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<td>Nomura, Kazumasa</td>
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Kyoto University
Spin Models Constructed from Hadamard matrices

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Tokyo Ikashika University Kazumasa Nomura

A new spin model $M$ is constructed from an arbitrary Hadamard matrix $H$ through a distance-regular graph which is called a Hadamard graph. F. Jaeger gives a formula for the link invariant of the model $M$, and V. F. R. Jones gives two links which have the same V-polynomial but different polynomials of $M$.

1 Definition of a Spin Model

The following definition is essentially due to V. F. R. Jones [8].

**Definition 1** Let $n$ be a positive integer, $D$ be one of the square roots of $n$. A spin model with loop variable $D$ is a pair $(X, w)$ of a finite non-empty set $X$ of size $n$, and a complex-valued symmetric function $w$ on $X \times X$ which satisfy the following equations for all $\alpha, \beta, \gamma \in X$:

$\frac{1}{n} \sum_{x \in X} \frac{w(\alpha, x)}{w(\beta, x)} = \delta_{\alpha, \beta}$ (1)

$\frac{1}{D} \sum_{x \in X} \frac{w(\alpha, x)w(\beta, x)}{w(\gamma, x)} = \frac{w(\alpha, \beta)}{w(\alpha, \gamma)w(\beta, \gamma)}$ (2)

Each element of $X$ is called a spin, and the function $w$ is called Boltzmann weight. The $(n \times n)$-matrix $W = (w(\alpha, \beta))$, is called the weight matrix of the spin model. The equation (2) is called star-triangle relation.

**Example** Let $X$ be a finite set of size $n = D^2 > 1$ and let $a, b$ be complex numbers such that

$b^2 + \frac{1}{b^2} + D = 0, \quad a = -\frac{1}{b^2}$.

Define a function $w$ by

$w(\alpha, \beta) = \begin{cases} a & \text{if } \alpha = \beta \\ b & \text{if } \alpha \neq \beta \end{cases}$

As easily shown, $(X, w)$ becomes a spin model with the weight matrix

$M = (a - b)I + bJ$.

This spin model is called Potts model.

**Remark 1** If $(X, w)$ is a spin model with $D = \sqrt{n}$, then $(X, \sqrt{-1}w)$ becomes a spin model with $D = -\sqrt{n}$. 
Remark 2 Under (1), the star-triangle relation (2) is equivalent to:

$$
\frac{1}{D} \sum_{x \in X} \frac{w(\alpha,x)}{w(\beta,x)w(\gamma,x)} = \frac{w(\alpha,\beta)w(\alpha,\gamma)}{w(\beta,\gamma)}.
$$

(3)

Remark 3 By putting $\beta = \gamma$ in 2, we get

$$
\frac{1}{D} \sum_{x \in X} w(\alpha, x) = \frac{1}{w(\beta,\beta)}.
$$

This shows $w(\beta,\beta)$ is independent on the choice of $\beta \in X$:

$$
w(\beta,\beta) = a
$$

is a constant called modulus of the model. Thus we have

$$
\frac{1}{D} \sum_{x \in X} w(\alpha, x) = \frac{1}{a}.
$$

From 3, we have

$$
\frac{1}{D} \sum_{x \in X} \frac{1}{w(\alpha,x)} = a.
$$

Remark 4 The equation (1) is equivalent to

$$
\sum_{x \in X} \frac{w(\alpha,x)}{w(\beta,x)} = 0 \quad \text{if } \alpha \neq \beta.
$$

