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A REPORT ON OCNEANU'S LECTURE

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Dear Readers:

This report supplies detailed proofs of the first chapter in "Quantum Symmetry, Differential Geometry of Finite Graphs and Classification of Subfactors"— seminary notes on Ocneanu's lectures prepared by Y. Kawahigashi—, particularly a proof of biunitarity of connections.

The essense of the proof is the Frobenius reciprocity, which turns out to be useful in the topics of Markov traces and towers of algebras construction as well.

In this respect, the proof of Frobenius reciprocity via Lemma 24 and the explanations after the heading "Markov Trace" might be new.

Compared to the treatment of string algebras, the notion of paragroups seems to be rather difficult of access, partly because we cannot read the original proofs yet.

I hope the present report contribute to the improvement of such a situation.

Notation and Convention

Let N be a von Neumann algebra. By a left N-module, we understand a Hilbert space X with a normal *-representation of N which defines a left action of N on X. We often write ${}_N X$ to emphasize X being a left N-module.

Similarly a right N-module X_N is defined to be a Hilbert space X with a normal *-representation of the opposite algebra of N. Let M be another von Neumann algebra. A Hilbert space X is called an M-N bimodule if X is a left M- and right N-module at the same time and these two actions commute each other.

For a left N-module X, we denote by $End({}_N X)$ the set of bounded linear operators in X which commute with the left action of N. If we denote by N'_X (or simply N' if X can be understood) the opposite algebra of $End({}_N X)$, then X is a right N'-module in an obvious way and X becomes an N-N' bimodule. Similarly, starting from a right N-module X_N, X is an ${}_X N'$-N bimodule with ${}_X N' = End(X_N)$.

The standard space $L^2(N)$ is a special but important example of an N-N bimodule. Note that, when N is semifinite with a faithful normal semifinite weight τ, $L^2(N)$ is identified with the GNS-construction associated with τ with left and right actions are induced from left and right multiplications in N.

Given a set I, let $Mat_I(N)$ be the $I \times I$-matrix amplification of N and ${}_N L^2(N)^{\oplus I}$ (resp. $\oplus_I L^2(N)_N$) be the direct sum of left N-modules ${}_N L^2(N)$ indexed by I as row vector (resp. column vector). The algebra $Mat_I(N)$ acts on $L^2(N)^{\oplus I}$ (resp. $\oplus_I L^2(N)$) from the right (resp. left) and $Mat_I(N)$ is identified with $N'_{L^2(N)^{\oplus I}}$ (resp. ${}_{\oplus_I L^2(N)} N'$).

Let X be a left N-module and e a projection in N'_X. Since Xe is N-invariant, it defines a left N-module ${}_N Xe$. As a special case, consider a projection e in $Mat_I(N)$. We have a left N-module ${}_N L^2(N)^{\oplus I} e$ and the commutant of the left action is naturally identified with $eMat_I(N)e$.

Dimension

(cf. 'Theory of Operator Algebras' by M.Takesaki)

Let N be a semifinite von Neumann algebra with a faithful normal trace τ. Let ${}_N X$ be a left N-module. Since to give a left N-module structure is nothing but to give a homomorphism from $N \subset \mathcal{B}(L^2(N))$ to $\mathcal{B}(X)$, by the structure theorem of normal homomorphisms, we can find a set I and a projection e in $End_N(L^2(N)^{\oplus} = L^2(N) \oplus \cdots$, direct sum of countable copies) such that

$${}_N X \cong {}_N(eL^2(N)^{\oplus I}).$$

If we set $H = \ell^2(I)$, $L^2(N)^{\oplus I} \cong L^2(N) \otimes H$. With this identification, the left action of N on $L^2(N)^{\oplus I}$ is identified with $N \otimes 1_H$. So we may assume that

$$e \in (N \otimes 1_H)' = N' \otimes \mathcal{B}(H) = N^{op} \otimes \mathcal{B}(H).$$

In other words, we can find a projection $p \in Mat_I(N) = N \otimes \mathcal{B}(H)$, $p = \{p_{ij}\}$, $p_{ij} \in N$

such that

$$\begin{aligned} e(\oplus_i x_i) &= \oplus_i(\sum_j x_j p_{ji}) \\ &= (x_1, x_2, \cdots) \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &= (\oplus_i x_i)p, \end{aligned}$$

for $\oplus_i x_i \in L^2(N)^{\oplus I}$.

Now we set

$$\dim_N X = (\tau \otimes tr)(p),$$

where tr denotes the usual trace in $\mathcal{B}(H) = Mat_I(\mathbb{C})$. The right hand side in this definition is independent of the choice of p. In fact, if $q \in Mat_I(N)$ is another projection satisfying ${}_NX \cong {}_N(L^2(N)^{\oplus I}p)$, then we can find a partial isometry $v : L^2(N)^{\oplus I}p \to L^2(N)^{\oplus I}q$ which commutes with the left action of N. Thus $\exists\, u \in Mat_I(N)$, $u^*u = q$, $uu^* = p$,

$$v\xi = \xi u, \quad \xi \in L^2(N)^{\oplus I}.$$

In other words, $p \sim q$ in $Mat_I(N)$ which shows that

$$(\tau \otimes tr)(p) = (\tau \otimes tr)(q).$$

Extending this construction, we can define a semi-finite trace $\tau'_{{}_NX}$ (or simply τ') on $N'_X \cong pMat_I(N)p$, which is called a **canonical trace** and plays significant roles in the later parts.

Lemma 1.
(i) $(\tau')' = \tau$ *if the representation* $N \to End(X)$ *is faithful.*
(ii) *For* $T \in Hom({}_NX, {}_NY)$,

$$\tau'(T^*T) = \tau'(TT^*).$$

(iii) *For a projection* $e \in N'_X$,

$$\dim_N^\tau(Xe) = \tau'(e).$$

(iv)

$$\dim_{N'}^{\tau'} X = \tau(1).$$

$\because$) (ii) follows from the definition of τ'. To see (iii), consider the trace $\tau'_{X\oplus Y}$ and its restrictions. (iv) is a consequence of (i) and (iii). To see (i), by the central decomposition, we may assume that N is a factor. Let ${}_NX \cong {}_NL^2(N)^{\oplus I}p$.

If $\tau(1) \leq \tau'(p)$, subtracting projections which are equivalent to

$$\begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

we may assume that

$$p = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & q \end{pmatrix},$$

where $q \in N$. Then

$$L^2(N') = \begin{pmatrix} L^2(N) & \dots & L^2(N) & L^2(N)q \\ \vdots & \ddots & \vdots & \vdots \\ L^2(N) & \dots & L^2(N) & L^2(N)q \\ qL^2(N) & \dots & qL^2(N) & qL^2(N)q \end{pmatrix}$$

and ${}_NX$ is equivalent to

$$\begin{pmatrix} L^2(N) & \dots & L^2(N) & L^2(N)q \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Since the action of N on X is expressed by

$$N \ni a \mapsto \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

in this realization, we have

$$\tau''(a) = (\tau \otimes tr) \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = \tau(a).$$

If $\tau'(p) < \tau(1)$, N' and hence N are finite. So, to check that $\tau'' = \tau$, it suffices to show that $\tau''(e) = \tau(e)$ for a suitable projection $e \neq 0 \in N$. This time, ${}_NX \cong {}_NL^2(N)e$ with $e \in N$. Since $eL^2(N)e_{N'}$ is equivalent to $L^2(eNe)_{N'}$,

$$\tau''(e) = \tau'(1_{N'}) = \tau(e).$$

□

Proposition 2.
(i) $\dim_N H_1 = \dim_N H_2$ *if* ${}_N H_1 \cong {}_N H_2$.
(ii) $\dim_N(\oplus_i H_i) = \sum_i \dim_N H_i$.
(iii) *If* $(N,\tau) = \int^{\oplus} d\lambda (N_\lambda, \tau_\lambda)$ *and* $\int^{\oplus} d\lambda H_\lambda$ *is the corresponding decomposition of* H, *then*

$$\dim_N^{\tau} H = \int d\lambda \dim_{N_\lambda}^{\tau_\lambda} H_\lambda.$$

(iv) *If* N *is a (semi-finite) factor,* $\dim_N H < +\infty \iff End_N(H)$ *is a finite von Neumann algebra, and the converse of (i) holds:* $\dim_N H_1 = \dim_N H_2 \Rightarrow {}_N H_1 \cong {}_N H_2$.

∵) (i), (ii) are clear from the defintion and (iv) is a refrase of a classical result. For the proof of (iii), including the measurability of $\lambda \mapsto \dim_{N_\lambda}^{\tau_\lambda} H_\lambda$, see the section of coupling constant (§ V.3 in 'Theory of Operator Algebras'). □

We present here two examples, abstract one and concrete one.

Example 3. *Let* N *be a finite factor with* τ *the normalized trace. By multiplication,* $L^2(N)$ *is an* N*-bimodule and* ${}_N L^2(N)$ *is a standard representation. So we can identify* $End({}_N L^2(N))$ *with the right multiplication algebra of* N. *Thus , for a projection* $e \in N$, $L^2(N)e$ *is a left* N*-submodule of* $L^2(N)$ *and we have*

$$\dim_N L^2(N)e = \tau(e).$$

Example 4. *Consider a *-algebra* $\mathcal{A}$ *generated by two unitaries* u, v *satisfying the relation* $uv = e^{2\pi i\theta} vu$. *On* $\mathcal{A}$, *we can define a trace* τ *by*

$$\tau(u^m v^n) = \begin{cases} 1 & \text{if } m = n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

There are several realizations of such an algebra.
1° *Group algebra.*
Consider a central extension G *of an additive group* $\mathbb{Z}^2$ *by the torus* $\mathbb{T}$;

$$1 \to \mathbb{T} \to G \to \mathbb{Z}^2 \to 0,$$

where the group structure of $G = \mathbb{Z}^2 \times \mathbb{T}$ *is defined by*

$$(m,n,z)(m',n',z') = (m+m', n+n', zz'e^{-2\pi i\theta m'n}).$$

Through the identification

$$u \iff (1,0,1)$$
$$v \iff (0,1,1)$$
$$z \iff (0,0,z),$$

the algebra $\mathcal{A}$ is realized (or represented) by a unitary representation π of G such that $\pi(0,0,z) = z1$. As an example of such a π, we take an induced representation $\mathrm{ind}_{\mathbb{T}}^{G}\chi$, $\chi(0,0,z) = z$. Its representation space H is given by

$$H = \{f : G \to \mathbb{C}; f(gz) = z^{-1}f(g), g \in G, z \in \mathbb{T}, \sum_{g \in G/\mathbb{T}} |f(g)|^2 < +\infty\}.$$

Taking the restriction of f to $\mathbb{Z}^2 \subset G$,

$$H \cong \ell^2(\mathbb{Z}^2).$$

We define an anti-unitary involution J on H by

$$(Jf)(g) = \overline{f(g^{-1})},\ g \in G,$$

and a unit vector δ by

$$\delta(m,n,z) = \begin{cases} z^{-1} & \text{if } m = n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then (H, δ, J, π) defines a standard representation of G with δ a tracial vector. In particular, the von Neumann algebra generated by $\pi(G)$ is finite. A bit more algebraic calculation shows that $\pi(G)''$ is a factor if θ is irrational.

2° *Crossed product.*

In the above representation H, consider the Fourier transform of $f \in \ell^2(\mathbb{Z}^2)$ with respect to the second variable:

$$\ell^2(\mathbb{Z}^2) \ni f \mapsto \hat{f} \in \ell^2(\mathbb{Z}) \otimes L^2(\mathbb{T})$$

$$\hat{f}(m,z) = \sum_n z^n f(m,n).$$

In this new realization of H,

$$(u\hat{f})(m,z) = \hat{f}(m-1,z)$$
$$(v\hat{f})(m,z) = e^{-2\pi i\theta m} z\hat{f}(m,z)$$
$$(J\hat{f})(m,z) = \overline{\hat{f}(-m, ze^{-2\pi i\theta m})}$$
$$\hat{\delta}(m,z) = \delta_{m,0}.$$

If we interpret v as a multiplication operator by the function $z \mapsto z$ in $L^\infty(\mathbb{T})$, then v generates $L^\infty(\mathbb{T})$. On $L^\infty(\mathbb{T})$, we define an automorphism α by

$$\alpha(z) = e^{2\pi i\theta}z.$$

Then the above realization is nothing but the regular representation of the crossed product algebra $L^\infty(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$.

Since

$$(JuJ\hat{f})(m,z) = \hat{f}(m+1, ze^{2\pi i\theta})$$
$$(JvJ\hat{f})(m,z) = \bar{z}\hat{f}(m,z),$$

(in this crossed product representation,) one sees that $\pi(G)'$ *contains* $1_{\ell^2(\mathbb{Z})} \otimes L^\infty(\mathbb{T})$ *and* $\forall$ *Borel subset* $B \subset \mathbb{T}$ *the characteristic function* $1_B \in L^\infty$ *defines a projection* $e \in \pi(G)'$ *by*

$$e = 1_{\ell^2(\mathbb{Z})} \otimes 1_B.$$

Since the natural trace τ' *on* $1_{\ell^2(\mathbb{Z})} \otimes L^\infty(\mathbb{T})$ *is given by*

$$\tau'(1_{\ell^2(\mathbb{Z})} \otimes f) = \int_{\mathbb{T}} dz\, f(z),$$

we finally have

$$\dim_{M_e} eH = |B|.$$

Definition 5. *Let* M *be a finite factor and* $N \subset M$ *be a finite subfactor. The* **index** *of* $N \subset M$ *is defined by* $[M : N] = \dim_N L^2(M)$ *where the dimension is measured with respect to the normalized trace of* N.

Proposition 6. $[M : N] \geq 1$ *and the equality holds if and only if* $M = N$.

$\because$) Since $L^2(N) \subset L^2(M)$ and $L^2(N) = L^2(M)$ if and only if $N = M$ (cf. the conditional expectation $E : M \to N$), this follows from the following calculation:

$$\begin{aligned}\dim_N L^2(M) &= \dim_N(L^2(N) \oplus (L^2(M) \ominus L^2(N))) \\ &= \dim_N L^2(N) + \dim_N(L^2(M) \ominus L^2(N)) \\ &= 1 + \dim_N(L^2(M) \ominus L^2(N)).\end{aligned}$$

□

Operator valued inner product

Operator valued inner product is an important notion when one considers relations between two algebras (see [Paschke], [Rieffel]).

