DOUBLE TRANSFERS AT THE PRIME 2

MITSUNORI IMAOKA

The $S^1$-transfer map $\tau : CP^0_n \rightarrow S^{-1}$ is a stable map defined using the transfer construction for the principal $S^1$–bundle over $CP^n$, where $CP^0_n = CP^n \cup \ast$ is the disjoint union of the complex projective space $CP^n$ and a base point. Then, $\tau_2 = \tau \wedge \tau : CP^0_0 \wedge CP^0_0 \rightarrow S^{-2}$ is called the double transfer map, and gives information of filtration 2 in the stable homotopy groups of spheres. Knapp [Kn] has given the first great step toward understanding the double transfer map, and the authors in [BC] have extended his results.

In [Hi], Hilditch has found a factorization $\bar{u}_2 : CP^\infty_0 \wedge CP^\infty_0 \rightarrow \Sigma^{-4}N_2$ of the double transfer map $\tau_2$ through $N_2$, the realization of the second stage of the chromatic filtration given by Ravenel [Ra], under the condition that spectra are localized at an odd prime $p$. His result extends that of Miller in [Mi], and enables the transfer images to be compared with the chromatic filtration.

In the case that spectra are localized at 2, just the same factorization of $\tau_2$ as above does not exist (cf. [Hi; Remark 3.20]). In this note, I show that the restriction of $\tau_2$ on $CP^\infty \wedge CP^\infty$ is factored through a map $\bar{u}_2 : CP^\infty \wedge CP^\infty \rightarrow \Sigma^{-4}N_2$ even if it is localized at $p = 2$. Although we limit the content to this point, we notice that such factorization enables us to calculate the image of $(\tau_2)_* : \pi_*^i(CP^\infty \wedge CP^\infty)_{(2)} \rightarrow \pi_*^i(S^0)_{(2)}$ to some extent, using the result of Shimomura [Sh] concerning a calculation of the Novikov-Adams spectral sequence at prime 2.
§1 Preliminaries.

First, we prepare some properties about the $K$ and $KO$-cohomology of relevant spaces.

Let $\xi$ be the canonical complex line bundle over $CP^m$ for $0 \leq m \leq \infty$, and $(CP^m)^{k\xi}$ the Thom space of $k\xi$ for an integer $k$. We denote the Thom space by $CP^m_{k} = (CP^m)^{k\xi}$, and $CP_k = (CP^\infty)^{k\xi}$. Also, we put $CP = CP_1$ as in the introduction. $k\xi$ is $K$-orientable for any integer $k$, and thus we have a $K$-Thom class $U^k \in K^{2k}(CP_k)$ ([AB]). Here and hereafter, we only discuss the cases in which $\lim^1$ of $K$ and $KO$-cohomology groups are $0$, and thus we may regard $K^i(CP_k)$ as $K^i(CP^m_{k})$ for some large $N$.

We remark that $2k\xi$ is $KO$-orientable but $(2k + 1)\xi$ is not $KO$-orientable. Since we need the information of $KO^*(CP)$ and $KO^*(CP_{-1})$ later, we recall the structure of $KO^*(CP_{2k+1})$ first. Let $r : K \to KO$ and $c : KO \to K$ be the realification and complexification respectively, and $t \in K_2$ the generator. We put $X = [\xi - 1] \in K^0(CP)$ and $Y = r(X) \in KO^0(CP)$. Then, the following is shown by Fujii [Fu] in the case of $n \geq 0$, and we can prove it even in the case of $n < 0$.

Lemma 1.1. Let $n = 2k + 1$ and $m \geq 1$. Then there is an element $\bar{u}_n \in KO^{2n}(CP_n)$ which satisfies the following:

1. $KO^{2n}(CP_{2m+n-1}) = Z\{\bar{v}_n Y^i | 0 \leq i \leq m - 1\}$.
2. $c(\bar{v}_n) = U^K_n (2 + X)/(1 + X)^{k+1}$.

Corollary 1.2.