2 Spin Models on Distance-Regular Graphs

A connected graph $\Gamma$ is said to be distance-regular if there are integers $b_i, c_i$ ($i \geq 0$) such that for any two vertices $u, x$ at distance $i = \partial(u,x)$, there are precisely $c_i$ neighbours of $x$ in $\Gamma_{i-1}(u)$ and $b_i$ neighbours of $x$ in $\Gamma_{i+1}(u)$. In particular, $\Gamma$ is regular of valency $k = b_0$. The sequence

$$
\iota(\Gamma) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\},
$$

where $d$ is the diameter of $\Gamma$, is called the intersection array of $G$. For two vertices $u, v$, the size

$$
p_{ij}^\alpha = |\Gamma_i(u) \cap \Gamma_j(v)|
$$

depends only on the distance $\alpha = \partial(u,v)$, rather than the individual vertices $u, v$ with $\partial(u,v) = \alpha$ (see [4] 4.1). In particular $k_i = |\Gamma_i(u)|$, which is called the $i$-th valency, does not depend on the choice of a vertex $u$. For three vertices $u, v, w$, put

$$
P_{ij}(u,v,w) = |\Gamma_i(u) \cap \Gamma_j(v) \cap \Gamma_l(w)|.
$$

More precise descriptions about distance-regular graphs will be found in [3], [4].

The following Proposition is obtained directly from the definition and remarks in the previous section.
Proposition 1 Let $\Gamma$ be a distance-regular graph of diameter $d$ with the vertex set $X$. Put $|X| = n$ and let $D$ be one of the square roots of $n$. Let $t_0, t_1, \ldots, t_d$ be non-zero complex numbers and let $w$ be the complex valued function on $X \times X$ defined by $w(u, v) = t_i$ where $i = \partial(u, v)$. Then $(X, w)$ becomes a spin model if and only if the following conditions hold:

\[(C1) \sum_{i=0}^{d} k_i t_i = D t_0^{-1},\]
\[(C2) \sum_{i=0}^{d} k_i t_i^{-1} = D t_0,\]
\[(C3) \sum_{i=0}^{d} \sum_{j=0}^{d} p_{ij} t_i t_j^{-1} = 0 \quad (\alpha = 1, 2, \ldots, d),\]
\[(C4) \text{For all vertices } u, v, w \text{ in } X,\]
\[\sum_{t=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} P_{ijt}(u, v, w) t_i t_j t_t^{-1} = D t_\alpha t_\beta t_\gamma^{-1},\]

where $\alpha = \partial(u, v), \beta = \partial(u, w), \gamma = \partial(v, w)$.

Remark 5 Though conditions (C1) and (C2) can be removed in the above, these are useful to find solutions of the equations.

3 Result

A distance-regular graph having the intersection array

\[\{4m, 4m-1, 2m, 1; 1, 2m, 4m-1, 4m\}\]

is called a Hadamard graph of order $4m$. There is a natural one-to-one correspondence between Hadamard graphs of order $4m$ and Hadamard matrices of order $4m$ (see [4] 1.8). Now our main result follows:

Theorem 2 Let $\Gamma$ be a Hadamard graph of order $4m$. Let $s, t_0, t_4$ be complex numbers such that

\[s^2 + 2(2m-1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{m}}{(4m-1)s+1}, \quad t_4^4 = 1.\]

Put $t_2 = st_0, t_3 = -t_1$ and $t_4 = t_0$. Then $t_0, \ldots, t_4$ satisfy the conditions in Proposition 1 with $D = 4\sqrt{m}$.

Theorem 2 can be described without using distance-regular graphs as follows:
Theorem 3 Let $H$ be a Hadamard matrix of order $n$, $n \equiv 0 \pmod{4}$, and let $M$ be the weight matrix of the Potts model of size $n$. Let $\omega$ be one of the 4-th roots of 1, $\omega^4 = 1$. Define a $4n \times 4n$-matrix $W$ as:

\[
W = \begin{pmatrix}
  M & M & \omega H & -\omega H \\
  M & M & -\omega H & \omega H \\
  \omega H^t & -\omega H^t & M & M \\
  -\omega H^t & \omega H^t & M & M
\end{pmatrix}
\]

Then $W$ becomes the weight matrix of a spin model having $4n$ spins.