Let N be a semifinite von Neumann algebra with a faithful trace τ. Given a left N-module X, we define the set of (N,τ)-**bounded** (or simply N-bounded if τ is fixed) elements by

$$X_0 \equiv \{x \in X; \exists\,\lambda > 0, \forall\,a \in N,\ \|ax\| \le \lambda\tau(aa^*)^{1/2}\}.$$

Lemma 7.
(i) *X_0 is an N-invariant dense linear subspace of X.*
(ii) *If ${}_NY$ is another N-module and $T : {}_NX \to {}_NY$ is an intertwiner, $T(X_0) \subset Y_0$. In particular, X_0 is $End({}_NX)$-invariant.*

∵) (i) N-invariance follows from the trace property of τ. To see the density, we may assume that $N \curvearrowright X$ is faithful(consider $X \oplus L^2(N)$). Then we can take a family $\{x_i\}_{i\in I} \subset X$ such that

$$\tau(a) = \sum_i (ax_i|x_i),\ \forall\,a \in N^+.$$

These x_i's are in X_0 since $\|ax_i\|^2 \le \tau(aa^*)$. Let e be the projection onto the closure of X_0 in X. Since $e \in N$ by the N'-invariance of X_0,

$$(1-e)x_i = 0,\ \forall\,i \Rightarrow \tau(1-e) = 0.$$

Since τ is faithful, $e = 1$.

(ii) This property is an immediate consequence of the definition. □

Example 8. *If $X = L^2(N)$, $X_0 = N \cap L^2(N)$.*

For $x \in X_0$, consider a positive linear functional $N \ni a \mapsto (ax|x)$. Since

$$(ax|x) = (a^{1/2}x|a^{1/2}x) \le \|a^{1/2}x\|^2 \le \lambda^2\tau(a),$$

the Radon-Nikodym theorem assures that there is a positive element $h \in N$ such that

$$\tau(ah) = (ax|x),\ \forall\,a \in N^+.$$

From this relation, h is in the definition ideal of τ. Moreover, given $x, y \in X_0$, the polar identity shows that we can find a (unique) $h \in N$ which is in the definition ideal of τ and satisfies

$$\tau(ah) = (ax|y), \quad \forall\,a \in N.$$

In the following, this h is denoted by

$${}_N[x,y].$$

Lemma 9.
(i) ${}_N[x,y]$ *is an N-valued sesqui-linear form on X_0.*
(ii) ${}_N[x,y]^* = {}_N[y,x]$.
(iii) ${}_N[ax,y] = a_N[x,y], \quad a \in N, \ x,y \in X_0$.

Remark. (i) The operator-valued inner product ${}_N[\ ,\ ]$ depends on the choice of a trace of N.

(ii) For a ${}_N[x,y] \in N$, let $[x,y]_2$ be its image in $L^2(N)$. Then for $x \in X_0$, $X_0 \ni y \mapsto [x,y]_2$ is continuously extended to a bounded linear map of X into $L^2(N)$.

Example 10. *Let M be a finite von Neumann algebra with a faithful trace τ, N be a von Neumann subalgebra of M, and $E : M \to N$ be a conditional expectation uniquely determined by τ. For the left N-module ${}_N L^2(M)$, we have*

$${}_N[[a]_2, [b]_2] = E(ab^*),$$

where a, $b \in M$ and $[a]_2$, $[b]_2$ are their images in $L^2(M)$.

Basis

Definition 11. *A family $\lambda = \{\lambda_i\}_{i\in I}$ in X_0 is called a* **basis** *if*
(i) $\sum_i N\lambda_i$ *(algebraic sum) is dense in X,*
(ii) $\forall x \in X, \forall i \in I$

$$(x|\lambda_i) = \sum_j (x|{}_N[\lambda_i, \lambda_j]\lambda_j) \text{ (absolutely convergent).}$$

Let $\Lambda = \{[\lambda_i, \lambda_j]\}_{i,j\in J} \in Mat_I(N)$.

Proposition 12.
(i) *Λ is a projection in $Mat_I(N)$ (in particular, it is bounded).*
(ii) *$\forall i \in I$, $\oplus_j[\lambda_i, \lambda_j]$ is in $L^2(N)^{\oplus I}$ and the map*

$$\sum_{i\in I} N\lambda_i \ni \sum_i a_i\lambda_i \mapsto \oplus_{j\in I}\left(\sum_i a_{iN}[\lambda_i,\lambda_j]\right) \in L^2(N)^{\oplus I}$$

is extended to an isometry from X to $L^2(N)^{\oplus I}\Lambda$.
(iii) $\forall x \in X_0, \forall y \in X$,

$$(x|y) = \sum_i([x,\lambda_i]\lambda_i|y) \text{ (absolutely convergent).}$$

∵) We first note that $\oplus_j[\lambda_i,\lambda_j] \in L^2(N)^{\oplus I}$:

$$(1) \qquad \sum_j \tau([\lambda_i,\lambda_j][\lambda_j,\lambda_i]) = \sum_j([\lambda_i,\lambda_j]\lambda_j|\lambda_i) = (\lambda_i|\lambda_i) < +\infty.$$

So, if we set $D = \oplus_{i\in I}L^2(N)$ (algebraic sum), the right multiplication of the operator matrix Λ defines a linear operator h with domain D. Then the reproducing kernel property of λ yields

$$(h\xi|h\xi) = (\xi|h\xi), \quad \forall\xi \in D,$$

which means that h is a projection (see the appendix below), proving (i).

The formula (1) also shows that the map in (ii) is an isometry. So the only thing we need to prove is that its image is equal to $L^2(N)^{\oplus I}\Lambda$. Since $N \cap L^2(N)$ is weakly dense in N, the image is the closure of $(\oplus_i N\cap L^2(N))\,\Lambda$, i.e., $\Lambda^2(N)^{\oplus I}\Lambda$.

(iii) Since $(L^2(N)^{\oplus I}\Lambda)_0 = (L^2(N)^{\oplus I})_0\,\Lambda$, we first examine $(L^2(N)^{\oplus I}\Lambda)_0$. This set, as can be easily seen, is identified with

$$\{\oplus_i a_i \in L^2(N)^{\oplus I}; a_i \in N\cap L^2(N) \text{ and } \sum_i a_i a_i^* \text{ is bounded}\}$$

and the N-valued inner product is given by

$$(2) \qquad [\oplus_i a_i, \oplus_i b_i] = \sum_i a_i b_i^*, \quad \oplus_i a_i, \oplus_i b_i \in (L^2(N)^{\oplus I}\Lambda)_0.$$

Here the convergence in the right hand side is with respect to weak operator topology. On the other hand, $[\oplus_i a_i, \oplus_i b_i]$ and $a_i b_i^*$ are in the definition ideal of τ and hence in $L^2(N)$. So we can talk about the convergence in $L^2(N)$ and, in fact, the relation (2) holds as elements in $L^2(N)$. To see this, we may assume $a_i = b_i$ by the polar identity. Let $F \subset I$ be a finite subset and consider

$$\begin{aligned}\tau\left((\sum_{i\notin F} a_i a_i^*)^2\right) &\le \tau\left((\sum_{i\notin F} a_i a_i^*)^{1/2}\|\sum_{i\notin F} a_i a_i^*\|(\sum_{i\notin F} a_i a_i^*)^{1/2}\right)\\ &= \|\sum_{i\notin F} a_i a_i^*\|\tau(\sum_{i\notin F} a_i a_i^*)\\ &\le \|\sum_{i\notin F} a_i a_i^*\|\sum_{i\notin F}\tau(a_i a_i^*).\end{aligned}$$

Since $\oplus_i a_i \in L^2(N)^{\oplus I}$, and $\sum_{i\notin F} a_i a_i^*$ is bounded, this converges to 0 as $F \nearrow I$.

Now let $x_0 = \oplus_i b_i = (\oplus_i a_i)\Lambda \in (L^2(N)^{\oplus I}\Lambda)_0\Lambda$, with $\oplus_i a_i \in (L^2(N)^{\oplus I}\Lambda)_0$. Since $\lambda_i \in X$ is identified with $\oplus_j[\lambda_i,\lambda_j]$ in $L^2(N)^{\oplus I}$, we have

$$[x_0,\lambda_i] = \sum_j b_j[\lambda_j,\lambda_i].$$

If we consider this relation in $L^2(N)$, the right hand side is equal to the i-th component of

$$(\oplus_i b_i)\Lambda = (\oplus_i a_i)\Lambda^2 = (\oplus_i a_i)\Lambda.$$

From this, we have

$$[x_0, \lambda_i] = \sum_j b_j [\lambda_j, \lambda_i] \quad \text{in } L^2(N).$$

Since L^2-sum and operator sum gives the same results for $\oplus_j a_j$ and $\oplus_j [\lambda_i, \lambda_j] \in \left(L^2(N)^{\oplus I}\right)_0$ (see the above argument), this relation holds as weak operator convergence.

Now we have

$$\begin{aligned}
[x_0, \lambda_i]\lambda_i &= \sum_j a_j [\lambda_j, \lambda_i](\oplus_k[\lambda_i, \lambda_k]) \\
&= \oplus_k \left(\sum_j \overbrace{a_j [\lambda_j, \lambda_i]}^{N} \overbrace{[\lambda_i, \lambda_k]}^{L^2} \right) \\
&= \oplus_k \left(\sum_j \overbrace{a_j}^{L^2} \overbrace{[\lambda_j, \lambda_i][\lambda_i, \lambda_k]}^{N} \right) \\
&= \oplus_k (b_i [\lambda_i, \lambda_k]).
\end{aligned}$$

From this expression, we see that $\sum_i [x_0, \lambda_i]\lambda_i$ is weakly convergent in $L^2(N)^{\oplus I}$ and

$$\begin{aligned}
\sum_i [x_0, \lambda_i]\lambda_i &= \oplus_k (\sum_i b_i [\lambda_i, \lambda_k]) \\
&= (\oplus_i b_i)\Lambda \\
&= \oplus_i b_i \quad (\text{since } \oplus_i b_i \in L^2(N)^{\oplus I}\Lambda) \\
&= x_0.
\end{aligned}$$

□

Corollary 13.
(i) $\forall a' \in N'_X$,

$$\tau'_N(a') = \sum_i (\lambda_i a' | \lambda_i).$$

In particular, $\dim_N X = \sum_{i \in I} (\lambda_i | \lambda_i)$.

(ii) $\forall x, y \in X_0$,

$$\sum_i {}_N[x, \lambda_i]_N[\lambda_i, y] = {}_N[x, y] \quad \textit{(weakly convergent).}$$

∵) (i) Since $a' \in N'_X$, a' has a matrix expression

$$A = (a_{ij}) \in \Lambda Mat_I(N)\Lambda \subset Mat_I(N).$$

We show the formula (i) for $A = \Lambda B \Lambda$, where $B \in Mat_I(N)$ is of finite size and its components are in the definition ideal of τ. Once this is proved, the formula of the general case follows by continuity (both sides in the above relation define normal weights). Thus we need to consider $A = \Lambda B \Lambda$. Using $B = (b_{kl})$, we have

$$a_{ij} = \sum_{kl} [\lambda_i, \lambda_k] b_{kl} [\lambda_l, \lambda_j]$$

(note that the summation $\sum_{kl}$ is taken over finite number of indices and b_{kl}'s are in the trace class of N), and then

$$\begin{aligned} \tau'_N(a') &= \sum_i \tau(a_{ii}) = \sum_i \sum_{kl} \tau([\lambda_i, \lambda_k] b_{kl} [\lambda_k, \lambda_i]) \\ &= \sum_{k,l} \tau(b_{kl} \sum_i [\lambda_l, \lambda_i][\lambda_i, \lambda_k]) \\ &= \sum_{k,l} \tau(b_{kl} [\lambda_l, \lambda_k]) \\ &= \sum_{k,l} (b_{kl} \lambda_l | \lambda_k). \end{aligned}$$

On the other hand, using the correspondence

$$\lambda_i \iff \oplus_j [\lambda_i, \lambda_j] \in \oplus_j L^2(N, \tau),$$

we have

$$\begin{aligned} \lambda_i a' &\iff (\oplus_j [\lambda_i, \lambda_j]) \Lambda B \Lambda \\ &= (\oplus_j [\lambda_i, \lambda_j]) B \Lambda \\ &= \oplus_j \sum_{k,l} [\lambda_i, \lambda_k] b_{kl} [\lambda_l, \lambda_j] \\ &= \oplus_j [\sum_{k,l} [\lambda_i, \lambda_k] b_{kl} \lambda_l, \lambda_j] \\ &\iff \sum_{k,l} [\lambda_i, \lambda_k] b_{kl} \lambda_l. \end{aligned}$$

Thus

$$\begin{aligned} \sum_i (\lambda_i a' | \lambda_i) &= \sum_i \sum_{k,l} ([\lambda_i, \lambda_k] b_{kl} \lambda_l | \lambda_i) \\ &= \sum_{k,l} (b_{kl} \lambda_l | \lambda_k) \\ &= \tau'_N(a'). \end{aligned}$$

(ii) By the polar identity, we may assume $x = y$. For $a \in N_+$,

$$\begin{aligned}\tau(a^{1/2}\sum_i {}_N[x,\lambda_i]_N[\lambda_i,x]a^{1/2}) &= \sum_i \tau(a^{1/2}[x,\lambda_i][\lambda_i,x]a^{1/2}) \\ &= \sum_i \tau(a[x,\lambda_i][\lambda_i,x]) \\ &= \sum_i (a[x,\lambda_i]\lambda_i|x) \\ &= (ax|x) \\ &= \tau(a[x,x]).\end{aligned}$$

Since this holds for any $a \in N_+$, we get the desired formula. □

Remark. The proof of (i) for the special case $a' = 1_X$ is much easier. We just calculate $\dim_N X = (\tau \otimes tr)(\Lambda) = \sum_{i\in I} \tau([\lambda_i,\lambda_i]) = \sum_{i\in I}(\lambda_i|\lambda_i)$.