1. For any odd integer $n$, $i^*(\bar{u}_n) = 2\iota$.
2. $c(\bar{v}_{-1}) = U^K_{-1}(2 + X)$.

Let $\sinh^{-1}(T)$ be the inverse of the formal power expansion on $T$ of the function $\sinh(T)$, and put $S(T) = (\sqrt{T}/2)/(\sinh^{-1}(\sqrt{T}/2))$. Then, we define an
element $G_n(Y) \in KO^{2n}(CP_n; Q)$ for odd $n$ as follows:

\begin{equation}
G_n(Y) = \frac{1}{2} S(Y)^{-n} \left( 1 + \frac{Y}{4} \right)^{-1/2} \tilde{v}_n.
\end{equation}

Let $ch : K^*(-; Q) \to H^{2*}(-; Q)$ be the Chern character, and $ph = ch \circ c : KO^*(-; Q) \to H^{2*}(-; Q)$ the Pontryagin character. We denote by $U_n^H \in H^{2n}(CP_n; Z)$ the Thom class of $n\xi$ in the ordinary cohomology group. Then we have the following lemma, which is implicit in [CK].

**Lemma 1.4.** For any odd $n$, we have $ph(G_n(Y)) = U_n^H$.

Consider the following commutative diagram:

\begin{equation}
\begin{array}{ccc}
\pi_i^{2n}(CP_n; Q) & \xrightarrow{h^{KO}} & KO^{2n}(CP_n; Q) \\
| & | & | \\
H^{2n}(CP_n; Q) & \xrightarrow{c} & K^{2n}(CP_n; Q)
\end{array}
\end{equation}

where $\pi_i^{2n}(-)$ denotes the stable cohomotopy group and $h^{KO}$ is the $KO$-Hurewicz map. In this diagram, the vertical homomorphisms are all isomorphisms and the horizontal two maps are inclusions. We put

\begin{equation}
u_1 = (h^H)^{-1}(U_n^H) \in \pi_i^{2n}(CP_n; Q).
\end{equation}

Then we can show the following, in which $h^K = co h^{KO}$ is the $K$-Hurewicz map.

**Lemma 1.7.**

\begin{enumerate}
\item $h^{KO}(u_1) = G_n(Y)$ for odd $n$.
\item $h^K(u_1) = U_n^K(\log(1 + X)/X)^n$ for any $n$.
\end{enumerate}

**Corollary 1.8.** We have $c(G_n(Y)) = U_n^K(\log(1 + X)/X)^n$. In particular, $c(G_1(Y)) = t^{-1} \log(1 + X) \in K^2(CP; Q)$.

We note that Corollary 1.8 can be also proved directly. We need in §3 the following corollary of Lemma 1.7 and Corollary 1.8.
Corollary 1.9. For odd $n = 2k - 1$, there is an element $V \in KO^{2n}(CP_{n+1}; Q)$ which is uniquely defined by the equation $j^*(V) = h^{KO}(u_1) - (1/2)v_n$ and satisfies

$$c(V) = U_{n+1}^{K}(\frac{(\log(1 + X))^{n}}{X^{n+1}} - \frac{1 + \frac{X}{2}}{X(1 + X)^{k}}).$$

§ 2 The cofiber of the transfer map

Stably, we can consider $CP$ as a subspace of $CP_0$, and we denote by $\bar{\tau} : CP \to S^{-1}$ the restriction of the $S^1$-transfer map $\tau : CP_0 \to S^{-1}$. Let $W = S^{-2} \cup_{f} C(\Sigma^{-1}CP)$ be the cofiber of $\bar{\tau} : \Sigma^{-1}CP \to S^{-2}$. Since the cofiber of $\tau : \Sigma^{-1}CP_0 \to S^{-2}$ is stably homotopy equivalent to $CP_{-1}$ (cf. [Mi],[Kn]), we have inclusion maps $i' : CP \to CP_0$ and $i' : W \to CP_{-1}$, and the following homotopy commutative diagram up to sign:

$$\begin{array}{ccc}
S^{-2} & \xrightarrow{i} & W \\
\downarrow & & \downarrow i' \\
S^{-2} & \xrightarrow{i} & CP_{-1}
\end{array}$$

(2.1)

$$\begin{array}{ccc}
CP & \xrightarrow{r} & S^{-1} \\
\downarrow & & \downarrow i' \\
CP_{0} & \xrightarrow{r} & S^{-1}
\end{array}$$

Then the following is obvious.