4 Proof of Theorem 2

Let $H$ be a Hadamard graph of order $4m$ and let $s, t_0, \ldots, t_4$ be complex numbers such that

\[ s^2 + 2(2m-1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{m}}{(4m-1)s + 1}, \]
\[ t_1^4 = 1, \quad t_2 = st_0, \quad t_3 = -t_1, \quad t_4 = t_0. \]

By $k_{i-1}b_{i-1} = k_i c_i$, we get

\[ k_0 = 1, \quad k_1 = 4m, \quad k_2 = 8m - 2, \quad k_3 = 4m, \quad k_4 = 1. \]

So (C1) becomes

\[ t_0 + 4mt_1 + (8m - 2)t_2 + 4mt_3 + t_4 = 4\sqrt{m} t_0^{-1}. \]

By $t_3 = -t_1$, $t_0 = t_4$ and $t_2 = st_0$, this becomes

\[ 2t_0 + (8m - 2)st_0 = 4\sqrt{m} t_0^{-1}. \]

Clearly this holds by the assumption $t_0^2 = 2\sqrt{m}((4m-1)s + 1)^{-1}$.

Condition (C2) becomes

\[ t_0^{-1} + 4mt_1^{-1} + (8m - 2)t_2^{-1} + 4mt_3^{-1} + t_4^{-1} = 4\sqrt{m} t_0, \]

and it becomes

\[ 2t_0^{-1} + (8m - 2)t_2^{-1} = 4\sqrt{m} t_0, \]
\[ 1 + (4m - 1)s^{-1} = 2\sqrt{m} t_0^2. \]

By the assumption $t_0^2 = 2\sqrt{m}((4m-1)s + 1)^{-1}$, it is equivalent to

\[ 1 + (4m - 1)s^{-1} = 2\sqrt{m} \cdot 2\sqrt{m} ((4m - 1)s + 1)^{-1}. \]
This is implied by the assumption \( s^2 + 2(2m - 1)s + 1 = 0 \).

Next consider condition (C3). The values of \( p_{ij}^\alpha \) are easily computed by the following formula ([4] 4.1.7).

\[
p_{j+1,t}^\alpha = \frac{1}{c_{j+1}}(p_{j-t-1}^\alpha b_{t-1} + p_{j-t+1}^\alpha c_{t+1} - p_{j-t-1}^\alpha b_{t-1}).
\]

Case \( \alpha = 1 \):

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>( p_{ij}^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1), (1, 0), (3, 4), (4, 3)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2), (2, 1), (2, 3), (3, 2)</td>
<td>4m - 1</td>
</tr>
</tbody>
</table>

Condition (C3) becomes

\[
t_0 t_1^{-1} + t_1 t_0^{-1} + t_3 t_4^{-1} + t_4 t_3^{-1} + (4m - 1) (t_1 t_2^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_2^{-1}) = 0.
\]

This holds by \( t_3 = -t_1 \) and \( t_0 = t_4 \).

Case \( \alpha = 2 \):

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>( p_{ij}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2), (2, 0), (2, 4), (4, 2)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 1), (1, 3), (3, 1), (3, 3)</td>
<td>2m</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>8m - 4</td>
</tr>
</tbody>
</table>

(C3) becomes

\[
t_0 t_2^{-1} + t_2 t_0^{-1} + t_4 t_4^{-1} + t_4 t_2^{-1} + 2m(t_1 t_1^{-1} + t_1 t_3^{-1} + t_3 t_1^{-1} + t_3 t_3^{-1}) + (8m - 4) = 0.
\]

This is implied by \( t_3 = -t_1 \), \( t_0 = t_4 \), \( t_2 = st_0 \) and \( s^2 + 2(2m - 1)s + 1 = 0 \).

Case \( \alpha = 3 \):

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>( p_{ij}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3), (3, 0), (1, 4), (4, 1)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2), (2, 1), (2, 3), (3, 2)</td>
<td>4m - 1</td>
</tr>
</tbody>
</table>

\[
t_0 t_3^{-1} + t_3 t_0^{-1} + t_1 t_4^{-1} + t_4 t_1^{-1} + (4m - 1) (t_1 t_2^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_2^{-1}) = 0.
\]

This holds by \( t_3 = -t_1 \) and \( t_0 = t_4 \).

