_1^{i_1}\cdots$. Take $\xi_{IJr} \in k$ such that

$$(a \oplus b)_r = a_r + b_r + \sum \xi_{IJr}a^I b^J.$$

If $x \in \mathrm{H}^1(G;k)$ and $\beta_1 x = \cdots = \beta_n x = 0$, then there are $x_1, \ldots, x_n$ such that $dx_r = \sum \xi_{IJr}x^I \otimes x^J$ for $1 \leq r \leq n$ and we can define

$$\beta_{n+1}x = \sum \xi_{IJn}x^I \otimes x^J.$$

QUESTION It is still an open question whether the higher Bockstein β_n can be extended to all of $\mathrm{H}^*(G;k)$.

We have the following nonvanishing lemma for the higher Bockstein.

Lemma 4.3 *Let G be an algebraic k-group. Consider the spectral sequence induced from a central extension $0 \to G_a \to G \xrightarrow{\pi} G' \to 1$. For any integer $n \geq 1$, if in the Hochschild-Serre's spectral sequence, $d_2 x(0) = \beta_n(x') \neq 0$ for $x(0) \in \mathrm{H}^1(G_a; k)$ and $x' \in \mathrm{H}^1(G'; k)$. Then $\beta_{n+1}(\pi^* x') \neq 0$ in $\mathrm{H}^*(G; k)$.*

Proof: Since $\beta_n(\pi^* x') = 0$ in $\mathrm{H}^*(G; k)$, there exists a map $q_n : G \to W(n)$ inducing a map of extensions

$$\begin{array}{ccccccccc} 0 & \to & G_a & \longrightarrow & G & \longrightarrow & G' & \to & 0 \\ & & \downarrow q_a & & \downarrow q_n & & \downarrow q_{n-1} & & \\ 0 & \to & G_a & \longrightarrow & W(n) & \longrightarrow & W(n-1) & \to & 0 \end{array}$$

with $q_{n-1}^* x_0(0) = x'$. Since $q_n^* y_{n-2}(1) = \beta_{n-1}(x') \neq 0$ in the E_2 term of the spectral sequence associated to the first extension, we know that $q_a^* x_{n-1}(0) \neq 0$ in $\mathrm{H}^*(G_a; k)$ since $d_2 x_{n-1}(0) = y_{n-2}(1) \in \mathrm{H}^2(W(n-1), k)$. Hence $q_a^* y_{n-1}(1) = q_a^* \beta x_{n-1}(0) \neq 0$ in $E_2^{*,*}$. Since $y_{n-1}(1)$ is permanent, so is $q_a^* y_{n-1}(1)$ which is $\beta_n(\pi^* x')$. □

5 The group $V(n)$

Every commutative affine algebraic group over k whose underlying varity is an affine n-space is isogeneous to a product of Witt groups. I.e. it is an extension of a product of Witt groups by a finite abelian group. Those groups that are of interest to us in this work are the ones that are isomorphic as algebraic group to a product $\prod_i^m W(n_i)$, when $n_i \leq n_{i+1}$ and $\sum n_i = n$. For $n = m$ we get the additive vector group G_a^n and for $m = 1$ we get $W(n)$. See [6].

For each integer $n \geq 2$, let $V(n)$ be the commutative linear algebraic group isomorphic to a subgroup of the unipotent group $U(n)$ consisting of $n \times n$ upper triangular matrices such that each entry along an off diagonal is constant. More precisely, a matrix $[a_{i,j}] \in V(n)$ if $a_{i,j} = \delta_{i,j}$ for $i \geq j$, and $a_{i,j} = a_{i+r,j+r}$ for $i < j$ and $0 \leq r \leq n-i$. The coordinate algebra $k[V(n)]$ is a polynomial algebra $k[a_1, \ldots, a_{n-1}]$ with comultiplication $\Delta a_i = \sum_{j=0}^{i} a_j \otimes a_{i-j}$, where, by convension, $a_0 = 1$. $V(n)$ is the so called big Witt group of length n, or Witt group at all prime simultaneously. It isomorphic as an algebraic group to a product of Witt groups.

$$V(n) \cong \prod_{p \nmid i} W(r_i), \tag{9}$$

where for each i, r_i is the smallest positive integer such that $p^{r_i} \geq n/i$. See [6] chapter 5. This decomposition, together with the rational cohomology of $W(n)$ computed in the previous section immediately yield $\mathrm{H}^*(V(n); k)$. However we can compute $\mathrm{H}^*(V(n); k)$ directly. Using the higher Bockstein operation we will prove (9) by showing that there is a tensor decomposition of $\mathrm{H}^*(V(n); k)$ in terms of $\mathrm{H}^*(W(m); k)$.