Lemma 2.2.

(1) $0 \to H^k(CP; Z) \xrightarrow{j^*} H^k(W; Z) \xrightarrow{i^*} H^k(S^{-2}; Z) \to 0$ is a split exact sequence for any $k$, and $(i')^* : H^k(CP_{-1}; Z) \to H^k(W; Z)$ is an epimorphism with the kernel $H^0(CP_{-1}; Z) = Z\{U_{-1}X\}$.

(2) $0 \to K^k(CP) \xrightarrow{j^*} K^k(W) \xrightarrow{i^*} K^k(S^{-2}) \to 0$ is a split exact sequence for any $k$, and $(i')^* : K^*(CP_{-1}) \to K^*(W)$ is an epimorphism with the kernel $K_*\{U_{-1}X\}$.

Concerning $KO^{-2}(W)$, we have the following:

Proposition 2.3. There is an element $w \in KO^{-2}(W)$ which satisfies the following:
(i) $KO^{-2}(W) = Z\{w\} \oplus j^*(KO^{-2}(CP))$ and $j^*: KO^{-2}(CP) \to KO^{-2}(W)$ is a monomorphism.
(ii) $(i')^*(\overline{v}_{-1}) = 2w$ for $(i')^*: KO^{-2}(CP_{-1}) \to KO^{-2}(W)$.
(iii) $i^*(w)$ is a generator of $KO^{-2}(S^{-2})$ for the inclusion $i: S^{-2} \to W$.
(iv) $c(w) = (i')^*(U_K^{-1})$ in $K^{-2}(W)$.

§3 Factorization

In this section, we show a factorization of $\tilde{\tau} \wedge \tilde{\tau}$, which is one of our main results. We will use the following notations: $S(G)$ denotes the Moore spectrum of a group $G$, and $S^i G = \Sigma^i S(G)$; $\psi = \psi^3 - 1 : KO_{(2)} \to KO_{(2)}$ is the stable Adams operation, and $Ad$ the fiber spectrum of $\psi$. Thus we have a cofibering $Ad \to KO_{(2)} \to KO_{(2)}$, and we put $Ad^i G = Ad \wedge S^i G$.

Let $N_i \xrightarrow{l_i} M_i \xrightarrow{f_i} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i$ be the cofiber sequence such that

$$\cdots \to \Sigma^{-2} N_2 \xrightarrow{\delta_2} \Sigma^{-1} N_1 \xrightarrow{\delta_1} S^0$$

is the geometrical realization of the chromatic filtration by Ravenel [Ra], where $l_i$ is the Bousfield localization [Bo] with respect to the $v_i^{-1}BP_*$-homology. Then $N_0 \xrightarrow{l_0} M_0 \xrightarrow{f_0} N_1$ is identified with $S^0 \xrightarrow{i} S^0 Q \xrightarrow{\rho_Z} S^0 Q/Z$ by definition. where $\rho_Z$ is the mod $Z$ reduction. By [Bo], the second cofiber sequence $N_1 \xrightarrow{l_1} M_1 \xrightarrow{j_1} N_2$ is canonically identified with

$$S^0 Q/Z \xrightarrow{Ad^0 Q/Z} S^0 Q/Z \xrightarrow{Ad^0 Q/Z}$$

where $Ad = Ad/S^0$. Hereafter, we denote $KO_{(2)}$ simply by $KO$, and put $KO^i G = KO \wedge S^i G$.

Let $u_1 \in \pi^{-2} (CP_{-1}; Q)$ be the element in (1.6) for $n = -1$. Then we have an element $\overline{u}_1 \in \pi^{-2} (CP; Q/Z)$ which makes the following diagram homotopy
commutative up to sign:

\[
\begin{array}{cccc}
S^{-2} & \overset{i}{\longrightarrow} & W & \overset{j}{\longrightarrow} \ CP & \overset{f}{\longrightarrow} S^{-1} \\
\| & \downarrow & \| & \downarrow & \| \\
S^{-2} & \overset{i}{\longrightarrow} & S^{-2}Q & \overset{\rho z}{\longrightarrow} S^{-2}Q/Z & \overset{\delta_1}{\longrightarrow} S^{-1},
\end{array}
\]

(3.3)

where the upper cofiber sequence is that of (2.1).