Case \( \alpha = 4 \):

<table>
<thead>
<tr>
<th>(i, j)</th>
<th>( p_{ij}^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 4), (4, 0)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3), (3, 1)</td>
<td>4m</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>8m - 2</td>
</tr>
</tbody>
</table>
\[ t_0t_4^{-1} + t_4t_0^{-1} + 4m(t_1t_3^{-1} + t_3t_1^{-1}) + (8m - 2)t_2t_2^{-1} = 0. \]

Clearly this holds.

Now we consider condition (C4). Since (C4) is symmetric in \( u, v \), we may assume \( \partial(u, w) \leq \partial(v, w) \). Fix three vertices \( u, v, w \). Put \( \partial(u, v) = \alpha, \partial(u, w) = \beta, \partial(v, w) = \gamma \) and \( P_{ij\ell} = P_{ij\ell}(u, v, w) \). If \( \beta = 0 \), we have \( u = w, \alpha = \gamma \), and \( P_{ij\ell} = 0 \) for \( i \neq \ell \). Therefore

\[ \sum_{i,j,\ell} P_{ij\ell} t_i t_j t_\ell^{-1} = \sum_{j} \sum_{i} P_{iji} t_j = \sum_{j} k_j t_j, \]

and (C4) is equivalent to (C1) in the case \( \beta = 0 \). So we must verify (C4) in each of the following cases of \( (\alpha, \beta, \gamma) \):

\[
\begin{array}{cccc}
(0,1,1) & (0,2,2) & (0,3,3) & (0,4,4) \\
(1,1,2) & (1,2,3) & (1,3,4) & \\
(2,1,1) & (2,1,3) & (2,2,2) & (2,2,4) & (2,3,3) \\
(3,1,2) & (3,1,4) & (3,2,3) & \\
(4,1,3) & (4,2,2) & \\
\end{array}
\]

In the case \( (\alpha, \beta, \gamma) \neq (2,2,2) \), the values of \( P_{ij\ell} \) are easily computed. We need the following Lemma for the case \( (\alpha, \beta, \gamma) = (2,2,2) \).

**Lemma 4** If \( \partial(u, v) = \partial(u, w) = \partial(v, w) = 2 \), then \( w \) has precisely \( m \) neighbours in \( \Gamma_1(u) \cap \Gamma_1(v) \).

**Proof.** Put \( D_j^i = \Gamma_i(u) \cap \Gamma_j(v) \). We have \( w \in D_2^2 \). Put \( e(w, D_1^1) = r, e(w, D_3^1) = s, e(w, D_3^3) = s', e(w, D_2^3) = r' \). Notice that every vertex \( x \in X \) has the unique opposite vertex \( x' \) such that \( \partial(x, x') = 4 \), since we have \( k_4 = 1 \). Since the opposite vertex \( x' \) of \( x \in D_1^1 \cap \Gamma_1(w) \) is in \( D_3^3 \), we get \( r' \leq |D_3^3| - r = 2m - r \). Similarly we get \( s' \leq 2m - s \). On the other hand, we have \( r + s = 2m \) since \( w \) has precisely \( 2m \) neighbours in \( \Gamma_1(u) \). We have also \( s + r' = 2m \) since \( w \) has \( 2m \) neighbours in \( \Gamma_3(v) \). These imply \( r = r' \). By the same reason, we get \( s = s' \). Therefore we must have \( r = s = r' = s' = m \).

**Case** \( (\alpha, \beta, \gamma) = (0,1,1) \):

<table>
<thead>
<tr>
<th>( (i,j,\ell) )</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0,0,1), (1,1,0), (3,3,4), (4,4,3) )</td>
<td>1</td>
</tr>
<tr>
<td>( (1,1,2), (2,2,1), (2,2,3), (3,3,2) )</td>
<td>( 4m - 1 )</td>
</tr>
</tbody>
</table>
So, condition (C4) becomes
\[ t_0^2t_1^{-1} + t_1^2t_0^{-1} + t_3^2t_4^{-1} + t_4^2t_3^{-1} + (4m - 1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) = Dt_0t_1^{-2}, \]
\[ 2t_1^2t_0^{-1} + (8m - 2)t_1^2t_2^{-1} = Dt_0t_1^{-2}. \]

By \( t_1^4 = 1 \), this is equivalent to (C2).

