

K_* -LOCALIZATIONS OF SPECTRA X WITH $KU_*X \cong \mathbb{Z} \oplus \mathbb{Z}$

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0. Introduction

Let E be an associative ring spectrum with unit. For any CW-spectra X, Y we say that X is quasi E_* -equivalent to Y if there exists an equivalence $h : E \wedge Y \rightarrow E \wedge X$ of E -module spectra (see [10] or [5]). Let KO and KU be the real and the complex K -spectrum respectively. We denote by S_K the K_* -localization of the sphere spectrum $S = \Sigma^0$. Since the K_* -localization of an arbitrary CW-spectrum X is obtained as the smash product $S_K \wedge X$ ([3] or [9]), we observe that two CW-spectra X and Y have the same K_* -local type if and only if X is quasi S_{K*} -equivalent to Y .

Let $\bar{\eta} : \Sigma^1 SZ/2 \rightarrow \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \rightarrow SZ/2$ be an extension and a coextension of the Hopf map $\eta : \Sigma^1 \rightarrow \Sigma^0$. Their cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ are quasi KO_* -equivalent to Σ^4 and Σ^7 respectively. In [12] we first dealt with CW-spectra X such that $KU_0X \cong \mathbb{Z}$ and $KU_1X = 0$ in order to study their K_* -local types. Such a CW-spectrum X with $X \wedge SQ = SQ$ is quasi KO_* -equivalent to either of Σ^0 and Σ^4 , and more precisely it has the same K_* -local type as either of Σ^0 and $C(\bar{\eta})$ (see [12, Theorem 1.2 i]) or [4, Proposition 10.6]). In particular, the cofibers $C(\bar{\eta})$ and $\Sigma^{-3}C(\tilde{\eta})$ have the same K_* -local type.

The purpose of this paper is to determine completely the K_* -local types of CW-spectra X with $KU_0X \oplus KU_1X \cong \mathbb{Z} \oplus \mathbb{Z}$ (see [13] for details). Such a CW-spectrum X with $X \wedge SQ = (\Sigma^t \vee \Sigma^0) \wedge SQ$

is quasi KO_* -equivalent to the wedge sum $\Sigma^t \vee \Sigma^0$, $\Sigma^{t+4} \vee \Sigma^0$, $\Sigma^t \vee \Sigma^4$, $\Sigma^{t+4} \vee \Sigma^4$ or the following cofiber $Y = \Sigma^{-1}C(\eta^2)$, $C(\eta)$ or $C(\eta^2)$ according as $t \equiv 1, 2$ or $3 \pmod{4}$. A CW-spectrum X is said to be a Wood spectrum if it is quasi KO_* -equivalent to $C(\eta)$, and an Anderson spectrum if it is quasi KO_* -equivalent to $C(\eta^2)$. In the previous paper [12, Theorems 1 and 2] we have determined the K_* -local types of Wood and Anderson spectra except when $t = -1$, applying the following powerful tool due to Bousfield [4] (see [12, Theorem 4]):

Theorem 1. Let Y be a certain CW-spectrum such that
 i) KU_*Y is either free or divisible and $[\Sigma^1 Y, Y \wedge SQ] = 0$, or
 ii) $KU_1 Y = 0$ (or $KU_0 Y = 0$). Assume that a CW-spectrum X is quasi KO_* -equivalent to Y and the stable real Adams operations ψ_R^k on $KO_* X \otimes Z[1/k]$ and $KO_* Y \otimes Z[1/k]$ behave as the same action. Then X has the same K_* -local type as Y .

1. Spectra with $KU_* X \cong Z \oplus Z$ and $X \wedge SQ = \langle \Sigma^t \vee \Sigma^0 \rangle \wedge SQ$, $t \neq \pm 1$

1.1. Recall that the homotopy groups $\pi_i S_K$ are given as follows ([9] or [3]): $Z \oplus Z/2$ for $i = 0$; $Z/2$ for $i \equiv 0 \pmod{8}$ with $i \neq 0$; $Z/2 \oplus Z/2$ for $i \equiv 1 \pmod{8}$; $Z/2$ for $i \equiv 2 \pmod{8}$; $Z/m(2s)$ for $i = 4s-1$ with $s \neq 0$; Q/Z for $i = -2$; and 0 otherwise where $m(2s)$ is the numerical function defined explicitly in [1] or [2]. We first realize generators of the homotopy groups $\pi_i S_K$ except $i = -2$ by constructing maps between small spectra. By the solution of the Adams conjecture we get an Adams' K_* -equivalence

$$(1.1) \quad A_r : \Sigma^{8r} SZ/m(4r) \rightarrow SZ/m(4r)$$

such that the composite map $j_{A_r} i : \Sigma^{8r-1} \rightarrow \Sigma^0$ is the generator