For \( V \in KO^{-2}(CP_0; Q) \) in Lemma 1.9 for \( n = -1 \), we put \( \tilde{V} = (i')^*(V) \in KO^{-2}(CP; Q) \). Then from Lemma 1.9, the following is clear.

**Lemma 3.4.** \( j^*(\tilde{V}) = h^{KO}(u_1 \circ i') - w, \rho_z \tilde{V} = h^{KO} \bar{u}_1 \) and \( c(\tilde{V}) = t(1/\log(1 + X) - 1/X) \), where \( w \) is the element in Proposition 2.3.

Let \( g_i \in KO_{4i} \) be the generator and \( a(i) = 1 \) (resp.2) if \( i \) is even (resp. odd). For the Bernoulli number \( B_i \in Q \) defined by the equation \( x/(e^x - 1) = \sum_{i \geq 0}(B_i/i!)x^i \), we consider the following \( K \)-theoretical Bernoulli numbers:

\[
\tilde{B}_{i}^{KO} = (B_{2i}/(2i)!)(g_i/a(i)) \in KO_{4i} \otimes Q \quad \text{and} \quad \tilde{B}_{i}^{K} = (B_i/i!)t^i \in K_{2i} \otimes Q.
\]

(3.5)

For \( CP_0 \wedge CP_0 \), we will denote its \( K \)-cohomology groups by \( KO^*(CP_0 \wedge CP_0) = KO_*[[Y_1, Y_2]] \) and \( K^*(CP_0 \wedge CP_0) = K_*[[X_1, X_2]] \), where \( Y_i \) and \( X_i \) denote the respective Euler classes of \( \xi \). We can consider as \( KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q) \subset KO^{-4}(CP_0 \wedge CP_0; Q) \), and define an element \( h(Y_1, Y_2) \in KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q) \) by

\[
h(Y_1, Y_2) = \sum_{i,j \geq 0} \frac{9^j - 1}{9^{i+j} - 1} \tilde{B}_i^{KO} \tilde{B}_j^{KO} G_1(Y_1)^{2i-1} \otimes G_1(Y_2)^{2j-1},
\]

where \( G_1(Y) \) is the element of (1.3) for \( n = 1 \). Using this element, we define

\[
\tilde{u} = \tilde{V} \otimes w + h(Y_1, Y_2) \in KO^{-4}(CP \wedge W; Q).
\]

(3.6)

Similarly as \( h(Y_1, Y_2) \), we can define \( h_C(X_1, X_2) = \sum_{i,j \geq 0}(3^i - 1)/(3^i+1 - 1) \tilde{B}_i^{K} \tilde{B}_j^{K} (t^{-1} \log(1 + X_1))^{i-1} \otimes (t^{-1} \log(1 + X_2))^{j-1} \). Then, by using Corollary 1.9 and Proposition 2.3, we have the following lemma, in which we denote by \( 1/X \) the element \( t^{-1}U_{-1}^{K} \in K^0(CP_{-1}) \).
Lemma 3.7.

1. \( c(h(Y_1, Y_2)) = h_C(X_1, X_2) \in K^{-4}(CP \wedge CP; Q) \).
2. \( c(\tilde{V} \otimes w) = (i' \wedge i')^*(t^2(1/\log(1+X_1)-1/X_1) \otimes 1/X_2) \),
   as an element of \( K^{-4}(CP \wedge CP; Q) \).

Let \( U(X_1, X_2) = t^2(1/\log(1+X_1)-1/X_1) \otimes 1/X_2 \in K^{-4}(CP_0 \wedge CP_{-1}; Q) \).
Then we have the following corollary.

Corollary 3.8. \( c(\tilde{u}) = (i' \wedge i')^*(U(X_1, X_2) + h_C(X_1, X_2)) \in K^{-4}(CP \wedge W; Q) \).

This corollary shows that, through \((i' \wedge i')^*\), \(c(\tilde{u})\) has just the same formula with that in the case of an odd prime \( p \) in [BC]. The following is crucial.