Case \((\alpha, \beta, \gamma) = (0, 2, 2)\):

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 2), (2, 2, 0), (2, 2, 4), (4, 4, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 1), (1, 1, 3), (3, 3, 1), (3, 3, 3))</td>
<td>2(m)</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>(8m - 4)</td>
</tr>
</tbody>
</table>

Then condition (C4) becomes
\[ 2(t_0^2t_2^{-1} + t_2^2t_0^{-1}) + (8m - 4)t_2 = Dt_0t_2^{-2}, \]
\[ s^{-1} + s^2 + (4m - 2)s = 2\sqrt{m}s^{-2}t_0^{-2}. \]

By the assumption \( t_0^2 = 2\sqrt{m}((4m - 1)s + 1)^{-1} \), this becomes
\[ s^{-1} + s^2 + (4m - 2)s = (4m - 1)s^{-1} + s^{-2}. \]

This is implied by the assumption \( s^2 + 2(2m - 1)s + 1 = 0 \).

Case \((\alpha, \beta, \gamma) = (0, 3, 3)\):

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 3), (1, 1, 4), (3, 3, 0), (4, 4, 1))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 2), (2, 2, 1), (2, 2, 3), (3, 3, 2))</td>
<td>(4m - 1)</td>
</tr>
</tbody>
</table>

(C4) becomes
\[ t_0^2t_3^{-1} + t_1^2t_4^{-1} + t_3^2t_0^{-1} + t_4^2t_1^{-1} + (4m - 1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) = Dt_0t_3^{-2}. \]

This is equivalent to Case \((\alpha, \beta, \gamma) = (0, 1, 1)\).

Case \((\alpha, \beta, \gamma) = (0, 4, 4)\):

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 4), (4, 4, 0))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 3), (3, 3, 1))</td>
<td>(4m)</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>(8m - 2)</td>
</tr>
</tbody>
</table>
$$t_0 t_4^{-1} + t_4 t_0^{-1} + 4m(t_3 t_3^{-1} + t_3 t_1^{-1}) + (8m - 2)t_2 t_2^{-1} = Dt_0 t_4^{-2}.$$  

Case $(\alpha, \beta, \gamma) = (1, 1, 2)$:

<table>
<thead>
<tr>
<th>$(i, j, \ell)$</th>
<th>$P_{ij\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1, 1), (1, 0, 2), (1, 2, 0), (3, 2, 4), (3, 4, 2), (4, 3, 3)$</td>
<td>1</td>
</tr>
<tr>
<td>$(2, 1, 1), (2, 3, 3)$</td>
<td>$2m - 1$</td>
</tr>
<tr>
<td>$(2, 1, 3), (2, 3, 1)$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$(1, 2, 2), (3, 2, 2)$</td>
<td>$4m - 2$</td>
</tr>
</tbody>
</table>

$$t_0 + t_4 + t_0t_1t_2^{-1} + t_1t_2t_0^{-1} + t_2t_3t_4^{-1} + t_3t_4t_2^{-1} + 2m(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) + (4m - 2)(t_1 + t_2 + t_3) = Dt_2^{-1}.$$  

Case $(\alpha, \beta, \gamma) = (1, 2, 3)$:

<table>
<thead>
<tr>
<th>$(i, j, \ell)$</th>
<th>$P_{ij\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1, 2), (1, 0, 3), (2, 1, 4), (2, 3, 0), (3, 4, 1), (4, 3, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 2, 3), (3, 2, 1)$</td>
<td>$2m - 1$</td>
</tr>
<tr>
<td>$(1, 2, 1), (3, 2, 3)$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$(2, 1, 2), (2, 3, 2)$</td>
<td>$4m - 2$</td>
</tr>
</tbody>
</table>

$$t_0t_1t_2^{-1} + t_0t_1t_3^{-1} + t_1t_2t_4^{-1} + t_2t_3t_0^{-1} + t_3t_4t_1^{-1} + (2m - 1)(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) + (4m - 2)(t_1 + t_3) + 4mt_2 = Dt_1t_2^{-1}t_3^{-1}.$$  