Appendix. *Let h be a hermitian operator with dense domain D such that*

$$(h\xi|h\xi) = (\xi|h\xi), \quad \forall \xi \in D.$$

Then h is a projection.

∵) We only need to check that h is bounded. This follows from

$$\begin{aligned}||h\xi|| &= (h\xi|h\xi)^{1/2} = (\xi|h\xi)^{1/2} \\ &\le ||\xi||^{1/2}||h\xi||^{1/2} \\ &= ||\xi||^{1/2}(h\xi|h\xi)^{1/4} = ||\xi||^{1/2}(\xi|h\xi)^{1/4} \\ &\le ||\xi||^{1/2}||\xi||^{1/4}||h\xi||^{1/4} \\ &\dots \\ &\le ||\xi||^{1/2+1/4+\cdots+1/2^n}||h\xi||^{1/2^n} \\ &\to ||\xi||.\end{aligned}$$

□

Change of Bases

Let $\theta = \{\theta_j\}_{j\in J}$ be another basis. Let V be an operator matrix given by $\{{}_N[\theta_j,\lambda_i]\}_{j\in J, i\in I}$. From Corollary (ii), V satisfies

$$V^*V = \Lambda, \quad VV^* = \Theta.$$

Since

$$[x, \lambda_i] = \sum_j [x, \theta_j][\theta_j, \lambda_i], \quad \forall i,\ \forall x \in X_0$$

both in operator sense and L^2 -sense, we have

$$\iota_\lambda(x) = \iota_\theta(x)V,$$

where ι_λ denotes the natural imbedding $X \to L^2(N)^{\oplus I}\Lambda$. Thus we have obtained the following commutative diagram of N-module isomorphisms:

$$\begin{array}{ccc} X & \xrightarrow{\iota_\lambda} & L^2(N)^{\oplus I}\Lambda \\ \Vert & & \uparrow V \\ X & \xrightarrow{\iota_\theta} & L^2(N)^{\oplus J}\Theta \end{array}$$

In this sense, V gives a matrix system for **change of bases**.

Existence of Basis

A basis $\{\lambda_i\}_{i\in I}$ is called **orthogonal** if $[\lambda_i, \lambda_j] = 0$ for $i \neq j$. Note that $[\lambda_i, \lambda_i]$, $i \in I$ is a projection in this case (because $\Lambda = \{[\lambda_i, \lambda_j]\}$ is a projection). Given an N-module X, we can construct a basis for X. To this end, we first present

Lemma 14.
(i) $\forall x \in X_0$,

$$[x] \equiv \inf\{p \in proj(N); px = x\}$$

is the support projection of ${}_N[x, x]$.
(ii) *If* $\tau([x]) < +\infty$,

$$\lambda = \lim_{\epsilon \searrow 0} ({}_N[x, x] + \epsilon)^{-1/2} x$$

exists in X, $\lambda \in X_0$, *and*

$${}_N[\lambda, \lambda] = [x].$$

(iii) *If* $\lambda \in X_0$ *and* ${}_N[\lambda, \lambda]$ *is a projection, then* $\forall y \in X_0 \cap \overline{N\lambda}$,

$${}_N[y, \lambda] = y.$$

$\because$) (i) If $px = x$, $p_N[x, x]p = {}_N[x, x]$. Conversely, if $p_N[x, x]p = {}_N[x, x]$, ${}_N[px - x, px - x] = 0$ and hence $(px - x|px - x) = \tau({}_N[px - x, px - x]) = 0$.
(ii)

$$\begin{aligned} \|([x, x] + \epsilon)^{-1/2}x - ([x, x] + \eta)^{-1/2}x\|^2 &= (([x, x] + \epsilon)^{-1}x|x) + (([x, x] + \eta)^{-1}x|x) \\ &\quad - 2(([x, x] + \epsilon)^{-1/2}x|([x, x] + \eta)^{-1/2}x) \\ &= \tau(([x, x] + \epsilon)^{-1}[x, x]) + \tau(([x, x] + \eta)^{-1}[x, x]) \\ &\quad - 2\tau(([x, x] + \epsilon)^{-1/2}([x, x] + \eta)^{-1/2}[x, x]). \end{aligned}$$

Since

$$([x,x]+\epsilon)^{-1/2}([x,x]+\eta)^{-1/2}[x,x] \nearrow [x],$$

$$\lim_{\epsilon,\eta\searrow 0}\|([x,x]+\epsilon)^{-1/2}x-([x,x]+\eta)^{-1/2}x\|^2=\tau([x])+\tau([x])-2\tau([x])=0.$$

Thus $\lambda\in X$ exists. $\lambda\in X_0$ follows from

$$\begin{aligned}\|a\lambda\| &= \lim_{\epsilon\searrow 0}\|a([x,x]+\epsilon)^{-1/2}x\|^2\\ &= \lim \tau(a([x,x]+\epsilon)^{-1}[x,x]a^*)\\ &= \tau(a[x]a^*)\\ &\le \tau(aa^*), \quad a\in N.\end{aligned}$$

The above calculation also shows that $[\lambda,\lambda]=[x]$, because $\|a\lambda\|^2=\tau(a[\lambda,\lambda]a^*)$.

(iii) Let $y=\lim_i a_i\lambda\in X_0\cap\overline{N\lambda}$. Then $\forall x\in X_0$,

$$\begin{aligned}([y,\lambda]\lambda|x) &= \tau([y,\lambda][\lambda,x])\\ &= \tau([\lambda,x][y,\lambda])\\ &= ([\lambda,x]y|\lambda)\\ &= \lim_i([\lambda,x]a_i\lambda|\lambda)\\ &= \lim_i \tau([\lambda,x]a_i[\lambda,\lambda])\\ &= \lim_i \tau(a_i[\lambda,\lambda][\lambda,x])\\ &= \lim_i(a_i[\lambda,\lambda]\lambda|x)\\ &= \lim_i(a_i\lambda|x) \quad ([\lambda,\lambda]=[\lambda])\\ &= (y|x).\end{aligned}$$

□

Returning to the construction of basis, we first decompose an N-module X into cyclic pieces; we can find a family $\{x_i\}\subset X_0$ such that

$$X=\oplus_i\overline{Nx_i} \quad \text{(orthogonal direct sum)}.$$

Cutting down x_i by a projection in N with finite trace, we may assume that $\tau([x_i])<+\infty$. Let

$$\lambda_i=\lim_{\epsilon\searrow 0}([x_i,x_i]+\epsilon)^{-1/2}x_i\in\overline{Nx_i}.$$

Since X_0 is invariant under N', $\forall x_0\in X_0$ is decomposed according to the above decomposition and the component-wise application of Lemma (iii) insures that

$$x_0=\sum_i[x_0,\lambda_i]\lambda_i.$$

The other properites for basis can be also checked by Proposition 12 (ii). Thus $\{\lambda_i\}_{i\in I}$ is a desired basis.

Proposition 15. *Let N be a semi-finite factor, X be a left N-module, and suppose that $\dim_N X < +\infty$. Then $\exists$ an orthogonal basis consisting of finite elements.*

$\because$) Suppose that N is properly infinite. If we write as

$$H \cong L^2(N)^{\oplus I} q, \qquad q = \{q_{ij}\} \in Mat_I(N),$$

then

$$\dim_N H = (\tau \otimes tr)(q) < +\infty$$

implies that there are at most countably many $i \in I$ such that

$$\tau(q_{ii}) \neq 0.$$

So we may assume that I is countable. This case, since N is properly infinite,

$${}_N L^2(N)^{\oplus I} \cong {}_N L^2(N)$$

and we can find a projection $q \in N$ such that

$${}_N H \cong {}_N L^2(N) q.$$

Then $\tau(q) = \dim_N H < +\infty$ and, if we denote by x an element in $L^2(N)$ which corresponds to q, then $x \in H_0$. Since

$$\tau(a[x,x]) = (ax|x) = \tau(qaq) = \tau(aq),$$

$[x,x] = q$ and therefore

$$[ax,x]x = aqx = ax.$$

Thus $\{x\}$ is a basis.

Suppose that N is finite and τ is normalized. Express as

$${}_N H \cong {}_N L^2(N)^{\oplus I} q.$$

Since $\tau \otimes tr(q) < +\infty$, $\exists$ projections $q_1, \dots, q_{n+1} \in Mat_I(N)$ such that

$$q = q_1 + \cdots + q_{n+1},$$

$$\tau \otimes (q_1) = \cdots = \tau \otimes (q_n) = 1, \ \tau \otimes (q_{n+1}) < 1.$$

Since q_i $(1 \leq i \leq n)$ are equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and q_{n+1} is equivalent to a sub-projection of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\exists$ $p \in N$

$${}_N H \cong \overbrace{{}_N L^2(N) \oplus \cdots \oplus {}_N L^2(N)}^{n \text{ times}} \oplus {}_N L^2(N) p.$$

Now if we set

$$x_i = 0 \oplus \cdots \oplus \overset{\text{i}}{1} \oplus 0 \oplus \cdots \in {}_N H \quad (1 \le i \le n)$$
$$x_{n+1} = 0 \oplus \cdots \oplus 0 \oplus p \in {}_N H,$$

$\{x_i\}_{1 \le i \le n+1}$ is a basis. □

Tensor Product

Let B be a von Neumann algebra with a faithful trace τ. Take a right and a left B-mudules X_B, ${}_BY$ and let X_0, Y_0 be the sets of B-bounded elements. As in the case of left modules, we define a B-valued inner product $[\ ,\ ]_B$ in X_0 by

$$\tau([x_1, x_2]_B b) = (x_2 b | x_1).$$

This time, $[\ ,\ ]_B$ is linear in the second variable.

The following lemma is essentially known in [Paschke] and [Rieffel].

Lemma 16. *Let $x_1, \ldots, x_n \in X_0$. Then the operator matrix $\{[x_i, x_j]_B\}_{1 \le i,j \le n} \in M_n(B)$ is positive and*

$$\{[ax_i, ax_j]_B\}_{1 \le i,j \le n} \le \|a\|^2 \{[x_i, x_j]_B\}_{1 \le i,j \le n}, \quad \forall a \in End(X_B).$$

∵) For a sequence $b_1, \ldots, b_n \in B \cap L^2(B)$,

$$\begin{aligned} \{\overleftarrow{b_i}\}\{[ax_i, ax_j]_B\}\{\overrightarrow{b_j}\} &= \sum_{i,j} \tau(b_i^*[ax_i, ax_j]_B b_j) \\ &= \sum_{i,j} (ax_i b_i | ax_j b_j) \\ &= (a \sum_i x_i b_i | a \sum_j x_j b_j) \end{aligned}$$
$$\le \|a\| \sum_{i,j} (x_i b_i | x_j b_j).$$

□

Now define an inner product $(\ |\ )$ in the algebraic tensor product $X_0 \otimes Y_0$ by

$$(\sum_i x_i \otimes y_i | \sum_j x'_j \otimes y'_j) = \sum_{i,j} \tau([x'_j, x_i]_B\, {}_B[y_i, y'_j]).$$

For $x'_j = x_j$, $y'_j = y_j$, the right hand side is the trace of the product of two positive operator matrices

$$\{[x_i, x_j]_B\}, \quad \{{}_B[y_i, y_j]\}$$

and hence non-negative. In this way, $X_0 \otimes Y_0$ becomes a (degenerate) pre-Hilbert space, and we denote by $X \otimes_B Y$ its completion. Note that, from the property of operator valued inner product, the above inner product is well-defined on the algebraic tensor product as B-modules. Also, the above lemma shows that for $a \in End\, X_B$,

$$x \otimes_B y \mapsto ax \otimes_B y$$

defines a bounded linear operator in $X \otimes_B Y$ and this correspondence gives a normal representation of $End\, X_B$ in $X \otimes_B Y$. A similar result holds for $End\,_B Y$. In particular, if X is an A-B module and Y is a B-C module, then $X \otimes_B Y$ is an A-C module in a natural way.

Remark. (i) If one takes a basis $\{\lambda_i\}_{i \in I} \subset X_0$ and $\{\mu_j\}_{j \in J} \subset Y_0$, then $X_0 \otimes_B Y_0 = \sum_{i,j} \lambda_i \otimes_B \mu_j$ gives an orthogonal decomposition and, from this, one sees that the inner product is non-degenerate on $X_0 \otimes_B Y_0$.

(ii) The inner product in $X \otimes_B Y$ depends on the choice of a trace τ.

Proposition 17.
(i) $(X_1 \oplus X_2) \otimes_B Y = (X_1 \otimes_B Y) \oplus (X_2 \otimes_B Y)$.
(ii) $X \otimes_B (Y \otimes_C Z) = (X \otimes_B Y) \otimes_C Z$.

∵) (i) is clear.
(ii) Since

$$\begin{aligned}(x \otimes_B y | x' \otimes_B y') &= \tau([x', x]_B\, {}_B[y, y']) \\ &= ([x', x]_B y | y'), \quad x, x' \in X_0,\ y, y' \in Y_0,\end{aligned}$$

this inner product is continuously extended to an inner product in $X_0 \otimes_B Y$. The above formula also shows that if Y' is a dense linear subspace in Y, then $X_0 \otimes_B Y'$ is again dense in $X \otimes_B Y$. With these observations, to see (ii), it suffices to show that $X_0 \otimes_B (Y \otimes_C Z_0) = (X_0 \otimes_B Y) \otimes_C Z_0$. Of course, the identification is given by

$$x_0 \otimes_B (y \otimes_C z_0) = (x_0 \otimes_B y) \otimes_C z_0.$$

To complete the proof, we nned to check the equality of the norms:

$$\begin{aligned}(x_0 \otimes_B (y \otimes_C z_0) | (x_0 \otimes_B y) \otimes_C z_0) &= ([x_0, x_0]_B y \otimes_C z_0 | y \otimes_C z_0) \\ &= (y | [x_0, x_0]_B y\, {}_C[z_0, z_0]) \\ &= ([x_0, x_0]_B y | y\, {}_C[z_0, z_0]) \\ &= (x_0 \otimes_B y | x_0 \otimes_B y\, {}_C[z_0, z_0]) \\ &= ((x_0 \otimes_B y) \otimes_C z_0 | (x_0 \otimes_B y) \otimes_C z_0).\end{aligned}$$

□

Remark. By the proof of (ii), the inner product in $X \otimes_B Y$ is given by

$$(\xi \otimes_B \eta | \xi' \otimes_B \eta') = (\xi_B[\eta, \eta'] | \xi') = (\eta | [\xi, \xi']_B \eta')$$

and this formula reveals that the linear maps

$$\xi \mapsto \xi \otimes_B \eta_0, \qquad \eta \mapsto \xi_0 \otimes_B \eta,$$

defined for $\xi_0 \in X_0$, $\eta_0 \in Y_0$ are bounded. This fact will be used later.