ρ_r of the J-image, where SZ/m denotes the Moore Spectrum of type Z/m . Set

$\bar{\rho}_r = jA_r : \Sigma^{8r-1}SZ/m(4r) \rightarrow \Sigma^0$ and $\tilde{\rho}_r = A_r i : \Sigma^{8r} \rightarrow SZ/m(4r)$ and consider their cofibers $C(\bar{\rho}_r)$ and $C(\tilde{\rho}_r)$ for which there exist K_* -equivalences

$$(1.2) \quad e_r : C(\bar{\rho}_r) \rightarrow \Sigma^0 \quad \text{and} \quad e'_r : \Sigma^{8r+1} \rightarrow C(\tilde{\rho}_r).$$

For $r \geq 1$ we denote by

$$(1.3) \quad \rho_{-r} : C(\bar{\rho}_r) \rightarrow \Sigma^{8r+1} \quad \text{and} \quad \rho'_{-r} : \Sigma^0 \rightarrow C(\tilde{\rho}_r)$$

the top cell projection and the bottom cell inclusion.

Let $\bar{\eta} : \Sigma^1 SZ/2m \rightarrow \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \rightarrow SZ/2m$ be an extension and a coextension of the Hopf map $\eta : \Sigma^1 \rightarrow \Sigma^0$. We construct the following maps of order 2 (cf. [12, (1.13) and (2.5)]):

$$(1.4) \quad \mu_r = \bar{\eta} A_r i : \Sigma^{8r+1} \rightarrow \Sigma^0, \quad \mu'_r = j A_r \tilde{\eta} : \Sigma^{8r+1} \rightarrow \Sigma^0, \\ \mu_{-r} = \bar{\eta} i_r : \Sigma^{-8r+1} C(\bar{\rho}_r) \rightarrow \Sigma^0, \quad \mu'_{-r} = j_r \tilde{\eta} : \Sigma^{-8r+1} \rightarrow \Sigma^{-8r-1} C(\tilde{\rho}_r)$$

where $i_r : C(\bar{\rho}_r) \rightarrow \Sigma^{8r} SZ/m(4r)$ and $j_r : SZ/m(4r) \rightarrow C(\tilde{\rho}_r)$ denote the canonical projection and inclusion respectively.

Choose an extension and a coextension

$$\bar{\xi}_r : \Sigma^{8r+3} SZ/m(8r+4) \rightarrow \Sigma^0 \quad \text{and} \quad \xi_r : \Sigma^{8r+4} \rightarrow SZ/m(8r+4)$$

of the generator ξ_r of the J-image, and then construct the following maps

$$(1.5) \quad \xi_{-r-1} = \bar{\xi}_r i_{2r+1} : \Sigma^{-8r-5} C(\bar{\rho}_{2r+1}) \rightarrow \Sigma^0 \quad \text{and} \\ \xi'_{-r-1} = j_{2r+1} \xi_r : \Sigma^{-8r-5} \rightarrow \Sigma^{-16r-9} C(\tilde{\rho}_{2r+1}).$$

Consider the cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ of the maps $\bar{\eta} : \Sigma^1 SZ/2 \rightarrow \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \rightarrow SZ/2$. The homotopy groups $\pi_i S_K \wedge C(\bar{\eta}) \cong \pi_{i+3} S_K \wedge C(\tilde{\eta})$ are given as follows: Z for $i = 0$; $Z/2$ for $i \equiv 4$ or $6 \pmod{8}$ with $i \neq -2$; $Z/2 \oplus Z/2$ for $i \equiv 5 \pmod{8}$; $Z/m(2s)$ for $i = 4s-1$ with $s \neq 0$; $Z/2 \oplus Q/Z$ for $i = -2$; and 0 otherwise. We next realize generators of the homotopy groups $\pi_i S_K \wedge C(\bar{\eta})$ except $i = -2$ by constructing maps between small spectra. Choose maps $\bar{\lambda} : C(\bar{\eta}) \rightarrow \Sigma^0$ and $\tilde{\lambda} : \Sigma^3 \rightarrow C(\tilde{\eta})$ satisfying $\bar{\lambda} \bar{i} = 4$ and $\tilde{j} \tilde{\lambda} = 4$ where $\bar{i} : \Sigma^0 \rightarrow C(\bar{\eta})$

and $\tilde{j} : C(\tilde{\eta}) \rightarrow \Sigma^3$ denote the bottom cell inclusion and the top cell projection. We now construct the following maps

$$(1.6) \quad \begin{aligned} s_r &= \tilde{\lambda} \rho_r : \Sigma^{8r-1} \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad , \quad s'_r = \rho_r \bar{\lambda} : \Sigma^{8r-1}C(\bar{\eta}) \rightarrow \Sigma^0 \quad , \\ s_{-r} &= \tilde{\lambda} \rho_{-r} : \Sigma^{-8r-1}C(\bar{\rho}_r) \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad \text{and} \\ s'_{-r} &= \rho'_{-r} \bar{\lambda} : \Sigma^{-8r-1}C(\bar{\eta}) \rightarrow \Sigma^{-8r-1}C(\bar{\rho}_r) \quad . \end{aligned}$$