Proposition 3.9. The element \( \tilde{u} \in KO^{-4}(CP \wedge W; Q) \) satisfies the following:

1. \((1 \wedge i)^*(\tilde{u}) = \tilde{V} \in KO^{-4}(CP \wedge S^{-2}; Q) \) and
2. \( \psi(\tilde{u}) \in \text{Im}[I : KO^{-4}(CP \wedge W) \to KO^{-4}(CP \wedge W; Q)] \), where
   \( \psi = \psi^3 - 1 : KO^0Q \to KO^0Q \) is the stable Adams operation.

Proof. (1) follows immediately from the definition of \( \tilde{u} \), because \((i)^*(w) = 1\) by
Proposition 2.3 (iii) and \((1 \wedge i)^*h(Y_1, Y_2) = h(Y_1, 0) = 0\). Also, we have (2) by a
direct calculation in \(KO\)-theory, but it is better to apply the complexification \(c\) once and consider it in the \(K\)-theory. Then, by Corollary 3.8 the calculation is just the same as that done in [Hi] or [BC] for the case of an odd prime, and
we have \(c\psi(\tilde{u}) = c((\psi^3(w) - w) \otimes \psi^3(w)) \in K^{-4}(CP \wedge W) \subset K^{-4}(W \wedge W).\)
Since \(c : KO^{-4}(CP \wedge W) \to K^{-4}(CP \wedge W)\) is a monomorphism, we have
\(\psi(\tilde{u}) = (\psi^3(w) - w) \otimes \psi^3(w) \in KO^{-4}(CP \wedge W),\) and thus we have (2).

Let \( \overline{u}_1 \) be the element in (3.3), \( \overline{j} : \overline{Ad} \to \overline{KO} \) the map induced from
\( j : Ad \to KO \) and \( \overline{\rho} : KO \to \overline{KO} \) the reduction.

Theorem 3.10. We have an element \( \overline{u}_2 \in \overline{Ad}^{-4}(CP \wedge CP; Q/Z) \) which satisfies
\( \delta_1(\overline{u}_2) = [\overline{u}_1 \circ (1 \wedge \overline{\tau})] \) and \((1 \wedge j)^*\overline{j}_*(\overline{u}_2) = \overline{\rho}\rho_2(\overline{u}).\)
Proof. Proposition 3.9 (2) means $\psi \circ \rho_Z \circ \tilde{u} \simeq 0$, and thus there is an element $u_2 \in \text{Ad}^{-4}(CP \land W; Q/Z)$ with $j_*(u_2) = \rho_Z(\tilde{u})$. Proposition 3.9 (1) and Corollary 1.9 yield $(1 \land i)^*\rho_Z(\tilde{u}) = \rho_Z(\tilde{V}) = h^{KO}(\tilde{u}) = j_*h^{Ad}(\tilde{u})$. Then these two equations give $[\tilde{u}_1 \circ h^{Ad}] = [u_2 \circ (1 \land i)]$, since $j_* : \text{Ad}^{-2}(CP; Q/Z) \to KO^{-2}(CP; Q/Z)$ is a monomorphism. Then it derives a map from the cofiber sequence $CP \land W \xrightarrow{1\land i} CP \land CP \xrightarrow{1\land \tilde{f}} CP \land S^{-1}$ to the cofiber sequence $\text{Ad}^{-4}Q/Z \xrightarrow{\delta_2} \overline{\text{Ad}}^{-4}Q/Z \xrightarrow{\delta_1} S^{-3}Q/Z$. Thus, we have $\tilde{u}_2 : CP\land CP \to \overline{\text{Ad}}^{-4}Q/Z$ with the required properties, and it completes the proof.

Since the chromatic filtration $\Sigma^{-2}N_2 \xrightarrow{\delta_2} \Sigma^{-1}N_1 \xrightarrow{\delta_1} S^0$ is equal to $\overline{\text{Ad}}^{-2}Q/Z \xrightarrow{\delta_2} S^{-1}Q/Z \xrightarrow{\delta_1} S^0$, we have the desired factorization of $\tilde{\tau} \land \tilde{\tau}$ as follows:

Corollary 3.11. $\tilde{\tau} \land \tilde{\tau} \simeq \delta_2\delta_1\tilde{u}_2$.

References