Example 18. *Let N be a semi-finite von Neumann algebra. Then*

$$L^2(N) \otimes_N L^2(N) \cong L^2(N).$$

The isomorphism is given by

$$N \cap L^2(N) \otimes_N N \cap L^2(N) \ni a \otimes_N b \mapsto ab \in L^2(N).$$

Lemma 19. *Let X_B and $_BY$ be bimodules. As described after Lemma 16, we can imbed $End(X_B)$ and $End(_BY)$ into $\mathcal{B}(X \otimes_B Y)$. Then, in $\mathcal{B}(X \otimes_B Y)$, we have*

$$End(X_B)' = End(_BY).$$

∵) Express X_B and $_BY$ as

$$X_B = pL^2(B)_B^{\oplus I}, \quad _BY = {_B}L^2(B)^{\oplus I} q,$$

with p, q projections in $Mat_I(B)$. Then

$$End(X_B) = pMat_I(B)p, \quad End(_BY)^{op} = qMat_I(B)q, \quad \text{on} \quad X \otimes_B Y = pL^2(Mat_I(B))q,$$

and the assertion is reduced to the usual relations of commutants in the induction and reduction operations. □

Remark. The results in this part including Lemma 19 can be formulated and proved without the assumption of the existence of traces but with much effort on modular operators. See [Sauvageot] for the details.

Bimodule

Let A, B be finite sums of finite factors. According to Popa (Connes ?), we call a bimodule $X = {_A}X_B$ being of **finite type** if $\dim_A X < +\infty$ and $\dim X_B < +\infty$ (note that this definition is independent of the choice of faithful traces on A and B).

In the rest of this part, we assume that A and B are finite factors and we consider dimensions with respect to the normalized traces τ_A and τ_B on A and B.

Proposition 20. *Let ${}_AX_B$ be a bimodule of finite type and ${}_BY$ be a finitely generated module (i.e.,* $\dim_B Y < +\infty$*). Then*

$$\dim_A X \otimes_B Y = \dim_A X \dim_B Y.$$

$\because$) By a decomposition of Y into cyclic submodules, the problem is reduced to the case ${}_BY = {}_BL^2(B)p$ with $p \in B$ a projection. This case, ${}_AX \otimes_B Y = {}_AXp$ and the left hand side is

$$\dim_A Xp = \tau'_A(p) = \frac{\tau'_A(p)}{\tau'_A(1)}\tau'_A(1) = \frac{\tau'_A(p)}{\tau'_A(1)}\dim_A X,$$

while $\dim_B Y = \tau_B(p) = \tau'_A(p)/\tau'_A(1)$ because $\tau'_A(1)^{-1}\tau'_A|_B$ is the normalized trace of B. □

For a bimodule $X = {}_AX_B$ (A and B are assumed to be factors), we define its **index** by

$$[X] = [{}_AX_B] = \dim_A X \dim X_B.$$

Corollary 21. *Let C be another finite factor.*

(i) *If ${}_BY_C$ is a bimodule of finite type, then ${}_AX \otimes_B Y_C$ is of finite type and*

$$[{}_AX \otimes_B Y_C] = [{}_AX_B][{}_BY_C].$$

(ii) *If ${}_AY_B$ is a bimodule of finite type,*

$$[{}_AX_B \oplus {}_AY_B] \geq [{}_AX_B] + [{}_AY_B].$$

(iii) *Any bimodule ${}_AX_B$ of finite type has index $[X] \geq 1$, with $[X] = 1$ if and only if $A' = B$ (i.e., B is realized as the commutant of A).*

(iv) *If ${}_AX_B$ and ${}_AY_B$ are bimodules of finite type, then $Hom({}_AX_B, {}_AY_B)$ is finite dimensional.*

$\because$) (ii) is immediate from the definition. (i) follows from Proposition 20.

(iii): Since A is a finite factor and $\dim_A X < +\infty$, $A' \subset \mathcal{B}(X)$ is a finite factor and τ'_A is a trace of A' with $\tau'_A(1) = \dim_A X$. Since $\tau''_A = \tau_A$,

$$\dim X_{A'} = (\tau'_A(1)^{-1}\tau'_A)'(1) = \tau'_A(1)^{-1}\tau_A(1) = 1/\dim_A X.$$

Using this formula and Proposition 20,

$$\begin{aligned}\dim_A X \dim X_B &= \dim_A X \dim(X \otimes_{A'} L^2(A')_B)\\ &= \dim_A X \dim X_{A'} \dim L^2(A')_B\\ &= \dim L^2(A')_B.\end{aligned}$$

Now the assertion follows from $[A' : B] \geq 1$ and $[A' : B] = 1 \iff A' = B$.

(iv): Since $Hom({}_AX_B, {}_AY_B)$ is a subspace of $End({}_AX_B \oplus {}_AY_B)$, it suffices to consider the case $X = Y$. Since any submodule of ${}_AX_B$ has index ≥ 1 by (iii), the inequality in (ii) shows that the number of components does not exceed $[X]$. □

Frobenius Reciprocity

Let H be an A-B module. The conjugate Hilbert space H^* is naturally a B-A module. We denote by x^* the element in H^* corresponding to $x \in H$.

Example 22. *$L^2(M)^* = L^2(M)$ as M-M module.*

In the rest of this paragraph, von Neumann algebras are assumed to be finite sums of finite factors.

Lemma 23. *Let ${}_AX_B$ be a bimodule of finite type.*
(i) *$({}_AX)_0 = (X_B)_0$ (i.e., an element $\xi \in X$ is A-bounded if and only if B-bounded). In the following, this set is denoted by X_0.*
(ii) *Let ${}_BY$ be a finitely generated module. For finite bases $\{\lambda_i\}$ for ${}_AX$ and $\{\mu_j\}$ for ${}_BY$, $\{\lambda_i \otimes_B \mu_j\}$ forms a basis for ${}_AX \otimes_B Y$.*

∵) (i) Decomposing ${}_AX_B$ into a direct sum, we may assume that A and B are factors. Then $A' = End_A X$ is a factor and, if we realize ${}_AX$ as ${}_AL^2(A) \oplus \cdots \oplus {}_AL^2(A) \oplus {}_AL^2(A)p$ with $p \in A$ a projection (see), then A' is realized by

$$\begin{pmatrix} A & \dots & A & Ap \\ \vdots & \ddots & \vdots & \vdots \\ A & \dots & A & Ap \\ pA & \dots & pA & pAp \end{pmatrix}.$$

From this form, it is easy to see that elements in $A \oplus \cdots \oplus A \oplus Ap \subset {}_AL^2(A) \oplus \cdots \oplus {}_AL^2(A) \oplus {}_AL^2(A)p$ are bounded with respect to the right action of A' and hence they are bounded with respect to $B \subset A'$ as well. In particular, the basis

$$1 \oplus 0 \oplus \cdots \oplus 0 \oplus 0$$

$$0 \oplus 1 \oplus \cdots \oplus 0 \oplus 0$$

$$\vdots$$

$$0 \oplus 0 \oplus \cdots \oplus 1 \oplus 0$$

$$1 \oplus 0 \oplus \cdots \oplus 0 \oplus p$$

for ${}_AX$ consists of B-bounded elements. Since any element ξ in $({}_AX)_0$ is an A-linear combination of this basis and since $(X_B)_0$ is A-invariant, ξ is B-bounded. Thus $({}_AX)_0 \subset (X_B)_0$. By the symmetry of the argument, we get the reverse inclusion as well.

(ii) Let $\varphi \in (X \otimes_B Y)_0$, $\lambda \in X_0$, and $\mu \in Y_0$. Since $(\otimes_B \mu)^* \in Hom({}_AX \otimes_B Y, {}_AX)$, $\varphi(\otimes_B \mu)^* \in X_0$ and we infer

$$\begin{aligned}(a\varphi|\lambda \otimes_B \mu) &= (\varphi(\otimes_B \mu)^*|a^*\lambda) \\ &= \tau_A(a_A[\varphi(\otimes_B \mu)^*, \lambda]),\end{aligned}$$

which shows that $\lambda \otimes_B \mu \in (X \otimes_B Y)_0$ and

$${}_A[\varphi, \lambda \otimes_B \mu] = {}_A[\varphi(\otimes_B)^*, \mu].$$

Now the following calculation gives a proof.

$$\begin{aligned}\sum_{ij}({}_A[\varphi,\lambda_i\otimes_B\mu_j]\lambda_i\otimes_B\mu_j|\xi\otimes_B\eta) &= \sum_{ij}({}_A[\varphi(\mu_j)^*,\lambda_i]\lambda_i|\xi_B[\eta,\mu_j]) \\ &= \sum_j(\varphi(\mu_j)^*|\xi_B[\eta,\mu_j]) \\ &= \sum_j(\varphi|\xi_B[\eta,\mu_j]\otimes_B\mu_j) \\ &= \sum_j(\varphi|\xi\otimes_B[\eta,\mu_j]\mu_j) \\ &= (\varphi|\xi\otimes_B\eta).\end{aligned}$$

□

Lemma 24. *Let ${}_AX_B$ and ${}_BY_A$ be bimodules of finite type. Then we have the following *-preserving (i.e., the adjoints of intertwiners correspond to conjugate vectors in Hilbert spaces) linear isomorphisms:*

$$\begin{array}{ccccc}(X\otimes_B Y)^A & \cong & Hom({}_AY^*_B,{}_AX_B) & \cong & (Y\otimes_A X)^B, \\ \varphi & \longleftrightarrow & \Phi & \longleftrightarrow & \psi\end{array}$$

where, for an A-A bimodule ${}_AZ_A$, Z^A is defined by

$$Z^A=\{\zeta\in Z; a\zeta=\zeta a\quad \forall a\in A\},$$

and φ, ψ, Φ are related by

$$(\xi\otimes_B\eta|\varphi)=(\xi|\Phi(\eta^*))=(\eta\otimes_A\xi|\psi).$$

∵) For $\varphi\in X\otimes_B Y$, define a linear map $\Phi: Y_0^*\to X$ by

$$\Phi(\eta^*)=\varphi(\otimes_B\eta)^*.$$

Note that $\otimes_B\eta$ denotes the bounded linear map $\xi\mapsto\xi\otimes_B\eta$. Since

$$(\xi|\Phi(\eta^*))=(\xi\otimes_B\eta|\varphi)=(\eta|(\xi\otimes_B)^*\varphi)\quad \text{for } \xi\in X_0,$$

the adjoint of Φ is densely defined and hence Φ is a closable operator. On the other hand, the A-invariance of φ shows that Φ is A-B equivariant. Using the irreducible decompositions of ${}_AX_B$ and ${}_AY^*_B$ (recall that $Hom({}_AX_B,{}_AY^*_B)$ is finite-dimensional) and the polar decomposition of the closure of Φ, one sees that Φ is a constant multiple of an isometry on each irreducible components. In particular, it is bounded and we conclude that $\Phi\in Hom({}_AX_B,{}_AY^*_B)$.

Conversely, suppose that $\Phi \in Hom({}_A X_B, {}_A Y_B^*)$. Take a finite basis $\{\mu_j\}$ for ${}_B Y$ and set

$$\varphi = \sum_j \Phi(\mu_j^*) \otimes_B \mu_j \in X \otimes_B Y.$$

Note that $\Phi(\mu_j^*) \in X_0$ since $\mu_j^* \in Y_0^*$. For $\xi \in X_0$, $\eta \in Y_0$,

$$\begin{aligned}(\xi \otimes_B \eta|\varphi) &= \sum_j (\xi|\Phi(\mu_j^*)_B[\mu_j, \eta]) \\ &= \sum_j (\xi|\Phi(\mu_j^*)[\mu_j^*, \eta^*]_B) \\ &= (\xi|\Phi(\sum_j \mu_j^*[\mu_j^*, \eta^*]_B)) \\ &= (\xi|\Phi(\eta^*)),\end{aligned}$$

from which we see that the correspondence $\varphi \mapsto \Phi$ is bijective. □

Corollary 25. *Let ${}_A X_B$, ${}_B Y_C$, ${}_A Z_C$ be bimodules of finite type. Then*

$$Hom({}_A X \otimes_B Y_C, {}_A Z_C) \cong Hom({}_B Y_C, {}_B X^* \otimes_A Z_C).$$

More precisely, an intertwiner $T \in Hom({}_A X \otimes_B Y_C, {}_A Z_C)$ and its Frobenius transform $S \in Hom({}_B Y_C, {}_B X^* \otimes_A Z_C)$ is related by

$$(\xi^* \otimes_A \zeta|S(\eta)) = (\zeta|T(\xi \otimes_B \eta)).$$

∵)

$$\begin{aligned}\text{l.h.s.} &\cong ((X \otimes_B Y) \otimes_C Z^*)^A \\ &= (X \otimes_B (Y \otimes_C Z^*))^A \\ &\cong ((Y \otimes_C Z^*) \otimes_A X)^B \\ &= (Y \otimes_C (Z^* \otimes_A X))^B \\ &\cong \text{r.h.s.}\end{aligned}$$