Take the composite maps $\bar{h} = \tilde{i}\pi : SZ/2m \rightarrow C(\tilde{\eta})$ and $\tilde{h} = \pi\bar{j} : C(\bar{\eta}) \rightarrow \Sigma^2SZ/2m$ where $\tilde{i} : SZ/2 \rightarrow C(\tilde{\eta})$ and $\bar{j} : C(\bar{\eta}) \rightarrow \Sigma^2SZ/2$ denote the canonical inclusion and the bottom cell collapsing respectively and the maps π are the canonical maps between Moore spectra. Then we can construct the following maps of order 2 (cf. [12, (1.13) and (2.5)]):

$$(1.7) \quad \begin{aligned} m_r &= \bar{h}A_r i : \Sigma^{8r-3} \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad , \quad m'_r = jA_r \tilde{h} : \Sigma^{8r-3}C(\bar{\eta}) \rightarrow \Sigma^0 \quad , \\ m_{-r} &= \bar{h}i_r : \Sigma^{-8r-3}C(\bar{\rho}_r) \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad \text{and} \\ m'_{-r} &= j_r \tilde{h} : \Sigma^{-8r-3}C(\bar{\eta}) \rightarrow \Sigma^{-8r-1}C(\bar{\rho}_r) \quad . \end{aligned}$$

Choose maps $\bar{n} : \Sigma^6SZ/16 \rightarrow C(\tilde{\eta})$ and $\tilde{n} : \Sigma^4C(\bar{\eta}) \rightarrow SZ/16$ with $\tilde{j}\bar{n} = 3\bar{v}$ and $\tilde{n}\bar{i} = 3\tilde{v}$, and moreover choose maps $\bar{u}_r : \Sigma^{8r+6}SZ/m \rightarrow C(\tilde{\eta})$ and $\tilde{u}_r : \Sigma^{8r+4}C(\bar{\eta}) \rightarrow SZ/m$ with $\tilde{j}\bar{u}_r = 8\bar{\xi}_r$ and $\tilde{u}_r\bar{i} = 8\tilde{\xi}_r$ where $v : \Sigma^3 \rightarrow \Sigma^0$ is the Hopf map and $m = m(4r+2)/8$.

Using these maps we can construct the following maps

$$(1.8) \quad \begin{aligned} n_r &= \bar{n}i_{2r+1} : \Sigma^{8r+3} \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad , \quad n'_r = j_{2r+1}\tilde{n} : \Sigma^{8r+3}C(\bar{\eta}) \rightarrow \Sigma^0 \quad , \\ n_{-r-1} &= \bar{n}i_{2r+1} : \Sigma^{-8r-5}C(\bar{\rho}_{2r+1}) \rightarrow \Sigma^{-3}C(\tilde{\eta}) \quad \text{and} \\ n'_{-r-1} &= j_{2r+1}\tilde{n} : \Sigma^{-8r-5}C(\bar{\eta}) \rightarrow \Sigma^{-16r-9}C(\bar{\rho}_{2r+1}) \quad . \end{aligned}$$

Using the composite maps $\sigma\mu_{-1} = \sigma\bar{n}i_1 : C(\bar{\rho}_1) \rightarrow \Sigma^0$ and $\mu'_{-1}\sigma = j_1\tilde{n}\sigma : \Sigma^9 \rightarrow C(\bar{\rho}_1)$ where $\sigma : \Sigma^7 \rightarrow \Sigma^0$ is the Hopf map, we introduce the following two maps

$$(1.9) \quad \begin{aligned} \theta_m &= 2^m e_1 + \sigma\mu_{-1} : C(\bar{\rho}_1) \rightarrow \Sigma^0 \quad \text{and} \\ \theta'_m &= 2^m e'_1 + \mu'_{-1}\sigma : \Sigma^0 \rightarrow \Sigma^{-9}C(\bar{\rho}_1) \end{aligned}$$

whose cofibers are respectively denoted by SZ_m and SZ'_m for $m \geq 1$. Denote by V_m and V'_m respectively the cofibers of the composite maps $i\bar{\eta} : \Sigma^1SZ/2 \rightarrow SZ/2^{m-1}$ and $\tilde{\eta}j : \Sigma^1SZ/2^{m-1} \rightarrow SZ/2$. The spectrum $\Sigma^{-2}V'_m$ is quasi KO_* -equivalent to Σ^4V_m ,

and more precisely it has the same K_* -local type as $V_m \wedge C(\bar{\eta})$ (use [12, Theorem 1.2 and (1.3)]). Consider the following maps