Case $(\alpha, \beta, \gamma) = (1, 3, 4)$:

<table>
<thead>
<tr>
<th>$(i, j, \ell)$</th>
<th>$P_{ij\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1, 3), (1, 0, 4), (3, 4, 0), (4, 3, 1)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 2, 2), (2, 1, 3), (2, 3, 1), (3, 2, 2)$</td>
<td>$4m - 1$</td>
</tr>
</tbody>
</table>

$$t_0t_1t_3^{-1} + t_0t_1t_4^{-1} + t_3t_4t_0^{-1} + t_3t_4t_1^{-1} + (4m - 1)(t_1 + t_3 + t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) = Dt_1t_3^{-1}t_4^{-1}.$$  

Case $(\alpha, \beta, \gamma) = (2, 1, 1)$:
<table>
<thead>
<tr>
<th>( (i, j, \ell) )</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2, 1), (2, 0, 1), (2, 4, 3), (4, 2, 3), (1, 1, 0), (3, 3, 4))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 2), (3, 3, 2))</td>
<td>(2m - 1)</td>
</tr>
<tr>
<td>((1, 3, 2), (3, 1, 2))</td>
<td>(2m)</td>
</tr>
<tr>
<td>((2, 2, 1), (2, 2, 3))</td>
<td>(4m - 2)</td>
</tr>
</tbody>
</table>

\[
t_1^2t_0^{-1} + t_3^2t_4^{-1} + 2(t_0t_2t_1^{-1} + t_2t_4t_3^{-1}) + (2m - 1)(t_1^2t_2^{-1} + t_3^2t_4^{-1}) + (4m - 2)(t_1^2t_4^{-1} + t_3^2t_3^{-1}) + 4mt_1t_3t_2^{-1} = Dt_4^{-1}.
\]

**Case \((\alpha, \beta, \gamma) = (2, 1, 3)\):**

<table>
<thead>
<tr>
<th>( (i, j, \ell) )</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2, 1), (2, 0, 3), (2, 4, 1), (4, 2, 3), (1, 3, 0), (3, 1, 4))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 3, 2), (3, 1, 2))</td>
<td>(2m - 1)</td>
</tr>
<tr>
<td>((1, 1, 2), (3, 3, 2))</td>
<td>(2m)</td>
</tr>
<tr>
<td>((2, 2, 1), (2, 2, 3))</td>
<td>(4m - 2)</td>
</tr>
</tbody>
</table>

\[
t_0t_2t_1^{-1} + t_0t_2t_3^{-1} + t_2t_4t_3^{-1} + t_2t_4t_3^{-1} + t_1t_3t_0^{-1} + t_1t_3t_4^{-1} + 2m(t_1^2t_3^{-1} + t_3^2t_1^{-1}) + 3m(t_1 + t_3) + (8m - 6)t_2 = Dt_4^{-1}.
\]

**Case \((\alpha, \beta, \gamma) = (2, 2, 2)\):**

<table>
<thead>
<tr>
<th>( (i, j, \ell) )</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2, 2), (2, 0, 2), (2, 2, 0), (2, 2, 4), (2, 4, 2), (4, 2, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3))</td>
<td>(m)</td>
</tr>
<tr>
<td>((3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3))</td>
<td>(m)</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>(8m - 6)</td>
</tr>
</tbody>
</table>

\[
t_1^2t_0^{-1} + t_3^2t_4^{-1} + 2(t_0 + t_4) + m(t_1^2t_3^{-1} + t_3^2t_1^{-1}) + 3m(t_1 + t_3) + (8m - 6)t_2 = Dt_4^{-1}.
\]

**Case \((\alpha, \beta, \gamma) = (2, 2, 4)\):**

<table>
<thead>
<tr>
<th>( (i, j, \ell) )</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 2, 2), (2, 0, 4), (2, 4, 0), (4, 2, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 3), (1, 3, 1), (3, 1, 3), (3, 3, 1))</td>
<td>(2m)</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>(8m - 4)</td>
</tr>
</tbody>
</table>

\[
t_0 + t_4 + t_0t_2t_4^{-1} + t_2t_4t_0^{-1} + 2m(t_1 + t_3 + t_1^2t_3^{-1} + t_3^2t_1^{-1}) + (8m - 4)t_2 = Dt_4^{-1}.
\]