□

Lemma 26. *Let ${}_A X_B$, ${}_B Y_C$, ${}_A Z_C$ be bimodules of finite type and suppose that ${}_B Y_C$ and ${}_A Z_C$ are irreducible. For $T \in Hom({}_A X \otimes_B Y_C, {}_A Z_C)$, its* **Frobenius transform** *$S \in Hom({}_B Y_C, {}_B X^* \otimes_A Z_C)$ satisfies*

$$\dim_B Y \|S\|^2 = \dim_A Z \|T\|^2.$$

$\because$) Let $\{\lambda_i\} \subset {}_AX$, $\{\mu_j\} \subset {}_BY$ $\{\zeta_k\} \subset {}_AZ$ be finite bases. We first claim

$$S(\mu_j) = \sum_i \lambda_i^* \otimes_A T(\lambda_i \otimes_B \mu_j).$$

In fact,

$$\begin{aligned}(\xi^* \otimes_A \zeta | S(\eta)) &= (\zeta | T(\xi \otimes_B \eta)) \\ &= (\zeta | T(\sum_i {}_A[\xi, \lambda_i]\lambda_i \otimes_B \eta)) \\ &= \sum_i (\zeta | [\xi^*, \lambda_i^*]_A T(\lambda_i \otimes_B \eta)) \\ &= \sum_i (\xi^* \otimes_A \zeta | \lambda_i^* \otimes_A T(\lambda_i \otimes_B \eta)).\end{aligned}$$

Since Y and Z are irreducible, TT^* and S^*S are scalar multiples of identity and hence

$$\begin{aligned}\dim_B Y \|S\|^2 &= \sum_j (S(\mu_j)|S(\mu_j)) \\ &= \sum_{ijk} (\lambda_i^* \otimes_A T(\lambda_i \otimes_B \mu_j) | \xi_k^* \otimes_A T(\xi_k \otimes_B \mu_j)) \\ &= \sum_{ijk} (T(\lambda_i \otimes_B \mu_j) | {}_A[\lambda_i, \lambda_k] T(\lambda_k \otimes_B \mu_j)) \\ &= \sum_{ij} (T(\lambda_i \otimes_B \mu_j) | T(\lambda_i \otimes_B \mu_j)) \\ &= \sum_{ijk} (T(\lambda_i \otimes_B \mu_j) | {}_A[T(\lambda_i \otimes_B \mu_j), \zeta_k]\zeta_k) \\ &= \sum_{ijk} \tau_A({}_A[\zeta_k, T(\lambda_i \otimes_B \mu_j)]\, {}_A[T(\lambda_i \otimes_B \mu_j), \zeta_k]) \\ &= \sum_{ijk} \tau_A({}_A[T^*(\zeta_k), \lambda_i \otimes_B \mu_j]\, {}_A[\lambda_i \otimes_B \mu_j, T^*(\zeta_k)]) \\ &= \sum_{ijk} \tau_A({}_A[T^*(\zeta_k), \lambda_i \otimes_B \mu_j]\lambda_i \otimes_B \mu_j | T^*(\zeta_k)]) \\ &= \sum_k (T^*(\zeta_k) | T^*(\zeta_k)) \text{ (since } \{\lambda_i \otimes \mu_j\} \text{ is a basis)} \\ &= \dim_A Z \|T\|^2.\end{aligned}$$

□

Remark. Both $\|S\|$ and $\|T\|$ depend on the choice of traces τ_A, τ_B, which is canceled by the dependence of $\dim_A Z$, $\dim_B Y$ on traces.

Notation. In the following, for an intertwiner T, its (linear) Frobenius transform changing the left actions of bimodules is denoted by T^l. Similarly the Frobenius transform changing the right actions is denoted by T^r. Furthermore, in the situation of Lemma (i.e., with the assumption on irreducibility), the normalization of the adjoint of the left Frobenius transform T^l (resp. the right Frobenius transform T^r) is denoted by $T^\sharp$ (resp. $T^\flat$). For example, with the notation of the above lemma, $T^l = S$ and

$$T^\sharp = \sqrt{\frac{\dim_B Y}{\dim_A Z}} S^*.$$

Lemma 27. *Let $T \in Hom({}_AX \otimes_B Y_C, {}_AZ_C)$ and $S \in Hom({}_AZ_C, {}_AW_C)$. Then*

$$(ST)^l = (1 \otimes_A S)T^l.$$

$\because$) Let $\{\zeta_i\}$ be an orthonormal basis in X. For $\xi \in X$, $\eta \in Y$, and $\omega \in W$

$$\begin{aligned}((ST)^l(\eta)|\xi^* \otimes_A \omega) &= (ST(\xi \otimes_B \eta)|\omega) \\ &= (T(\xi \otimes_B \eta)|S^*\omega) \\ &= \sum_i (T(\xi \otimes_B \eta)|\zeta_i)(\zeta_i|S^*\omega) \\ &= \sum_i (T^l(\eta)|\xi^* \otimes_A \zeta_i)(\zeta_i|S^*\omega) \\ &= (T^l(\eta)|\xi^* \otimes_A \sum_i \zeta_i(\zeta_i|S^*\omega)) \\ &= (T^l(\eta)|\xi^* \otimes_A S^*\omega) \\ &= (T^l(\eta)|(1 \otimes_A S^*)(\xi^* \otimes_A \omega)) \\ &= ((1 \otimes_A S)T^l(\eta)|\xi^* \otimes_A \omega).\end{aligned}$$

□

Paragroup

Now we introduce a group-like structure (called paragroup) for a pair $N \subset M$ of factor and subfactor with finite index. For A, $B = M$ or N, let $\mathcal{G}_{A,B}$ be the set of equivalence classes of A-B bimodules which appear in ${}_AL^2(M) \otimes_N \cdots \otimes_N L^2(M)_B$ (note that $End({}_AL^2(M) \otimes_N \cdots \otimes_N L^2(M)_B)$ is finite-dimensional due to Corollary 21). Let $\mathcal{G}$ be a bipartite graph having even vertices $\mathcal{G}^{(0)}_{even} = \mathcal{G}^{(0)}_{M,M} \cup \mathcal{G}^{(0)}_{N,N}$ and odd vertices $\mathcal{G}^{(0)}_{odd} = \mathcal{G}^{(0)}_{M,N} \cup \mathcal{G}^{(0)}_{N,M}$ with n edges between $E \in \mathcal{G}^{(0)}_{even}$ and $O \in \mathcal{G}^{(0)}_{odd}$ if, a suitable M-action being restricted to N, one of E or O appears in the other with multiplicity n. For example, if $E = {}_ME_M$ and $O = {}_NO_M$, then $n = \dim Hom({}_NO_M, {}_NE_M)$. Note that, by Frobenius reciprocity, this number can be also interpreted as $\dim Hom({}_M L^2(M) \otimes_N O_M, {}_M E_M)$. The class of the identity bimodule ${}_M L^(M)_M$ (resp. ${}_N L^2(N)_N$) is distinguished and denoted by $\star_M$ (resp. $\star_N$). $\mathcal{G}$ also admits $*$-**operation** induced by taking adjoint bimodules.

$\mathcal{G}$ has 4 distinguished subgraphs. Let $\mathcal{G}_M$ (resp. $\mathcal{G}_N$) be two of them, which is obtained by reducing the vertex set $\mathcal{G}^{(0)}$ to $\mathcal{G}^{(0)}_{M,M} \cup \mathcal{G}^{(0)}_{M,N}$ (resp. to $\mathcal{G}^{(0)}_{N,M} \cup \mathcal{G}^{(0)}_{N,N}$). The other two are given by $\mathcal{G}^*_M$ and $\mathcal{G}^*_N$.

Lemma 28. *Graphs $\mathcal{G}_M$, $\mathcal{G}_N$ are connected and locally finite.*

∵) Local finiteness follows from the finite dimensionality of the space of intertwiners. By the definition, any vertex in $\mathcal{G}^{(0)}_M$ (resp. $\mathcal{G}^{(0)}_N$) is connected with the irreducible bimodule ${}_M L^2(M)_N$ (resp. ${}_N L^2(M)_M$). □

At this stage, edges with same end points are not distinguished; we have only fixed the number of edges between two vertices and not defined edges themselves. Now we give a concrete meaning to edges: Take an even vertex X and a odd vertex Y. To be specific, suppose that $X = {}_M X_M$, $Y = {}_N Y_M$ for example, and set $n = \dim Hom({}_N Y_M, {}_N X_M)$. Since n is the multiplicity of the irreducible bimodule ${}_N Y_M$ in the bimodule ${}_N X_M$, we can find pair-wise orthogonal n isometric intertwiners $T_1, \ldots, T_n$ from ${}_N Y_M$ into ${}_N X_M$. We regard such a choice of intertwiners represent n-edges between X and Y, and call it a **representation** of $\mathcal{G}$.

It is easy to see that we can find a representation of $\mathcal{G}$ so that $*$-operation keeps it invariant. Such representations are called $*$-representations.

Haar Measure

Definition 29. *Let Γ be a connected unoriented graph with a distinguished vertex $\star$. For a function μ on the set of vertex $\Gamma^{(0)}$, we set*

$$(\Delta\mu)(x) = \sum_{y \in \Gamma^{(0)}} \Delta(x,y)\mu(y), \quad x \in \Gamma^{(0)},$$

where $\{\Delta(x,y)\}_{x,y \in \Gamma^{(0)}}$ denotes the incidence matrix of Γ (i.e., $\Delta(x,y)$ = the number of edges between x and y). Δ is called the **Laplacian** *of Γ.*

A **Haar measure** *is, by definition, an eigen-function μ of Δ with positive entries which is 'normalized' in the sense*

$$\mu(\star) = 1.$$

Note that the eigenvalue of a Haar measure is positive.

Lemma 30. *Let $\mathcal{G}$ be a graph introduced above and define functions μ_0, μ on the set of vertices $\mathcal{G}^{(0)}$ by*

$$\mu_0(V) = \text{dimension of } V \text{ as left module},$$

$$\mu(V) = \begin{cases} [M:N]^{1/2}\mu_0(V) & \text{if } V \in \mathcal{G}^{(0)}_{M,N} \\ [M:N]^{-1/2}\mu_0(V) & \text{if } V \in \mathcal{G}^{(0)}_{N,M} \\ \mu_0(V) & \text{otherwise.} \end{cases}$$

(Note that we are measuring dimensions with respect to the unique normalized trace.)

Then restrictions of μ define Haar mesures for 4 graphs $\mathcal{G}_M$, $\mathcal{G}^*_M$, $\mathcal{G}_N$, $\mathcal{G}^*_N$.

∵) We consider the case for $\mathcal{G}_M$. The other cases are checked in a similar way. Let ${}_MX_M \in \mathcal{G}^{(0)}_{M,M}$, ${}_MY_N \in \mathcal{G}^{(0)}_{M,N}$, and T be a representation of an edge between X and Y. With these notations, we have

$$\begin{aligned} {}_MY \otimes_N L^2(M)_M &= \oplus_{X,T} T^\flat({}_MX_M) \\ {}_MX_N &= \oplus_{Y,T} T({}_MY_N), \end{aligned}$$

from which, taking dimensions, we deduce that

$$\begin{aligned} [M:N]\mu_0(Y) &= \sum_{X,T} \mu_0(X) = (\Delta\mu_0)(Y) \\ \mu_0(X) &= \sum_{Y,T} \mu_0(Y) = (\Delta\mu_0)(X). \end{aligned}$$

Now the assertion follows from this. □

Connection

Here we introduce the notion of 'connection' for a representation of $\mathcal{G}$. A sequence of 4 edges, each taken from $\mathcal{G}_M$, $\mathcal{G}^*_M$, $\mathcal{G}_N$, $\mathcal{G}^*_N$ and making a closed path in the graph $\mathcal{G}$, is called a **cell**. Given a representation of $\mathcal{G}$, a **connection** W is a $\mathbb{C}$-valued function on the set of cells defined by

$$W(C)1_Z = S_2^* T_2^* T_1 S_1,$$

where, for a cell C, $T_1 : {}_N(Y_1)_M \to {}_NL^2(M) \otimes_M X_M$, $T_2 : {}_M(Y_2)_N \to {}_MX \otimes_M L^2(M)_N$, $S_1 : {}_NZ_N \to {}_NY_1 \otimes_M L^2(M)_N$, and $S_2 : {}_NZ_N \to {}_NL^2(M) \otimes_M (Y_2)_N$ are intertwiners representing 4 edges of C. Note that $S_2T_2T_1^*S_1^* \in End({}_NZ_N)$ is a scalar multiple of the identity because ${}_NZ_N$ is irreducible. Of course, W has a cohomological ambiguity coming from the choice of vertices (realization of irreducible bimodules) and edges of graphs (realization of intertwiners). In other words, W is unique up to cohomology-like gauge.

Let ${}_M X_M \in \mathcal{G}^{(0)}_{M,M}$ and ${}_N Z_N \in \mathcal{G}^{(0)}_{N,N}$ and suppose that there is a cell having X and Z as a part of vertices. In the vector space $Hom({}_N Z_N, {}_N X_N)$, we introduce a positive definite inner product by

$$(U|V)1_Z = V^* U, \quad U, V \in Hom({}_N Z_N, {}_N X_N).$$

It is easy to see that the set of intertwiners

$$\{T_1 S_1\}$$

indexed by left halves of cells, forms an orthonormal basis of $Hom({}_N Z_N, {}_N X_N)$. Similarly, we have an orthonormal basis

$$\{T_2 S_2\}$$

indexed by right halves of cells.

In this way, for a fixed pair of vertices ${}_M X_M$, ${}_N Z_N$, W can be interpreted as a unitary matrix connecting two orthonormal bases.

Theorem 31. *For a fixed pair of $Y_1 \in \mathcal{G}^{(0)}_{N,N}$ and $Y_2 \in \mathcal{G}^{(0)}_{M,N}$,*

$$\sqrt{\frac{\mu(X)\mu(Z)}{\mu(Y_1)\mu(Y_2)}} W\begin{pmatrix} & X & \\ Y_1 & & Y_2 \\ & Z & \end{pmatrix}$$

forms a unitary matrix indexed by upper halves and lower halves of cells containing Y_1 and Y_2.

Before giving the proof of the theorem, we present a lemma.