Like in $W(n)$, there exist the Frobenius, the restriction and the shift homomorphisms for $V(n)$ and we will also denote them by F, R and V respectively. These maps induce various extensions, in particular

$$0 \to G_a \to V(n+1) \to V(n) \to 0. \tag{10}$$

with the associated Hochschild-Serre's spectral sequence

$$E_2^{p,q}(n+1) = \mathrm{H}^p(V(n); \mathrm{H}^q(G_a;k)) \Longrightarrow \mathrm{H}^{p+q}(V(n+1);k). \tag{11}$$

Let us denote by $S(n)$ (resp. $E(n)$) the symmetric algebra $S(\oplus k y_n(i+1))$ (resp. exterior algebra $E(\oplus k x_n(i))$). For $p=2$, let $y_n(i+1) = x_n(i)^2$.

Theorem 5.1 *For all $n \geq 2$,*

(a) $V(n) \cong \prod_{p\nmid i} W(r_i)$

(b) $\mathrm{H}^*(V(n);k) \cong \bigotimes_{p\nmid i=1}^{n-1} S(p^{r_i-1}i) \otimes E(i)$,

where r_i is the smallest integer such that $p^{r_i}i \geq n$ and $\beta_{r_i}x_i(j) = y_{p^{r_i-1}i}(j+1)$.

The proof of the Theorem follows from the following Lemmas which may be useful for other results. Let G be a unipotent algebraic group obtained from an extension of a product of Witt groups by G_a.

$$0 \to G_a \to G \to \prod_{i=1}^{m} W(s_i) \to 0. \tag{12}$$

If we write $k[W(s_i)] = k[x_{i,0},\ldots,x_{i,s_i-1}]$ and $k[G_a] = k[x]$, then their cohomologies are $\mathrm{H}^*(W(s_i);k) = \otimes_{j=0}^{\infty} k[y_{i,s_i-1}(j+1)] \otimes E(x_{i,0}(j))$, and $\mathrm{H}^*(G_a;k) = \otimes_{j=0}^{\infty} k[y(j+1)] \otimes E(x(j))$ respectively, by Theorem 3.2.

Lemma 5.2 *In the spectral sequence induced from the extension (12);*

(1) If $d_2 x(0) = 0$, then $G \cong \left(\prod W(s_i)\right) \times G_a$,

(2) If $d_2 x(0) = y_{j,s_j-1}(1)$ for some $1 \leq j \leq m$, then $G \cong \left(\prod_{j\neq i} W(s_i)\right) \times W(s_j+1)$.

Proof: Consider the coextension associated to the extension (12).

$$k[x_n] \leftarrow k[G] \leftarrow \otimes k[W(s_i)].$$

If $d_2x(0) = 0$, then $0 \neq x(0) \in \mathrm{H}^1(G;k)$ induces a map

$$\pi : G \to \left(\prod W(s_i)\right) \times G_a,$$

which induces an epimorphism in the coordinate algebras. Since $k[G]$ is polynomial, it also induces an isomorphism of groups by dimension counting argument.

Next consider the case $d_2x(0) = y_{j,s_j-1}(1)$. Since $y_{j,s_j-1}(1) = \beta_{s_j}x_{j,0}(0)$, by Lemma 4.3 $\beta_{s_j+1}x_{j,0}(0) \neq 0$ in $\mathrm{H}^2(G;k)$. Let $\psi : G \to W(s_j+1)$ be the map defining $\beta_{s_j+1}x_{j,0}(0)$. We get

$$G \xrightarrow{\pi} \left(\prod_{i\neq j} W(s_i)\right) \times W(s_j+1).$$

Since $d_2x(0) = y_{j,s_j-1}(1) = \beta_{s_j}x_{j,0}(0)$. In the cobar complex $C^2(G)$ we have

$$\partial^1 x = \pi^2(\beta_{s_j}x_{j,0}) = \pi^2(\partial^1 x_{j,s_j}) = \partial^1\pi^1 x_{j,s_j}.$$

Therefore $\partial^1(x - \pi^1 x_{j,s_j}) = 0$ but $d_2x(0) \neq 0$. Hence $x = i^1\pi^1 x_{j,s_j}$ in $k[G_a]$, for $i : G_a \to G$. This means π^* is surjective and hence π is an isomorphism of groups. □