$$(1.10) \quad \begin{aligned} \varphi_m &= i\bar{\eta}(e_1 \wedge 1) + \tilde{\eta}j(\sigma\mu_{-1} \wedge 1) : \Sigma^1 C(\bar{\rho}_1) \wedge SZ/2 \rightarrow SZ/2^{m-1} \\ \varphi'_m &= (e'_1 \wedge 1)i\bar{\eta} + (\mu'_{-1}\sigma \wedge 1)\tilde{\eta}j : \Sigma^1 SZ/2 \rightarrow \Sigma^{-9} C(\tilde{\rho}_1) \wedge SZ/2^{m-1} \\ \psi_m &= \tilde{\eta}j(e_1 \wedge 1) + i\bar{\eta}(\sigma\mu_{-1} \wedge 1) : \Sigma^1 C(\bar{\rho}_1) \wedge SZ/2^{m-1} \rightarrow SZ/2 \\ \psi'_m &= (e'_1 \wedge 1)\tilde{\eta}j + (\mu'_{-1}\sigma \wedge 1)i\bar{\eta} : \Sigma^1 SZ/2^{m-1} \rightarrow \Sigma^{-9} C(\tilde{\rho}_1) \wedge SZ/2 \end{aligned}$$

whose cofibers are respectively denoted by ΔV_m , $\Delta V''_m$, $\Delta V'_m$ and $\Delta V''_m$. For convenience' sake we set $V_1 = \Sigma^2 SZ/2$, $V'_1 = SZ/2$, $\Delta V_1 = \Sigma^2 SZ_1$ and $\Delta V'_1 = SZ'_1$. As is easily seen,

$$(1.11) \text{ i) } \psi_R^3 = -1 \text{ on } KO_2 SZ_1 \cong KO_2 SZ'_1 \cong Z/4, \text{ and } \psi_R^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

on $KO_2 SZ_m \cong KO_2 SZ'_m \cong Z/2 \oplus Z/2$ whenever $m \geq 2$.

$$\text{ii) } \psi_R^3 = 1 + 2^m \text{ on } KO_4 \Delta V_m \cong KO_4 \Delta V''_m \cong KO_2 \Delta V'_m \cong KO_2 \Delta V''_m \cong Z/2^{m+1} \text{ for } m \geq 1.$$

Applying Theorem 1 by means of (1.11) we obtain the following result (cf. [4, Proposition 10.5]).

Theorem 2. Let X be a CW-spectrum such that $KU_0 X \cong Z/2^m$ on which $\psi_C^k = 1$ and $KU_1 X = 0$. Then X has the same K_* -local type as one of the following small spectra: $SZ/2^m$, $SZ/2^m \wedge C(\bar{\eta})$, SZ_m , $SZ_m \wedge C(\bar{\eta})$, V_m , $\Sigma^{-2} V'_m$, ΔV_m and $\Sigma^{-2} \Delta V'_m$.

1.2. Let X be a CW-spectrum with $KU_0 X \cong Z \oplus Z$ and $KU_1 X = 0$. For such a CW-spectrum X we may assume that $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some integer $t \geq 0$. When t is odd, X is quasi KO_* -equivalent to the wedge sum $\Sigma^{2t} \vee \Sigma^0$, $\Sigma^{6t} \vee \Sigma^0$, $\Sigma^{2t} \vee \Sigma^4$, $\Sigma^{6t} \vee \Sigma^4$ or the cofiber $C(\eta)$. The Adams operation ψ_C^k on $KU_0 X \otimes Z[1/k]$ is represented by either of the following two matrices as left action (cf. [2, Theorem 7.18]):

$$A_{k,t,0} = \begin{pmatrix} 1/k^t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{k,t,1} = \begin{pmatrix} 1/k^t & 0 \\ 1-k^t/2k^t & 1 \end{pmatrix}.$$

When t is even, X is quasi KO_* -equivalent to the wedge sum

$\Sigma^0 \vee \Sigma^0$, $\Sigma^4 \vee \Sigma^0$, $\Sigma^0 \vee \Sigma^4$ or $\Sigma^4 \vee \Sigma^4$. If X is quasi KO_* -equivalent to either $\Sigma^0 \vee \Sigma^0$ or $\Sigma^4 \vee \Sigma^4$, then ψ_C^k on $KU_0 X \otimes \mathbb{Z}[1/k]$ behaves as the following matrix

$$A_{k,t,m} = \begin{pmatrix} 1/k^t & 0 \\ m(1-k^t)/m(t)k^t & 1 \end{pmatrix}$$

for some m , $0 \leq m < m(t)$. If X is quasi KO_* -equivalent to either $\Sigma^4 \vee \Sigma^0$ or $\Sigma^0 \vee \Sigma^4$ then ψ_C^k on $KU_0 X \otimes \mathbb{Z}[1/k]$ behaves as the matrix $A_{k,t,2m}$ for some m , $0 \leq m < m(t)$. Applying Theorem 1 we easily obtain

Theorem 3. Let X be a CW-spectrum such that $KU_0 X \cong \mathbb{Z} \oplus \mathbb{Z}$, $KU_1 X = 0$ and $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some even integer t . Then X has the same K_* -local type as the following small spectrum Y :

- i) $Y = C(m\rho_r)$, $C(m\rho_r) \wedge C(\bar{\eta})$, $C(ms_r)$ or $C(ms'_r)$ for some m , $0 \leq m < m(4r)$ when $t = 4r$;
- ii) $Y = C(m\xi_r)$, $C(m\xi_r) \wedge C(\bar{\eta})$, $C(mn_r)$ or $C(mn'_r)$ for some m , $0 \leq m < m(4r+2)$ when $t = 4r+2$.