**Case \((\alpha, \beta, \gamma) = (2, 3, 3)\):**
\( t_1^{t_4^{-1}} + t_2^{t_3^{-1}} + 2(t_0t_2^{-1}t_3^{-1} + t_2t_4^{-1}t_1^{-1}) + (2m-1)(t_1t_2^{-1} + t_3t_2^{-1}) \\
+ (4m-2)(t_1^{t_4^{-1}} + t_2^{t_3^{-1}}) + 4mt_1t_3^{-1} = Dt_2t_3^{-2}. \)

**Case** \( (\alpha, \beta, \gamma) = (3, 1, 2) \):

\[
\begin{array}{ll}
(i, j, \ell) & P_{ij\ell} \\
\hline
(0, 3, 1), (1, 2, 0), (3, 0, 2), (1, 4, 2), (3, 2, 4), (4, 1, 3) & 1 \\
(2, 1, 3), (2, 3, 1) & 2m - 1 \\
(2, 1, 1), (2, 3, 3) & 2m \\
(1, 2, 2), (3, 2, 2) & 4m - 2 \\
\end{array}
\]

\[
t_0t_3t_1^{-1} + t_0t_3t_4^{-1} + t_1t_2t_0^{-1} + t_1t_4t_3^{-1} + (4m - 1)(t_1 + t_3) \\
+ (4m - 1)(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) = Dt_3t_1^{-1}t_2^{-1}. 
\]

**Case** \( (\alpha, \beta, \gamma) = (3, 1, 4) \):

\[
\begin{array}{ll}
(i, j, \ell) & P_{ij\ell} \\
\hline
(0, 3, 1), (3, 0, 4), (1, 4, 0), (4, 1, 3) & 1 \\
(1, 2, 2), (2, 1, 3), (2, 3, 1), (3, 2, 2) & 4m - 1 \\
\end{array}
\]

\[
t_0t_3t_1^{-1} + t_0t_3t_4^{-1} + t_1t_4t_0^{-1} + t_1t_4t_3^{-1} + (4m - 1)(t_1 + t_3) \\
+ (4m - 1)(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) = Dt_3t_1^{-1}t_4^{-1}. 
\]

**Case** \( (\alpha, \beta, \gamma) = (3, 2, 3) \):

\[
\begin{array}{ll}
(i, j, \ell) & P_{ij\ell} \\
\hline
(0, 3, 2), (2, 1, 4), (3, 0, 3), (1, 4, 1), (2, 3, 0), (4, 1, 2) & 1 \\
(1, 2, 1), (3, 2, 3) & 2m - 1 \\
(1, 2, 3), (3, 2, 1) & 2m \\
(2, 1, 2), (2, 3, 2) & 4m - 2 \\
\end{array}
\]

\[
t_0 + t_4 + t_0t_3t_2^{-1} + t_2t_3t_0^{-1} + t_1t_2t_4^{-1} + t_1t_4t_2^{-1} + 2m(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) \\
+ (4m - 2)(t_1 + t_2 + t_3) = Dt_2^{-1}. 
\]

**Case** \( (\alpha, \beta, \gamma) = (4, 1, 3) \):
$t_0t_4t_1^{-1} + t_0t_4t_3^{-1} + t_1t_3t_0^{-1} + t_1t_3t_4^{-1} + (4m-1)(t_2^2t_1^{-1} + t_2^2t_3^{-1})$
$+ (8m - 2)t_1t_3t_2^{-1} = Dt_4t_1^{-1}t_3^{-1}$.

Case $(\alpha, \beta, \gamma) = (4,2,2)$:

$t_2^2t_0^{-1} + t_2^2t_4^{-1} + 2t_0t_4t_2^{-1} + 4m(t_1 + t_3) + (8m - 4)t_2 = Dt_4t_2^{-2}$.

### 参考文献