Lemma 5.3 *Let G be a commutative unipotent group defined in (12). Then in the associated spectral sequence*

$$d_2x(0) = \sum \mu_i(s)y_{i,s_i-1}(s), \qquad \mu_i(s) \in k.$$

Proof: Suppose $p \geq 3$. Write

$$d_2x(0) = \sum \lambda_{i,j}(k,l)x_{i,0}(k)x_{j,0}(l) + \sum \mu_i(s)y_{i,s_i-1}(s),$$

for $\lambda_{i,j}(k,l)$ and $\mu_i(s) \in k$. This means that there is an element $a \in C^1(G)$ such that a belongs to the ideal $(x_{i,j})$, i.e. the image of a in $C^1(G_a) = 0$ and

$$\partial^1(x-a) = \sum \lambda_{i,j}(k,l)x_{i,0}^{p^k}x_{j,0}^{p^l} + \sum \mu_i(s)(\beta_{s_i-1}x_{i,0})^{p^s},$$

in $C^2(G)$. Since G is a commutative group, the coboundary ∂^1 must be cocommutative. This implies that $\partial^1(x-a)$ is invariant under the twist, $\tau(c \otimes d) = d \otimes c$, in $C^2(G)$. Therefore $\lambda_{i,j}(k,l) = \lambda_{j,i}(l,k)$. But $x_{i,0}(k)x_{j,0}(l) = -x_{j,0}(l)x_{i,0}(k)$ in $\mathrm{H}^2(\prod W(s_i);k)$, which forces $\lambda_{i,j}(k,l) = 0$ for all i,j,k,l. Hence $d_2x(0) = \sum \mu_i(s)y_{i,s_i-1}(s)$.

The case $p = 2$ is proved by replacing $y_{i,0}(k+1)$ by $x_{i,0}(k) \otimes x_{i,0}(k)$ and use similar argument as in the case $p > 2$. □

Proof of Theorem 5.1. Assume $p \geq 3$. It is clear that the lemma is true for $n = 2$. Assume true for $n \geq 2$ and induct on n. The group $V(n+1)$ can be obtained from $V(n)$ by extension by G_a, i.e. it is the extension (12) with the following replacements: $G \rightsquigarrow V(n+1)$, $s_i \rightsquigarrow r_i$, $p \nmid i$, r_i the smallest positive integer such that $p^{r_i} i \geq n-1$, $x_{i,j} \rightsquigarrow x_{ip^j}$ and $x \rightsquigarrow x_n$. Recall that the weight $w(x_i(j)) = w(y_i(j)) = ip^j$, which, of course, is preserved by the differential. From Lemma 5.3, we have

$$d_2 x_n(0) = \begin{cases} \mu y_{\frac{n}{p}}(1) & \text{if } p|n, \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

because the other elements of the same degree are also of the same weight, hence they are all of the form $y_{\frac{n}{p^s}}(s)$ for some $s \geq 2$. But these elements do not appear in the assumption (b) for $n-1$.

We will now show that $\mu \neq 0$. First take $n = p$, we will show that $V(p+1) \not\cong G_a^p = G_a \times \cdots \times G_a$. For simplicity in the notation, we denote a matrix $[a_{ij}] \in V(n)$ by its first row entries: $[a_{i,j}] = (1, a_1, \ldots, a_{n-1})$. For $n+1 = p+1$ consider the matrix $A = [a_{ij}] = (1,1,0,\ldots,0) \in V(p+1)$. Then $A^p = (1,0,\ldots,0,1) \neq I$, with the non trivial entries in position 1 and $p+1$. Hence $V(p+1)$ is not a product of G_a. Now, if $\mu = 0$, by induction and Lemma 4.3 implies that $V(p+1)$ is a product of G_a, which leads to a contradiction.

Let $n+1 = mp+1$ and let $\iota : V(p+1) \hookrightarrow V(mp+1)$ be an inclusion of $V(p+1)$ into $V(mp+1)$ defined as

$$\iota(a_1, \ldots, a_p) = (1, \underbrace{0, \ldots, 0, a_1}_{m}, \underbrace{0, \ldots, 0, a_2}_{m}, \ldots, \underbrace{0, \ldots, 0, a_p}_{m})$$

By the naturality with respect to ι of the spectral sequences, $d_2 x_p(0) = y_1(1)$ induces $d_2 x_{mp}(0) = y_m(1)$. The Frobenius F^* then implies

$$d_2 x_n(i) = \begin{cases} y_{\frac{n}{p}}(i+1) & \text{if } p|n, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

This proves Theorem 5.1 (b) for the case $n+1$. The Bockstein is given by Lemma 4.3. Part (a) follows from Lemma 4.3. The case $p = 2$ is proved similarly by replacing $y_j(i+1)$ with $x_j(i)^2$. □

Remark 5.4 The subalgebra $k[x_0, x_1^{p^s}] \subset k[x_0, x_1] = k[W(2)]$ is a Hopf subalgebra. Hence there is a group $W_s(2)$ isogenic to $W(2)$. For the extension.

$$0 \to G_a \to W_s(2) \to G_a \to 0$$

the differential of the induced spectral sequence is $d_2x_1(0) = y_0(s+1)$. And hence

$$\mathrm{H}^*(W_s(2);k) \cong \Big(\otimes_{i=1}^{s} S(y_0(i)) \otimes_{j=s+1}^{\infty} S(y_1(j))\Big) \otimes_{k=0}^{\infty} E(x_0(k)),$$

with $y_0(i) = \beta x_0(i-1)$ and $y_1(i) = \beta_2 x_0(j-1)$.