Theorem 4. Let X be a CW-spectrum such that $KU_0 X \cong \mathbb{Z} \oplus \mathbb{Z}$, $KU_1 X = 0$ and $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some odd integer t .

- i) If X is quasi KO_* -equivalent to the wedge sum $\Sigma^{2t} \vee \Sigma^0$, then X has the same K_* -local type as the following small spectrum Y : $Y = \Sigma^{8r+2} \vee \Sigma^0$ or $C(\mu'_{r-1} n\sigma)$ if $t = 4r+1$; and $Y = \Sigma^{8r-2} C(\bar{\eta}) \vee \Sigma^0$ or $C(m'_{r-1} n\sigma)$ if $t = 4r-1$.

- ii) If X is a Wood spectrum, then X has the same K_* -local type as the cofiber $C(\mu'_r)$ or $C(m'_r)$ according as $t = 4r+1$ or $4r-1$.

Let X be a CW-spectrum with $KU_0 X \cong KU_1 X \cong \mathbb{Z}$. For such a CW-spectrum X we may assume that $X \wedge SQ = (\Sigma^{2t+1} \vee \Sigma^0) \wedge SQ$

for some odd integer t . Thus $\psi_C^k = 1$ on $KU_0 X \otimes \mathbb{Z}[1/k]$ and $\psi_C^k = 1/k^t$ on $KU_1 X \otimes \mathbb{Z}[1/k]$. Then X is quasi KO_* -equivalent to the wedge sum $\Sigma^3 \vee \Sigma^0$, $\Sigma^7 \vee \Sigma^0$, $\Sigma^3 \vee \Sigma^4$, $\Sigma^7 \vee \Sigma^4$ or the cofiber $C(\eta^2)$. Similarly to Theorems 1.2 and 1.3 we obtain

Theorem 5. Let X be a CW-spectrum such that $KU_0 X \cong KU_1 X \cong \mathbb{Z}$ and $X \wedge SQ = (\Sigma^{2t+1} \vee \Sigma^0) \wedge SQ$ for some odd integer $t \neq -1$.

i) If X is quasi KO_* -equivalent to the wedge sum $\Sigma^3 \vee \Sigma^0$, then X has the same K_* -local type as the wedge sum $\Sigma^{8r+3} \vee \Sigma^0$ or $\Sigma^{8r-1} C(\bar{\eta}) \vee \Sigma^0$ according as $t = 4r+1$ or $4r-1$.

ii) If X is quasi KO_* -equivalent to the wedge sum $\Sigma^{-1} \vee \Sigma^0$ then X has the same K_* -local type as the following small spectrum Y : $Y = \Sigma^{8r+3} C(\bar{\eta}) \vee \Sigma^0$ or $\Sigma^{8r+3} C(m_{-r-1}(\sigma \wedge 1))$ if $t = 4r+1$; and $Y = \Sigma^{8r-1} \vee \Sigma^0$ or $\Sigma^{8r-1} C(\mu_{-r-1} \sigma)$ if $t = 4r-1$.

iii) If X is an Anderson spectrum, then X has the same K_* -local type as the cofiber $C(\mu'_r \eta)$ or $C(m'_r(\eta \wedge 1))$ according as $t = 4r+1$ or $4r-1$.

2. Spectra with $KU_0 X \cong KU_1 X \cong \mathbb{Z}$ and $X \wedge SQ = \langle \Sigma^{-1} \vee \Sigma^0 \rangle \wedge SQ$

2.1. we here realize elements of the homotopy groups $\pi_{-2} S_K \cong \mathbb{Q}/\mathbb{Z}$ and $\pi_{-2} S_K \wedge C(\bar{\eta}) \cong \mathbb{Z}/2 \oplus \mathbb{Q}/\mathbb{Z}$ by constructing maps between small spectra. Given an integer $n \geq 1$ we take the minimum $r = r(n) \geq 1$ such that $m(4r)$ is divisible by n , and then set