Lemma 32. *Let p be the orthogonal projection onto an N-invariant closed subspace $Z \cong 1 \otimes_N Z \subset L^2(M) \otimes_N Z$. Then for $T \in End({}_M L^2(M) \otimes_N Z)$, we have*

$$\tau'_M(T) = \tau'_N(pTp).$$

$\because$) If $\{\lambda_i\}$ is a basis for ${}_N Z$, then $\{1 \otimes_N \lambda_i\}$ is a basis for ${}_M L^2(M) \otimes_N Z$ and we have

$$\begin{aligned} \tau'_M(T) &= \sum_i (T(1 \otimes_N \lambda_i)|1 \otimes_N \lambda_i) \\ &= \sum_i (Tp(1 \otimes_N \lambda_i)|p(1 \otimes_N \lambda_i)) \\ &= \tau'_N(pTp). \end{aligned}$$

□

Proof of the theorem. The idea of the proof is 'similar' to the case of the left-right unitarity of W. Here we consider the vector space $Hom({}_M(Y_2)_N, {}_M L^2(M) \otimes_N (Y_1)_M)$ and make it a finite dimensional Hilbert space as described above. Let $T_1 \in Hom({}_N(Y_1)_M, {}_N X_M)$, $T_2 \in Hom({}_M(Y_2)_N, {}_M X_N)$, $S_1 \in Hom({}_N Z_N, {}_N(Y_1)_N)$, and $S_2 \in Hom({}_N Z_N, {}_N(Y_2)_N)$ describe a cell. Since ${}_N(Y_1)_M$ and ${}_M X_M$ are irreducible,

$$T_1 \mapsto T_1^\sharp \in Hom({}_M X_M, {}_M L^2(M) \otimes_N (Y_1)_M)$$

is an isometry. So $T_1^\sharp$'s are orthogonal to each other and

$${}_M L^2(M) \otimes_N (Y_1)_M = \oplus_{X,T_1} T_1^\sharp(X)$$

gives an irreducible decomposition. Similarly, we have an irreducible decomposition

$${}_M Y_2 \otimes_N L^2(M)_M = \oplus_{X,T_2} T_2^\flat(X).$$

Thus $\{T_1^\sharp(T_2^\flat)^*\}_{X,T_1,T_2}$ forms a basis for $Hom({}_M Y_2 \otimes_N L^2(M)_M, {}_M L^2(M) \otimes_N (Y_1)_M)$. Then, by Frobenius reciprocity, $\{(T_1^\sharp(T_2^\flat)^*)^r\}$ is a basis of $Hom({}_M(Y_2)_N, {}_M L^2(M) \times_N (Y_1)_N)$. Since

$$(T_1^\sharp(T_2^\flat)^*)^r = (1 \otimes_M T_1^\sharp)(T_2^{\flat *})^r = T_1^\sharp(T_2^{\flat *})^r,$$

they form an orthogonal basis and its normalization is given by

$$T_1^\sharp T_2.$$

Similarly, for the lower half of a cell,

$$(1 \otimes_N S_1)S_2^\sharp$$

forms an orthonormal basis for $Hom({}_M(Y_2)_N, {}_M L^2(M) \otimes_N (Y_1)_N)$. Thus we have obtained a unitary matrix

$$((1 \otimes_N S_1)S_2^\sharp | T_1^\sharp T_2).$$

To relate this quantity with the connection W, we calculate as follows: Since

$$((1 \otimes_N S_1)S_2^\sharp | T_1^\sharp T_2)1_{Y_2} = T_2^*(T_1^\sharp)^*(1 \otimes_N S_1)S_2^\sharp$$

and since $\tau'_M(1_{Y_2}) = \dim_M Y_2$ (here traces on M and N are assumed to be normalized),

$$\begin{aligned}\mu_0(Y_2)((1 \otimes_N S_1)S_2^\sharp | T_1^\sharp T_2) &= \tau'_M(T_2^*(T_1^\sharp)^*(1 \otimes_N S_1)S_2^\sharp) \\ &= \tau'_M(S_2^\sharp T_2^*(T_1^\sharp)^*(1 \otimes_N S_1)) \\ &= \tau'_N(pS_2^\sharp T_2^*(T_1^\sharp)^*(1 \otimes_N S_1)p) \\ &= \sqrt{\frac{\mu_0(Y_2)}{\mu_0(Z)}}\sqrt{\frac{\mu_0(X)}{\mu_0(Y_1)}}\tau'_N(p(S_2^l)^* T_2^* T_1^l(1 \otimes_N S_1)p).\end{aligned}$$

Since, for $\zeta_1, \zeta_2 \in Z$,

$$\begin{aligned}(1 \otimes_N \zeta_2 | (S_2^l)^* T_2^* T_1^l (1 \otimes_N S_1)(1 \otimes_N \zeta_1)) \\ &= (T_2 S_2^l (1 \otimes_N \zeta_2) | T_1^l (1 \otimes_N S_1(\zeta_1))) \\ &= (T_2 S_2(\zeta_2) | T_1 S_1(\zeta_1)),\end{aligned}$$

$$p(S_2^l)^* T_2^* T_1^l (1 \otimes_N S_1) p = S_2^* T_2^* T_1 S_1 = W(C) 1_Z$$

and hence

$$\begin{aligned}\mu_0(Y_2)((1 \otimes_N S_1) S_2^\sharp | T_1^\sharp T_2) &= \sqrt{\frac{\mu_0(Y_2)}{\mu_0(Z)}} \sqrt{\frac{\mu_0(X)}{\mu_0(Y_1)}} W(C) \tau_N'(1_Z) \\ &= \sqrt{\frac{\mu_0(Y_2)}{\mu_0(Z)}} \sqrt{\frac{\mu_0(X)}{\mu_0(Y_1)}} W(C) \mu_0(Z),\end{aligned}$$

which proves the desired assertion (note that $\mu_0(Y_1)\mu_0(Y_2) = \mu(Y_1)\mu(Y_2)$, $\mu_0(X) = \mu(X)$, and $\mu_0(Z) = \mu(Z)$).

$$\begin{array}{ccc} {}_M L^2(M) \otimes_N (Y_1)_M & \xrightarrow{T_1^l} & {}_M X_M \\ {\scriptstyle 1\otimes 1}\uparrow & & \uparrow{\scriptstyle T_2} \\ {}_M L^2(M) \otimes_N Z_N & \xrightarrow[S_2^l]{} & {}_M (Y_2)_N \end{array}$$

□

Example 33.

Let N be a finite factor, G be a finite group, and $\alpha : G \to Aut(N)$ be an outer action. Let $M = N \rtimes_\alpha G$ and denote by H an N-M module ${}_N M_M$. Since $N' \cap M = \mathbb{C}1$ (cf. the appendix below), H is irreducible. Note that

$${}_N M_N = \sum_{g \in G} N \lambda_g$$

gives an irreducible decomposition and $\{N\lambda_g\}$'s are not equivalent to each others. It is clear that this gives an orthogonal decomposition and each component is invariant under N-N action. Since $\overline{N\lambda_g}^{L^2}$ is equivalent to $L^2(N)$ as left N-module, $(\overline{N\lambda_g}^{L^2})_0 = N\lambda_g$. Let $T : \overline{N\lambda_g} \to \overline{N\lambda_h}$ be an intertwiner of N-N modules. Since T commutes with the left action, $N\lambda_g = (\overline{N\lambda_g}^{L^2})_0$ is mapped into $N\lambda_h = (\overline{N\lambda_h}^{L^2})_0$. Thus $\exists\, a \in N$ such that

$$T(\lambda_g) = a\lambda_h.$$

For $b \in N$,

$$\begin{aligned}T(\lambda_g b) &= T(\alpha_g(b)\lambda_g) = \alpha_g(b) T(\lambda_g) = \alpha_g(b) a \lambda_h, \\ T(\lambda_g b) &= T(\lambda_g) = a\lambda_h b = a\alpha_h(b)\lambda_h,\end{aligned}$$

which means that

$$\alpha_g(b)a = a\alpha_h(b), \quad \forall\, b \in N.$$

From the outerness of the action,

$$a = \begin{cases} \text{scalar} & \text{if } g = h \\ 0 & \text{otherwise,} \end{cases}$$

proving the assertion.

Now we consider the decomposition of $M \otimes_N M = H^* \otimes_N H$. For $\rho \in \hat{G}$, set

$$\xi^\rho_{ij} = \sum_g \rho_{ij}(g^{-1})\rho^*_g \otimes_N \lambda_g \in H^* \otimes_N H.$$

Since

$$\begin{aligned}(a\lambda_g \otimes \lambda_h | a'\lambda_{g'} \otimes \lambda_{h'}) &= \tau([a'\lambda_{g'}, a\lambda_g]_N\, {}_N[\lambda_h, \lambda_{h'}]) \\ &= \tau(\alpha_{g'^{-1}}(a')^*[\lambda_{g'}, \lambda_g]_N \alpha_{g^{-1}}(a)\, {}_N[\lambda_h, \lambda_{h'}]) \\ &= \delta_{g,g'}\, \delta_{h,h'}\, \tau(a'^* a), \quad a, a' \in N,\end{aligned}$$

we have

$$\begin{aligned}(a\lambda_h \xi^\rho_{ij} | b\lambda_k \xi^\sigma_{kl}) &= \delta_{h,k}\tau(ab^*) \sum_g \rho_{ij}(g^{-1})\overline{\sigma_{kl}(g^{-1})} \\ &= \delta_{h,k}\tau(ab^*)\delta_{\rho,\sigma}\delta_{i,k}\delta_{j,l}|G|.\end{aligned}$$

Thus

$$\sum_{\rho,i,j} M\xi^\rho_{i,j}$$

is an orthogonal sum in $H^* \otimes_N H$. By the Peter-Weyl's theorem, $\{\rho_{ij}(g^{-1})\}_{\rho,i,j}$ forms a basis in $\ell^2(G)$. In particular, $\forall\, h \in G$ we can find a sequence $\{c^\rho_{ij} \in \mathbb{C}\}$ such that

$$\sum_{\rho,i,j} c^\rho_{ij}\rho_{ij}(g^{-1}) = \delta_{g,h}$$

which means that $\sum_{\rho,i,j} M\xi^\rho_{i,j}$ contains $\lambda^*_h \otimes_N \lambda_h$. Since $\sum_{\rho,i,j} M\xi^\rho_{i,j}$ is apparently left M-invariant, it contains $N\lambda_a\lambda^*_h \otimes_N \lambda_h$, $a, h \in G$, which shows that $\sum_{\rho,i,j} M\xi^\rho_{i,j}$ is dense in $H^* \otimes_N H$. Thus we have obtained an orthogonal decomposition

$$H^* \otimes_N H = \sum_{\rho,i,j} M\xi^\rho_{i,j}.$$

Since, by an easy computation,

$$\begin{aligned}\xi^\rho_{ij} x &= x\xi^\rho_{ij}, \quad x \in N \\ \xi^\rho_{ij}\lambda_h &= \lambda_h \sum_k \rho_{ik}(h)\xi^\rho_{kj}, \quad h \in G,\end{aligned}$$

$X_j^\rho \equiv \sum_k M\xi_{kj}^\rho$ is an M-M sub-module.

To see that these are irreducible, let $T : X_j^\rho \to X_k^\sigma$ be an intertwiner of M-M modules. Since $\overline{M\xi_{ij}^\rho}$ is isomorphic to $L^2(M)$ as left M-module, $(\overline{M\xi_{ij}^\rho})_0 = M\xi_{ij}^\rho$ and

$$(X_j^\rho)_0 = (\sum_i \overline{M\xi_{ij}^\rho})_0 = \sum_i (\overline{M\xi_{ij}^\rho})_0.$$

Since T maps $(X_j^\rho)_0$ into $(X_k^\sigma)_0$, $\exists\, a_{il} \in M$ such that

$$T(\xi_{ij}^\rho) = \sum_l a_{il}\xi_{lk}^\sigma.$$

Futhermore, since T commutes with left action of N, we have

$$a_{il} \in N' \cap M = \mathbb{C}1.$$

If we consider the right action of G on ξ_{ij}^ρ, we can show that the matrix $A = (a_{il})$ intertwines ρ and σ. Thus, by Schur's lemma,

$$a_{il} = \begin{cases} a\delta_{il} & \text{if } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$X_j^\rho \not\cong X_l^\sigma \quad \text{if } \rho \neq \sigma$$
$$X_j^\rho \cong X_l^\rho, \quad \forall j, k.$$

Since $N' \cap M = \mathbb{C}1$, $H = {}_N L^2(M)_M$ and $H^* = {}_M L^2(M)_N$ are irreducible modules.

Next, we show that all the irreducible components in $L^2(M) \otimes_N \cdots \otimes_N L^2(M)$ are contained in the ones already listed. To see this, we first consider the decomposition of ${}_N L^2(M) \otimes_N L^2(M)_N$. Using $M = \sum_{g\in G} N\lambda_g$, we have an orthogonal decomposition

$$L^2(M) \otimes_N L^2(M) = \sum_{g,h\in G} \overline{N\lambda_g \otimes_N \lambda_h}.$$

Since

$$\overline{N\lambda_g \otimes_M \lambda_h} \ni a\lambda_g \otimes_N \lambda_h \mapsto a\lambda_{gh} \in \overline{N\lambda_{gh}}$$

gives an isomorphism as N-N module, there appears no new irreducible component in $L^2(M) \otimes_N L^2(M)$. Repeating these arguments, we find that $L^2(M) \otimes_N \cdots \otimes_N L^2(M)$ does not generate new irreducible modules.

In this way, we have described the vertices. Let us now determine the edges. Since

$${}_N H_N = \sum_{g\in G} \overline{N\lambda_g},$$

there is one edge between (the vertex determined by) H and (the vertex of) each irreducible N-N module. On the other hand, the orthogonal decomposition

$$_N(X_j^\rho)_M = \sum_i {}_N(\overline{\xi_{ij}^\rho M})_M$$

and the fact $_N(\overline{\xi_{ij}^\rho M})_M \cong {}_N H_M$ shows that there are $n = \dim \rho$ edges between X_j^ρ and H.

Markov Trace

In this part, von Neumann algebras are assumed to be finite sums of finite factors. For a von Neumann algebra A of this type, let $\{p_i\}_{1 \le i \le m}$ be the set of minimal central projections in A. Then a trace τ of A is determined by the values on $\{p_i\}$. In the following, we use the notation $\tau(p)$ to express the row vector $(\tau(p_1), \ldots, \tau(p_m))$.

Let $_AX_B$ be a bimodule of finite type. Let $\{p_i\}$ (resp. $\{q_j\}$) be the set of minimal central projections in A (resp. B). Define a bipartite graph having vertices $\{p_i\}$ and $\{q_j\}$ with edges if $p_i X q_j \neq 0$. The bimodule $_AX_B$ is called **connected** if the graph obtained in this way is connected.