6 Frobenius Kernel and the Steenrod Algebra

Let r be a positive integer and let G_r be the r^{th} Frobenius kernel of an algebraic k-group G, i.e. it is the kernel of the r^{th} power of the Frobenius homomorphism

$$0 \longrightarrow G_r \longrightarrow G \xrightarrow{F^r} G \longrightarrow 0.$$

It is easy to obtain the similar results as in Sections 3 to 5 for the rational cohomology $\mathrm{H}^*(G_r;k)$. For example

$$\mathrm{H}^*(W(n)_r;k) \cong \bigotimes_{i=0}^{r-1} k[y_{n-1}(i+1)] \otimes E(x_0(i)),$$

and $\beta_n(x_0(i)) = y_{n-1}(i+1)$.

Let $G(n)$ be the subgroup of the unipotent group $U(n)$ such that a matrix $[a_{i,j}] \in G(n)$ if $a_{i,j} = \delta_{i,j}$ for $i \geq j$ and $a_{i,j}^{p^r} = a_{i+r,j+r}$ for $i < j$ and $0 \leq r \leq n-i$. The coordinate ring $k[G(n)]$ is a polynomial algebra $k[a_1,\ldots,a_{n-1}]$ with the comultiplication

$$\Delta a_i = \sum_{j=0}^{i} a_j \otimes a_{i-j}^{p^j}.$$

On the otherhand, let $P(n)$ be the finite dimensional subalgebra of the Steenrod algebra generated by the reduced powers $P^{p^0},\ldots,P^{p^n}$. Its dual Hopf algebra is

$$P(n)^* \cong k[\xi_1,\ldots,\xi_{n+1}]/(\xi_1^{p^{n+1}}, \xi_2^{p^n} \ldots, \xi_{n+1}^{p}),$$

with $\Delta\xi_i = \sum_{j=0}^{i} \xi_j \otimes \xi_{i-j}^{p^j}$. There is a Hopf algebra epimorphism by (3.3) in [4].

$$k[G(n)_{n-1}] \to P(n-2)^*.$$

Therefore $\mathrm{H}^*(G(n)_{n-1};k)$ is important in homotopy theories. However the computations seem difficult except for $p=2$ and $n \leq 3$ which we now show.

Consider the spectral sequence arises from the extension $1 \to G_{a2} \to G(3)_2 \to G_{a2} \to 1$

$$E_2^{*,*} \cong k[x_1(0), x_2(0), x_1(1), x_2(1)],$$

with $d_2x_2(0) = x_1(0)x_1(1)$ and $d_2x_2(1) = x_1(1)x_1(2) = 0$. Therefore we have

$$E_3^{*,*} \cong k[x_1(0), x_1(1)]/(x_1(0)x_1(1)) \otimes k[x_2(0)^2, x_2(1)].$$

The next differential is (see A1, 5.2 in [5])

$$\begin{aligned} d_3x_2(0)^2 &= d_3\widetilde{Sq}^1 x_2(0) = \widetilde{Sq}^1(x_1(0)x_1(1)) \\ &= \widetilde{Sq}^1 x_1(0)\widetilde{Sq}^0 x_1(1) + \widetilde{Sq}^0 x_1(0)\widetilde{Sq}^1 x_1(1) \\ &= x_1(1)^3. \end{aligned}$$

Therefore we get

$$E_4^{*,*} \cong k[x_2(0)^4, x_2(1)] \otimes \left(k[x_1(0), x_1(1)]/(x_1(0)x_1(1), x_1(1)^3) \oplus k[x_1(0)]x_1(0)x_2(0)^2\right),$$

and this is isomorphic to $E_\infty^{*,*}$. This result is essentially obtained by Liuevicius. See for example, 3.1.24 in [5], where their notation is the following $h_{10} = x_1(0)$, $h_{11} = x_1(1)$, $w = x_2(0)^4$ and $v = x_1(0)x_2(0)^2$, and

$$\mathrm{H}^*(G(3)_2; k) \cong \mathrm{Ext}_{P(1)^*}(k; k) \otimes k[x_2(1)].$$

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