$$(2.1) \quad \begin{aligned} \tau_{1/n} &= m \rho_r \rho_{-r} : \Sigma^{-2} C(\bar{\rho}_r) \rightarrow \Sigma^0 \\ \tau'_{1/n} &= m \rho'_{-r} \rho_r : \Sigma^{-2} \rightarrow \Sigma^{-8r-1} C(\tilde{\rho}_r) \\ t_{1/n} &= \tilde{\lambda} \tau_{1/n} : \Sigma^{-2} C(\bar{\rho}_r) \rightarrow \Sigma^{-3} C(\tilde{\eta}) \\ t'_{1/n} &= \tau'_{1/n} \tilde{\lambda} : \Sigma^{-2} C(\bar{\eta}) \rightarrow \Sigma^{-8r-1} C(\tilde{\rho}_r) \end{aligned}$$

in which $r = r(n)$ and $m = m(4r)/n$. Whenever $f_{1/n}$ is one

of the maps given in (2.1) we simply write $f_\alpha = \ell f_{1/n}$ for each rational number $\alpha = \ell/n$. Notice that the cofibers $C(\tau_\alpha)$ and $C(\tau'_\alpha)$ have the same K_* -local type, and $C(t_\alpha)$ and $C(t'_\alpha) \wedge C(\bar{\eta})$ have also the same K_* -local type. For simplicity we hereafter set $D_\alpha = C(\bar{\rho}_r)$ and $D'_\alpha = \Sigma^{-8r-1}C(\tilde{\rho}_r)$ with $\alpha = \ell/n \in \mathbb{Q}$ and $r = r(n)$ both of which have the same K_* -local type as Σ^0 . The exact sequences

$$\begin{aligned} 0 &\rightarrow S_{K-1}C(\tau'_\alpha) \rightarrow S_{K0}\Sigma^0 \rightarrow S_{K-2}D'_\alpha \rightarrow S_{K-2}C(\tau'_\alpha) \rightarrow 0 \\ 0 &\rightarrow S_{K-1}C(t'_\alpha) \rightarrow S_{K0}C(\bar{\eta}) \rightarrow S_{K-2}D'_\alpha \rightarrow S_{K-2}C(t'_\alpha) \rightarrow 0 \end{aligned}$$

are respectively given in the following forms:

$$(2.2) \quad \begin{aligned} 0 &\rightarrow Z \oplus Z/2 \xrightarrow{\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus Z/2 \xrightarrow{(\alpha \ 0)} Q/Z \xrightarrow{n} Q/Z \rightarrow 0 \\ 0 &\rightarrow Z \xrightarrow{n} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \rightarrow 0 \end{aligned}$$

where n is the denominator of $\alpha \in \mathbb{Q}$. Then we can show

Proposition 6. Let α and β be rational numbers.

- i) If the cofibers $C(\tau'_\alpha)$ and $C(\tau'_\beta)$ have the same K_* -local type, then $\alpha \equiv \beta \pmod{Z}$.
- ii) If the cofibers $C(t'_\alpha)$ and $C(t'_\beta)$ have the same K_* -local type, then $\alpha \equiv \beta \pmod{Z}$.

For any rational number $\alpha = \ell/n$ we now construct the following two maps

$$(2.3) \quad \begin{aligned} w_\alpha &= i_1 e_r + \ell t_{1/4n} : \Sigma^{-2}C(\bar{\rho}_r) \rightarrow \Sigma^{-3}C(\bar{\eta}) \quad \text{and} \\ w'_\alpha &= e'_r j j + \ell t'_{1/4n} : \Sigma^{-2}C(\bar{\eta}) \rightarrow \Sigma^{-8r-1}C(\tilde{\rho}_r) \end{aligned}$$

where $r = r(n)$ is the integer used in (2.1) and e_r and e'_r are the K_* -equivalences given in (1.2). Notice that the cofibers $C(w_\alpha)$ and $C(w'_\alpha) \wedge C(\bar{\eta})$ have the same K_* -local type. As is easily seen, the exact sequence

$$0 \rightarrow S_{K-1}C(w_\alpha) \rightarrow S_{K0}D_\alpha \rightarrow S_{K1}C(\bar{\eta}) \rightarrow S_{K-2}(w_\alpha) \rightarrow 0$$

is given in the following form:

$$(2.4) \quad 0 \rightarrow Z \xrightarrow{\begin{pmatrix} 2n \\ \ell \end{pmatrix}} Z \oplus Z/2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ \ell/4n & 1/2 \end{pmatrix}} Z/2 \oplus Q/Z \xrightarrow{\begin{pmatrix} \ell/2 & 2n \end{pmatrix}} Q/Z \rightarrow 0$$

where $\alpha = \ell/n \in \mathbb{Q}$. By virtue of (2.4) we can show

Proposition 7. Let α and β be rational numbers. If the cofibers $C(w_\alpha)$ and $C(w_\beta)$ have the same K_* -local type, then $\alpha \equiv \beta \pmod{2\mathbb{Z}}$.

Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong \mathbb{Z}$ and $X \wedge SQ \cong (\Sigma^{-1} \vee \Sigma^0) \wedge SQ$. If X is quasi KO_* -equivalent to either of $\Sigma^3 \vee \Sigma^0$ and $C(\eta^2)$, then $S_{K-1}X \cong \mathbb{Z}$ and $S_{K-2}X \cong Q/Z$. For such a CW-spectrum X the exact sequence

$$0 \rightarrow S_{K-1}X \rightarrow KO_{-1}X \rightarrow F_{-2}X \rightarrow S_{K-2}X \rightarrow 0$$

is written into the following form:

$$(2.5) \quad 0 \rightarrow Z \xrightarrow{n} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \rightarrow 0$$

with some $\alpha = \ell/n \in \mathbb{Q}$ where F denotes the fiber of the K_* -localized unit $i_R : S_K \rightarrow KO$. Then the cofiber sequence $S_{K(p)} \rightarrow Ad_{(p)} \rightarrow \Sigma^{-1}SQ \rightarrow \Sigma^1 S_{K(p)}$ induces an exact sequence

$$0 \rightarrow S_{K-1}^{X \otimes Z}(p) \rightarrow Ad_{(p)-1}X \rightarrow SQ_0X \rightarrow S_{K-2}^{X \otimes Z}(p) \rightarrow 0$$

given in the following form for each prime p :

$$(2.6) \quad 0 \rightarrow Z_{(p)} \xrightarrow{\begin{pmatrix} n \\ -\ell_p \end{pmatrix}} Z_{(p)} \oplus Z_{(p)} \xrightarrow{\begin{pmatrix} \ell_p/n & 1 \end{pmatrix}} Q \xrightarrow{n} Z/p^\infty \rightarrow 0$$

with some $\ell_p \in Z_{(p)}$ satisfying $\ell_p \equiv \ell \pmod{nZ_{(p)}}$.

For the rational number $\alpha = \ell/n$ obtained in (2.5) we set $\beta = \alpha$ or $\alpha+1$ according as $(\ell_2 - \ell)/n \in Z_{(2)}$ is even or odd, where ℓ_2 is given in (2.6). Then we obtain the following result as one of our main results.

Theorem 8. Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong \mathbb{Z}$ and $X \wedge SQ = (\Sigma^{-1} \vee \Sigma^0) \wedge SQ$.

i) If X is quasi KO_* -equivalent to the wedge sum $\Sigma^3 \vee \Sigma^0$,

then X has the same K_* -local type as the cofiber $C(t'_\alpha)$ for some $\alpha \in \mathbb{Q}$ with $0 \leq \alpha < 1$.

ii) If X is an Anderson spectrum, then X has the same K_* -local type as the cofiber $C(w_\beta)$ for some $\beta \in \mathbb{Q}$ with $0 \leq \beta < 2$.

2.2. Let $\eta_{4m,2} : \Sigma^2 SZ/4m \rightarrow SZ/2$ be a map satisfying $\eta_{4m,2}^i = \tilde{\eta}$ and $j\eta_{4m,2} = \bar{\eta}$. Using the composite map $\tilde{\mu}_{-1} = \eta_{240,2}^i : C(\bar{\rho}_1) \rightarrow \Sigma^6 SZ/2$ we introduce the following map

$$(2.7) \quad i\eta e_1 + (\sigma \wedge 1)\tilde{\mu}_{-1} : \Sigma^1 C(\bar{\rho}_1) \rightarrow SZ/2$$

whose cofiber is denoted by ΔM_1 . The map $2(\rho_r \wedge 1) :$

$\Sigma^{8r-1} C(\bar{\rho}_1) \rightarrow C(\bar{\rho}_1)$ is factorized through ΔM_1 as $2(\rho_r \wedge 1) = j_M v_r$ for a suitable map $v_r : \Sigma^{8r+1} C(\bar{\rho}_1) \rightarrow \Delta M_1$ where $j_M : \Delta M_1 \rightarrow \Sigma^2 C(\bar{\rho}_1)$ denotes the canonical projection, because $\sigma^2 \eta = 0$

and $\sigma \rho_r = 0$ for any $r \geq 2$. For any integer $n \geq 1$ we set

$$(2.8) \quad v_{1/n} = m v_r (\rho_{-r} \wedge 1) : C(\bar{\rho}_r) \wedge C(\bar{\rho}_1) \rightarrow \Delta M_1$$

where $r = r(n)$ and $m = m(4r)/n$. For any rational number $\alpha = \ell/n$ we now introduce the following map

$$(2.9) \quad \omega_\alpha = i_M i(e_r \wedge e_1) + \ell v_{1/4n} : C(\bar{\rho}_r) \wedge C(\bar{\rho}_1) \rightarrow \Delta M_1$$

where $i_M : SZ/2 \rightarrow \Delta M_1$ denotes the canonical inclusion. Note

that the cofiber $C(\omega_\alpha)$ is quasi KO_* -equivalent to the wedge sum $\Sigma^1 v \Sigma^2$. For simplicity we hereafter set $D_{\alpha,1} =$