Definition 34. *Let $_AX_B$ be a bimodule of finite type and τ_A, τ_B be traces on A, B respectively. τ_A and τ_B are called* **Markov traces** *(or balanced traces) if $\exists\, a > 0$, $b > 0$ such that*

$$\tau_A = a\tau_B'|_A, \quad \tau_B = b\tau_A'|_B.$$

Proposition 35. *Let $_AX_B$ be a bimodule of finite type. If $_AX_B$ is connected, there is a pair of balanced traces τ_A on A and τ_B on B, which is unique up to positive multiplication.*

Before the proof of the proposition, we introduce some useful notations.

Definition 36. *Let $\{p_i\}$, $\{q_j\}$ the set of minimal central projections in von Neumann algebras A, B, respectively. For a bimodule of finite type $_AX_B$, define matrices $L(X) = (L_{ij}(X))$ and $R(X) = (R_{ij}(X))$ by*

$$L_{ij}(X) = \dim_{Ap_i}(p_i X q_j), \quad R_{ij}(X) = \dim(p_i X q_j)_{Bq_j},$$

where dimensions are measured with respect to the normalized traces. (Note that Ap_i and Bq_j are factors.)

Lemma 37.
(i) *Let τ_A, τ_B be traces on A, B respectively. Then we have*

$$\tau_A'(q) = \tau_A(p)L(X), \quad \tau_B'(p) = \tau_B(q)^t R(X).$$

(ii) *If ${}_AX_B$ and ${}_BY_C$ are bimodules of finite type, then*

$$L(X \otimes_B Y) = L(X)L(Y), \quad R(X \otimes_B Y) = R(X)R(Y).$$

$\because$) (i) From Lemma 1 (iii),

$$\begin{aligned}\tau_A'(q_j) &= \dim_A^{\tau_A} X_{q_j} \\ &= \sum_i \dim_A^{\tau_A}(p_i X q_j) \\ &= \sum_i \tau_A(p_i)L_{ij}(X).\end{aligned}$$

Similarly for $\tau_B'(p_i)$.
(ii) is an immediate consequence of Corollary 21 (i). □

Going back to the proof of Proposition 35, for a trace τ of A, let $\tau' = \tau'_{{}_AX}$ and $\tau_B = \tau'|_B$. Then τ is a balanced trace if and only if $\exists\, \alpha > 0$,

$$(\tau_B)'|_A = \alpha\tau.$$

Since $\tau_B'(p) = \tau_B(q)^t R(X)$ and $\tau_B(q) = \tau(p)L(X)$ from the above lemma, this condition is equivalent to

$$\tau(p)L(X)^t R(X) = \alpha\tau(p).$$

Now we can apply the Perron-Frobenius theory because the positve matrix $L(X)^t R(X)$ is irreducible by the connectedness assumption on X. □

Remark. If τ_A and τ_B are balanced through ${}_AX_B$ and $\tau_A = a\tau_B'|_A$, $\tau_B = b\tau_A'|_B$, then

$$ab = \|L(X)^t R(X)\|.$$

Proposition 38. *Let ${}_AX_B$, ${}_BY_C$ be bimodules of finite type and τ_A, τ_B, τ_C be traces on A, B, C.*
(i) *If τ_A, τ_B are balanced through ${}_AX_B$ and τ_B, τ_C are balanced through ${}_BY_C$, then τ_A, τ_C are balanced through ${}_AX \otimes_B Y_C$.*
(ii) *If either ${}_AX_B$ or ${}_BY_C$ is connected, then ${}_AX \otimes_B Y_C$ is connected.*

$\because$) (i) We may suppose that $\tau_A = \tau_B'|_A$, $\tau_C = \tau_B'|_C$ with $\tau_B = a\tau_A'|_B$, $\tau_B = c\tau_C'|_B$ for suitable $a > 0$ and $c > 0$. Let $\{p_i\}$, $\{q_j\}$, $\{r_k\}$ be the set of minimal central projections in A, B, C, respectively. Then by a repetition of Lemma 36 (i), we have

$$\begin{aligned}\tau_A(p) &= ac\tau_A(p)L(X)L(Y)^t R(Y)^t R(X) \\ \tau_C(r) &= ac\tau_C(r)^t R(Y)^t R(X)L(X)L(Y).\end{aligned}$$

Now Lemma 37 (ii) shows that τ_A and τ_C are balanced (cf. the proof of Proposition 35).

(ii) Suppose that ${}_AX \otimes_B Y_C$ is not connected, i.e., minimal central projections in A and in C split into two parts. Then the set of minimal central projections in B splits into two parts and hence both ${}_AX_B$ and ${}_BY_C$ are not connected. □

Now we extend Proposition 20.

Proposition 39. *Let A, B finite sums of finite factors with normalized traces τ_A, τ_B which are balanced through a bimodule ${}_AX_B$ of finite type. If ${}_BY$ is a left B-module, then*

$$\dim_A^{\tau_A} X \otimes_B Y = \dim_A^{\tau_A} X \dim_B^{\tau_B} Y.$$

∵) A careful reading of the proof of Proposition 20 gives the result. □

The index of a bimodule introduced before is now extended to non-factor case. Let ${}_AX_B$ be a connected bimodule. The index $[X] = [{}_AX_B]$ is defined to be

$$[X] = \dim_A^{\tau_A}(X) \dim(X)_B^{\tau_B},$$

where τ_A and τ_B denote unique normalized balanced traces.

Remark. $[X] = \|L(X)^t R(X)\|$.

Corollary 40. *Let ${}_AX_B$, ${}_BY_C$ be bimodules of finite type and suppose that there exist traces τ_A, τ_B, τ_C of A, B, C such that τ_A, τ_B and τ_B, τ_C are balanced through X and Y respectively. Then*

$$[{}_AX \otimes_B Y_C] = [{}_AX_B][{}_BY_C].$$

Jones Construction

In this part, as an application of (the existence of) Markov traces, we shall construct so called Jones projections via Frobenius transform.

Lemma 41. *Let ${}_AX_B$ be a bimodule of finite type with τ_A, τ_B traces on A, B respectively, and $\{\mu_j\}$ be a basis for $(X)_B^{\tau_B}$. If $\tau'_B|_A = a\tau_A$ for some $a > 0$, then*

$$\sum_j {}_A[\mu_j, \mu_j] = \dim(X)_B^{\tau_B} 1_A.$$

∵) Let $\{p_i\}$ be the set of minimal central projections in A. Since

$$\begin{aligned} a(\sum_j {}_A[\mu_j, \mu_j]) &= \sum_j {}_A[a\mu_j, \mu_j] \\ &= \sum_j {}_A[\sum_k [\mu_k, a\mu_j]_B, \mu_j] \\ &= \sum_{j,k} {}_A[\mu_k, \mu_j [a\mu_j, \mu_k]_B] \\ &= \sum_{j,k} {}_A[\mu_k, \mu_j [\mu_j, a^* \mu_k]_B] \\ &= \sum_k {}_A[\mu_k, a^* \mu_k] \\ &= (\sum_k {}_A[\mu_k, \mu_k])a, \quad \forall a \in A, \end{aligned}$$

$p_i \sum_j {}_A[\mu_j, \mu_j]$ is a scalar multiple of p_i. To determine the scalar, we first remark that $\{p_i \mu_j\}_j$ is a basis for $(p_i X)_B$, which implies that

$$\begin{aligned} \tau_A(p_i \sum_j {}_A[\mu_j, \mu_j]) &= \tau_A(\sum_j {}_A[p_i\mu_j, p_i\mu_j]) = \sum_j (p_i\mu_j | p_i\mu_j) \\ &= \dim(p_i X)_B^{\tau_B} = \tau'_B(p_i) \end{aligned}$$

and hence

$$p_i(\sum_j {}_A[\mu_j, \mu_j]) = \frac{\tau'_B(p_i)}{\tau_A(p_i)} p_i = a p_i,$$

by the assumption $\tau'_B|_A = a\tau_A$. Summing up these relations over i, we get the assertion. □

Now suppose that ${}_A X_B$ is connected and let τ_A, τ_B be the normalized balanced traces. Let $\epsilon_X \in (X \otimes_B X^*)^A$ be a vector corresponding to $1_X \in End({}_A X_B)$ by Lemma 24, and call it the **standard vector**.

Lemma 42.
(i) $(\epsilon_X | \xi' \otimes_B \xi^*) = (\xi | \xi')$ *for* $\xi, \xi' \in X_0$.
(ii) *For a basis* $\{\mu_j\}$ *of* $(X)_B^{\tau_B}$,

$$\epsilon_X = \sum_j \mu_j \otimes_B \mu_j^*.$$

(iii) $\forall \varphi \in (X \otimes_B Y)_0$,

$${}_A[\varphi, \epsilon_X] = [\epsilon_X, \varphi]_A.$$

(iv) ${}_A[\xi' \otimes_B \xi^*, \epsilon_X] = {}_A[\xi', \xi]$, *for* ξ, $\xi' \in X_0$.

(v) ${}_A[\epsilon_X, \epsilon_X] = \dim X_B 1_A$.

∵) (i) and (ii) follow from the correspondence $\epsilon_X \iff 1_X$ given in Lemma 24. (iii) is a consequence of the A-invariance of ϵ_X. (iv) results from (i) and the definition of operator-valued inner product. (v) is a combination of (ii) and Lemma 41. □

Remark. From (ii) and the proof of Lemma 23 (ii), $\epsilon_X \in (X \otimes_B X^*)_0$.

Again, by Lemma 24, $\epsilon_X \otimes_A \epsilon_X \in ((X \otimes_B X^*) \otimes_A (X \otimes_B X^*))^A$ defines an intertwiner $e_X^0 \in End({}_A X \otimes_B X_A^*)$. Its concrete form is given by

$$e_X^0(\xi' \otimes_B \xi^*) = \epsilon_X[\epsilon_X, \xi' \otimes_B \xi^*]_A = \epsilon_X {}_A[\xi', \xi], \quad \xi, \xi' \in X_0.$$

Since the adjoint of e_X^0 is associated with $(\epsilon_X \otimes_A \epsilon_X)^* = \epsilon_X^* \otimes_A \epsilon_X^* = \epsilon_X \otimes_A \epsilon_X$ (see Lemma 24), e_X^0 is self-adjoint. Now the calculation

$$\begin{aligned} e_X^0 e_X^0(\xi' \otimes_B \xi^*) &= e_X^0(\epsilon_X {}_A[\xi', \xi]) = e_X^0(\epsilon_X)_A[\xi', \xi] \\ &= \sum_j e_X^0(\mu_j \otimes_B \mu_j^*)_A[\xi', \xi] \\ &= \epsilon_X \sum_j {}_A[\mu_j, \mu_j]_A[\xi', \xi] \\ &= \dim(X)_B^{\tau_B} \epsilon_X {}_A[\xi', \xi] \qquad \text{(by Lemma 41)} \\ &= \dim(X)_B^{\tau_B} e_X^0(\xi' \otimes_B \xi^*), \end{aligned}$$

shows that

$$e_X = \frac{1}{\dim(X)_B^{\tau_B}} e_X^0$$

is a projection in $End({}_A X \otimes_B X_A^*)$, which is called the **Jones projection**.

Lemma 43. *In $X \otimes_B X^* \otimes_A X$, we have the relation*

$$(1 \otimes e_{X^*})(e_X \otimes 1)(1 \otimes e_{X^*}) = [X]^{-1}(1 \otimes e_{X^*}).$$

$\because$) Take bases $\{\lambda_i\}$ for ${}_AX$ and $\{\mu_j\}$ for X_B. For ξ_1, ξ_2, $\xi_3 \in X_0$,

$$
\begin{aligned}
&(\dim X_B)(\dim {}_AX)^2(1 \otimes e_{X^*})(e_X \otimes 1)(1 \otimes e_{X^*})(\xi_1 \otimes_B \xi_2^* \otimes_A \xi_3) \\
&= (1 \otimes e^0_{X^*})(e^0_X \otimes 1)(1 \otimes e^0_{X^*})(\xi_1 \otimes_B \xi_2^* \otimes_A \xi_3) \\
&= \sum_i (1 \otimes e^0_{X^*})(e^0_X \otimes 1)(\xi_1 \otimes \lambda_i^* \otimes \lambda_i[\xi_2, \xi_3]_B) \\
&= \sum_{i,j} (1 \otimes e^0_{X^*})(\mu_j \otimes \mu_j^* {}_A[\xi_1, \lambda_i] \otimes \lambda_i[\xi_2, \xi_3]_B) \\
&= \sum_{i,j} \mu_j \otimes \epsilon_{X^*}[{}_A[\lambda_i, \xi_1]\mu_j, \lambda_i[\xi_2, \xi_3]_B]_B \\
&= \sum_{i,j} \mu_j \otimes \epsilon_{X^*}[\mu_j, {}_A[\xi_1, \lambda_i]\lambda_i[\xi_2, \xi_3]_B]_B \\
&= \sum_j \mu_j \otimes \epsilon_{X^*}[\mu_j, \xi_1[\xi_2, \xi_3]_B]_B \\
&= \sum_j \mu_j[\mu_j, \xi_1[\xi_2, \xi_3]_B]_B \otimes \epsilon_{X^*} \\
&= \xi_1[\xi_2, \xi_3]_B \otimes \epsilon_{X^*} \\
&= (1 \otimes e^0_{X^*})(\xi_1 \otimes \xi_2^* \otimes \xi_3) \\
&= \dim {}_AX((1 \otimes e_{X^*})(\xi_1 \otimes \xi_2^* \otimes \xi_3).
\end{aligned}
$$

□

Now we define towers of algebras.

Definition 44. *Define two sequences of algebras by*

$$A_1 = End(X_B) \subset A_2 = End(X \otimes_B X_A^*) \subset A_3 = End(X \otimes_B X^* \otimes_A X_B) \subset \dots,$$

$$B_1 = End({}_AX)^{opp} \subset B_2 = End({}_BX^* \otimes_A X)^{opp} \subset B_3 = End({}_AX \otimes_B X^* \otimes_A X)^{opp} \subset \dots,$$

which are called the **towers of algebras** *associated with a bimodule ${}_AX_B$.*

On A_i and B_j, we introduce normalized traces τ_{A_i} and τ_{B_j} as the normalization of canonical traces associated with defining modules of each algebras.

In $V \equiv \cdots \otimes_A X \otimes_B X^* \otimes_A X \otimes_B \dots$ (a tensor product of finite number of bimodules), mark one of $\otimes$'s and split the tensor product into two parts, and then consider the algebras of intertwiners in the left and right sides of the splitting: For example, if we split at a point in which A acts, then the left and right algebras in the tensor product are defined by

$$End(\cdots \otimes_B X_A^*), \qquad End({}_AX \otimes_B \dots).$$

They are one of algebras in the towers of algebras and, by Lemma 19, commutants of each other. In particular, the original tensor product V admits a structure of bimodule.

Similarly, if we mark two points in the tensoring operations and divide V into three parts, then the left and the right parts give rise to algebras on V which commute each other. In this way, we have many kinds of bimodule structures in V. Say, $V = {}_{A_i}V_{B_j}$ in this fashion. Then, by Proposition 38 (i), τ_{A_i} and τ_{B_j} are balanced through V.

Remark. The sets of left-(or right-)bounded elements are the same for any bimodule structures described above (cf. Lemma 23 (i)).

Lemma 45. *Let A_1 act on $X \otimes_B X^*$ from left and right and regard it an A_1-A_1 bimodule.*
(i) $\epsilon_X \in (X \otimes_B X^*)^{A_1}$.
(ii) $(a_1\epsilon_X|\epsilon_X) = \dim X_B \tau_{A_1}(a_1)$, *for* $a_1 \in A_1$.
(iii) $A_1 \ni a_1 \mapsto a_1\epsilon_X$ *extends to an isometric isomorphism* $L^2(A_1) \to X \otimes_B X^*$, *and* $(X \otimes_B X^*, \epsilon_X)$ *gives a standard representation of* A_1 *(here the conjugation* $*$ *stands for the* J*-involution in the standard representation).*

$\because$) (i) For $a_1 \in A_1 = End(X_B)$, and ξ, $\xi' \in X_0$,

$$\begin{aligned}((a_1 \otimes_B 1)\epsilon_X|\xi' \otimes_B \xi^*) &= (a_1\xi|\xi') \quad \text{(by Lemma 42 (i))}\\ &= (\epsilon_X|\xi' \otimes_B \xi^* a_1^*) \quad \text{(by Lemma 42 (i))}\\ &= (\epsilon_X|(\xi' \otimes_B \xi^*)(1 \otimes_B a_1)^*)\\ &= (\epsilon_X(1 \otimes_B a_1)|\xi' \otimes_B \xi^*).\end{aligned}$$

(ii) For a basis $\{\mu_j\}$ of X_B,

$$\begin{aligned}(a_1\epsilon_X|\epsilon_X) &= \sum_j (a_1\mu_j \otimes_B \mu_j^*|\epsilon_X) \quad \text{(by Lemma 42 (ii))}\\ &= \sum_j (a_1\mu_j|\mu_j) \quad \text{(by Lemma 42 (i))}\\ &= \tau'_{X_B}(a_1) \qquad \text{(by Corollary 13 (i))}\\ &= \tau'_{X_B}(1)\tau_{A_1}(a_1)\\ &= \dim X_B \tau_{A_1}(a_1).\end{aligned}$$

(iii) By (ii) just proved, we need to show the following two facts; ϵ_X is cyclic for A_1 and $(a_1\epsilon_X)^* = \epsilon_X a_1^*$ for $a_1 \in A_1$.

For the proof of cyclicity, we express X_B as $L^2(B)_B \oplus \cdots \oplus L^2(B)_B \oplus pL^2(B)_B$ and use the matrix form of $End(X_B)$. Then it is easy to see that $\{(a_1 \otimes_B 1)\epsilon_X; a_1 \in A_1\}$ (ϵ_X should be written in terms of the basis associated with the above expression) is a dense linear subspace of

$$X \otimes_B X^* \cong (L^2(B)_B \oplus \cdots \oplus L^2(B)_B \oplus pL^2(B)_B) \otimes_B (L^2(B)_B \oplus \cdots \oplus L^2(B)_B \oplus pL^2(B)_B).$$

The property related with the involution $*$ follows from

$$\begin{aligned}((a_1\epsilon_X)^*|\xi'\otimes_B\xi^*) &= (\xi\otimes_B\xi'^*|a_1\epsilon_X)\\ &= (a_1^*\xi\otimes_B\xi'^*|\epsilon_X)\\ &= (a_1^*\xi|\xi')\\ &= (\epsilon_X|\xi'\otimes_B\xi^* a_1)\\ &= (\epsilon_X a_1^*|\xi'\otimes_B\xi^*).\end{aligned}$$

□

Corollary 46. ${}_{A_1}[\varphi,\epsilon_Z]\epsilon_X = (\dim X_B)\varphi$ *for* $\varphi\in(X\otimes_B X^*)_0$.

∵) This follows from the fact that $(\dim X_B)^{-1/2}\epsilon_X$ corresponds to 1_{A_1} in $L^2(A_1)$ (cf. Example 10). □

Proposition 47. *The representation of* A_i *on* $(X\otimes_B X^*)\otimes_A\cdots\otimes_A(X\otimes_B X^*)$ *(i-times) is a standard representation with* $*$ *the canonical involution.*

∵) Instead of ${}_AX_B$, use ${}_{A_i}(X\otimes_B X^*\otimes_A X\otimes_B\cdots\otimes(X^*\text{ or }X))_{(A\text{ or }B)}$ in Lemma 45 (iii). □

Definition 48. *Define a sequence of projections* $e_1,e_2,\dots$ *in* $X\otimes_B X^*\otimes_A X\otimes_B\dots$ *(finitely many tensor product) by*

$$\begin{aligned}e_1 &= e_X\otimes_A 1_X\otimes_B 1_{X^*}\otimes_A\dots\\ e_2 &= 1_X\otimes_B e_{X^*}\otimes_B 1_{X^*}\otimes_A\dots\\ e_3 &= 1_X\otimes_B 1_{X^*}\otimes_A e_X\otimes_A\dots\\ &\ \vdots\end{aligned}$$

Lemma 49. *We have*

$$(1_X\otimes_B End(X_B)^{opp})\cap\{e_X\}' = 1_X\otimes_B A \quad \text{in } End({}_AX\otimes_B X^*).$$

∵) Let $S\in End(X_B)$ and suppose that the right action of $1_X\otimes_B S$ commutes with e_X. From

$$(e_X(\xi'\otimes_B\xi^*))(1\otimes_B S) = ({}_A[\xi',\xi]\epsilon_X)(1\otimes_B S) = {}_A[\xi',\xi]\epsilon_X(1\otimes_B S)$$

and

$$e_X((\xi'\otimes_B\xi^*)(1\otimes_B S)) = e_X(\xi'\otimes_B(S^*\xi)^*) = {}_A[\xi',S^*\xi]\epsilon_X,$$

we have

$${}_A[\xi',\xi]\epsilon_X(1\otimes_B S) = {}_A[\xi',S^*\xi]\epsilon_X.$$

Taking ${}_A[\epsilon_X, \cdot]$ in this relation (note that $\epsilon_X(1 \otimes_B S) \in (X \otimes_B X^*)_0$ since $\epsilon_X \in (X \otimes_B X^*)_0$) and then applying $\tau_A(\cdot)$, we have

$$({}_A[\epsilon_X, \epsilon_X(1 \otimes_B S)]\xi|\xi') = ({}_A[\epsilon_X, \epsilon_X]S^*\xi|\xi').$$

Since ${}_A[\epsilon_X, \epsilon_X] = \dim X_B 1_A$ (Lemma 42 (v)), this implies

$$S(\xi') = (\dim X_B)^{-1}{}_A[\epsilon_X(1 \otimes_B S), \epsilon_X]\xi',$$

and hence S is expressed as a left action of an element in A. □

Lemma 50. *Let ${}_AX_B$, ${}_BY_C$ be bimodules with balanced traces and set ${}_AZ_C = {}_AX \otimes_B Y_C$. If $A = End(X_B)$, then*

$$e_Z = 1_X \otimes_B e_Y \otimes_B 1_{X^*} \quad \text{in } X \otimes_B Y \otimes_C Y^* \otimes_B X^*.$$

∵) Since $A = B'$, if we use Lemma 19 in $Z \otimes_C Z^* = X \otimes_B Y \otimes_C Y^* \otimes_B X^*$, we have

$$End({}_AZ \otimes_C Z^*) = 1_X \otimes_X End({}_BY \otimes_C Y^* \otimes_B X^*)$$
$$End(Z \otimes_C Z^*_A) = End(X \otimes_B Y \otimes_C Y^*_B) \otimes_B 1_{X^*},$$

and then

$$End({}_AZ \otimes_C Z^*_A) = 1_X \otimes_B End({}_BY \otimes_C Y^*_B) \otimes_B 1_{X^*}.$$

Since $e_Z \in End({}_AZ \otimes_C Z^*_A)$, we can set

$$e_Z^0 = 1_X \otimes_B T \otimes_B 1_{X^*},$$

with $T \in End({}_BY \otimes_C Y^*_B)$. Then, for a basis $\{\lambda_i\}$ of ${}_AX$ and $\eta_1, \eta_2 \in Y_0$,

$$\sum_{i,j}(\lambda_i \otimes_B \eta_1 \otimes_C \eta_1^* \otimes_B \lambda_j^*|e_Z^0(\lambda_i \otimes_B \eta_2 \otimes_C \eta_2^* \otimes_B \lambda_j^*))$$
$$= \sum_{i,j}(\lambda_i \otimes_B \eta_1 \otimes_C \eta_1^* \otimes_B \lambda_j^*|\lambda_i \otimes_B T(\eta_2 \otimes_C \eta_2^*) \otimes_B \lambda_j^*).$$

Since $e_Z^0(\zeta' \otimes_C \zeta^*) = \epsilon_{ZA}[\zeta', \zeta]$, the left hand side is calculated as

$$\begin{aligned}
&\sum_{i,j}(\lambda_i \otimes_B \eta_1 \otimes_C \eta_1^* \otimes_B \lambda_j^* | \epsilon_{ZA}[\lambda_i \otimes_B \eta_2, \lambda_j \otimes_B \eta_2]) \\
&\quad = \sum_{i,j}(\lambda_i \otimes_B \eta_1 | {}_A[\lambda_i \otimes_B \eta_2, \lambda_j \otimes_B \eta_2]\lambda_j \otimes_B \eta_1) \\
&\quad = \sum_{i,j}(\lambda_i \otimes_B \eta_1 | {}_A[\lambda_i, \lambda_j\, {}_B[\eta_2, \eta_2]]\lambda_j \otimes_B \eta_1) \\
&\quad = \sum_j((\sum_i {}_A[\lambda_j\, {}_B[\eta_2, \eta_2], \lambda_i]\lambda_i) \otimes_B \eta_1 | \lambda_j \otimes_B \eta_1) \\
&\quad = \sum_j(\lambda_j\, {}_B[\eta_2, \eta_2] \otimes_B \eta_1 | \lambda_j \otimes_B \eta_1) \\
&\quad = \sum_j(\lambda_j\, {}_B[\eta_2, \eta_2]\, {}_B[\eta_1, \eta_1] | \lambda_j) \\
&\quad = \sum_j \tau_B({}_B[\eta_2, \eta_2]\, {}_B[\eta_1, \eta_1][\lambda_j, \lambda_j]_B) \\
&\quad = \dim {}_A X\, \tau_B({}_B[\eta_2, \eta_2]\, {}_B[\eta_1, \eta_1]) \quad \text{(Lemma 41)} \\
&\quad = \dim {}_A X(\eta_1 \otimes_C \eta_1^* | e_Y^0(\eta_2 \otimes_C \eta_2^*)).
\end{aligned}$$

On the other hand, the right hand side is calculated as

$$\begin{aligned}
&\sum_{i,j}(\lambda_i \otimes_B \eta_1 \otimes_C \eta_1^* \otimes_B \lambda_j^* | \lambda_i \otimes_B T(\eta_2 \otimes_C \eta_2^*) \otimes_B \lambda_j^*) \\
&\quad = \sum_{i,j}([\lambda_i, \lambda_i]_B \eta_1 \otimes_C \eta_1^* | T(\eta_2 \otimes_C \eta_2^*)[\lambda_j, \lambda_j]_B) \\
&\quad = (\dim {}_A X)^2(\eta_1 \otimes_C \eta_1^* | T(\eta_2 \otimes_C \eta_2^*)).
\end{aligned}$$

Comparing the results (note that $\dim Z_C = \dim X_B \dim Y_C$, and $\dim {}_A X \dim X_B = 1$ since $A = B'$), we have

$$e_Y = (\dim Z_C)^{-1} T,$$

i.e.,

$$e_Z = 1_X \otimes_B e_Y \otimes_B 1_{X^*}.$$

□

Proposition 51.
(i) $\{e_i\}_{i \geq 1}$ *satisfies the following relations:*

$$\begin{aligned}
e_{i\pm1} e_i e_{i\pm1} &= [X]^{-1} e_{i\pm1}, \\
e_i e_j &= e_j e_i, \qquad |i - j| \geq 2.
\end{aligned}$$

(ii) $A_i = \langle A_1, e_1, \ldots, e_{i-1}\rangle$ $(i \geq 2)$.

$\because$) (i) is a consequence of Lemma 43.

(ii) We shall prove by the induction on i. When $i = 2$, this is a consequence of Lemma 49. Suppose that this is valid for some i. To be specific, consider the case $i = 2k$. Applying Lemma 49 to

$$Z = {}_{A_{2k-1}}(\overbrace{X \otimes_B X^* \otimes_A \cdots \otimes_A X}^{2k-1 \text{ times}}) \otimes_B X^*_A,$$

We have

$$\langle A_i \otimes_B 1_{Z^*}, e_Z\rangle = A_{i+1} \otimes_A 1.$$

Since $e_Z = e_i \otimes_A 1$ by Lemma 50, the induction proceeds. $\square$

References

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