$C(\bar{\rho}_r) \wedge C(\bar{\rho}_1)$, which has the same K_* -local type as Σ^0 via the

K_* -equivalence $e_r \wedge e_1$. Since the induced homomorphism $\omega_{\alpha*} :$

$S_{K0} D_{\alpha,1} \rightarrow S_{K0} \Delta M_1$ is expressed as the matrix $\begin{pmatrix} 1 & 0 \\ \ell/4n & 1/2 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}/2$

$\rightarrow \mathbb{Z}/2 \oplus \mathbb{Q}/\mathbb{Z}$ and $\omega_{\alpha*} : S_{K1} D_{\alpha,1} \rightarrow S_{K1} \Delta M_1$ is an epimorphism, the

exact sequence

$$0 \rightarrow S_{K1} C(\omega_\alpha) \rightarrow S_{K0} D_{\alpha,1} \rightarrow S_{K0} \Delta M_1 \rightarrow S_{K0} C(\omega_\alpha) \rightarrow 0$$

is written into the same form as (2.4).

Similarly to Proposition 7 we can show

Proposition 9. Let α and β be rational numbers. If the cofibers $C(\omega_\alpha)$ and $C(\omega_\beta)$ have the same K_* -local type, then $\alpha \equiv \beta \pmod{2Z}$.

Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X \wedge SQ = (\Sigma^{-1} \vee \Sigma^0) \wedge SQ$. If X is quasi KO_* -equivalent to the wedge sum $\Sigma^{-1} \vee \Sigma^0$, then $S_{K-2}X \cong Q/Z$ and $S_{K-1}X \cong Z \oplus Z/2$ or Z according as $\psi_R^3 = 1$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $KO_1X \cong Z/2 \oplus Z/2$. For such a CW-spectrum X the exact sequence

$$S_{K-1}X \rightarrow KO_{-1}X \rightarrow F_0X \rightarrow S_{K-2}X \rightarrow 0$$

is written into the following form:

$$(2.10) \quad \begin{array}{ccccccc} Z \oplus Z/2 & \xrightarrow{\begin{pmatrix} n & 0 \\ -\ell_2 & 0 \end{pmatrix}} & Z & \xrightarrow{\alpha} & Q/Z & \xrightarrow{n} & Q/Z \rightarrow 0 \quad \text{if } S_{K-1}X \cong Z \oplus Z/2 \\ & & Z & \xrightarrow{n} & Z & \xrightarrow{\alpha} & Q/Z \xrightarrow{n} Q/Z \rightarrow 0 \quad \text{if } S_{K-1}X \cong Z \end{array}$$

with some $\alpha = \ell/n \in Q$. Then the exact sequence

$$0 \rightarrow S_{K-1}X \otimes Z_{(p)} \rightarrow Ad_{(p)-1}X \rightarrow SQ_0X \rightarrow S_{K-2}X \otimes Z_{(p)} \rightarrow 0$$

is given in the following form for each prime p :

(2.11) i) In the case when $S_{K-1}X \cong Z \oplus Z/2$ and $p = 2$,

$$0 \rightarrow Z_{(2)} \oplus Z/2 \xrightarrow{\begin{pmatrix} n & 0 \\ -\ell_2 & 0 \\ 0 & 1 \end{pmatrix}} Z_{(2)} \oplus Z_{(2)} \oplus Z/2 \xrightarrow{(\ell_2/n \ 1 \ 0)} Q \xrightarrow{n} Z/2^\infty \rightarrow 0$$

ii) In the case when $S_{K-1}X \cong Z$ or p is odd,

$$0 \rightarrow Z_{(p)} \xrightarrow{\begin{pmatrix} n \\ -\ell_p \end{pmatrix}} Z_{(p)} \oplus Z_{(p)} \xrightarrow{(\ell_p/n \ 1)} Q \xrightarrow{n} Z/p^\infty \rightarrow 0.$$

Here $\ell_p \in Z_{(p)}$ satisfies $\ell_p \equiv \ell \pmod{nZ_{(p)}}$ for each prime p .

Similarly to Theorem 2.3 we obtain the following result as another of our main results.

Theorem 10. Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X \wedge SQ = (\Sigma^{-1} \vee \Sigma^0) \wedge SQ$. Assume that X is quasi KO_* -equivalent to the wedge sum $\Sigma^{-1} \vee \Sigma^0$.

i) If $\psi_R^3 = 1$ on $KO_1X \cong Z/2 \oplus Z/2$, then X has the same

K_* -local type as the cofiber $C(\tau'_\alpha)$ for some $\alpha \in \mathbb{Q}$ with $0 \leq \alpha < 1$.

ii) If $\psi_R^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $KO_1 X \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, then X has the same K_* -local type as the cofiber $\Sigma^{-2}C(\omega_\beta)$ for some $\beta \in \mathbb{Q}$ with $0 \leq \beta < 2$.

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