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<td>Nomura, Kazumasa</td>
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Spin Models Constructed from Hadamard matrices

Tokyo Ikashika University  Kazumasa Nomura

A new spin model $M$ is constructed from an arbitrary Hadamard matrix $H$ through a distance-regular graph which is called a Hadamard graph. F. Jaeger gives a formula for the link invariant of the model $M$, and V. F. R. Jones gives two links which have the same V-polynomial but different polynomials of $M$.

1 Definition of a Spin Model

The following definition is essentially due to V. F. R. Jones [8].

**Definition 1** Let $n$ be a positive integer, $D$ be one of the square roots of $n$. A spin model with loop variable $D$ is a pair $(X, w)$ of a finite non-empty set $X$ of size $n$, and a complex-valued symmetric function $w$ on $X \times X$ which satisfy the following equations for all $\alpha, \beta, \gamma \in X$:

\[
\frac{1}{n} \sum_{x \in X} \frac{w(\alpha, x)}{w(\beta, x)} = \delta_{\alpha, \beta} \tag{1}
\]

\[
\frac{1}{D} \sum_{x \in X} \frac{w(\alpha, x)w(\beta, x)}{w(\gamma, x)} = \frac{w(\alpha, \beta)}{w(\alpha, \gamma)w(\beta, \gamma)} \tag{2}
\]

Each element of $X$ is called a spin, and the function $w$ is called Boltzmann weight. The $(n \times n)$-matrix $W = (w(\alpha, \beta))$, is called the weight matrix of the spin model. The equation (2) is called star-triangle relation.

**Example** Let $X$ be a finite set of size $n = D^2 > 1$ and let $a, b$ be complex numbers such that

\[
b^2 + \frac{1}{b^2} + D = 0, \quad a = -\frac{1}{b^2}.
\]

Define a function $w$ by

\[
w(\alpha, \beta) = \begin{cases} 
a & \text{if } \alpha = \beta \\ 
b & \text{if } \alpha \neq \beta
\end{cases}
\]

As easily shown, $(X, w)$ becomes a spin model with the weight matrix

\[
M = (a - b)I + bJ.
\]

This spin model is called Potts model.

**Remark 1** If $(X, w)$ is a spin model with $D = \sqrt{n}$, then $(X, \sqrt{-1}w)$ becomes a spin model with $D = -\sqrt{n}$. 
Remark 2 Under (1), the star-triangle relation (2) is equivalent to:

\[ \frac{1}{D} \sum_{x \in X} \frac{w(\alpha, x)}{w(\beta, x)w(\gamma, x)} = \frac{w(\alpha, \beta)w(\alpha, \gamma)}{w(\beta, \gamma)}. \]  (3)

Remark 3 By putting \( \beta = \gamma \) in (2), we get

\[ \frac{1}{D} \sum_{x \in X} w(\alpha, x) = \frac{1}{w(\beta, \beta)}. \]

This shows \( w(\beta, \beta) \) is independent on the choice of \( \beta \in X \):

\[ w(\beta, \beta) = a \]

is a constant called modulus of the model. Thus we have

\[ \frac{1}{D} \sum_{x \in X} w(\alpha, x) = \frac{1}{a}. \]

From 3, we have

\[ \frac{1}{D} \sum_{x \in X} \frac{1}{w(\alpha, x)} = a. \]

Remark 4 The equation (1) is equivalent to

\[ \sum_{x \in X} \frac{w(\alpha, x)}{w(\beta, x)} = 0 \quad \text{if} \quad \alpha \neq \beta. \]

2 Spin Models on Distance-Regular Graphs

A connected graph \( \Gamma \) is said to be distance-regular if there are integers \( b_i, c_i \) (\( i \geq 0 \)) such that for any two vertices \( u, x \) at distance \( i = \partial(u, x) \), there are precisely \( c_i \) neighbours of \( x \) in \( \Gamma_{i-1}(u) \) and \( b_i \) neighbours of \( x \) in \( \Gamma_{i+1}(u) \). In particular, \( \Gamma \) is regular of valency \( k = b_0 \). The sequence

\[ \iota(\Gamma) = \{ b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d \}, \]

where \( d \) is the diameter of \( \Gamma \), is called the intersection array of \( G \). For two vertices \( u, v \), the size

\[ p_{ij}^\alpha = |\Gamma_i(u) \cap \Gamma_j(v)| \]

depends only on the distance \( \alpha = \partial(u, v) \), rather than the individual vertices \( u, v \) with \( \partial(u, v) = \alpha \) (see [4] 4.1). In particular \( k_i = |\Gamma_i(u)| \), which is called the \( i \)-th valency, does not depend on the choice of a vertex \( u \). For three vertices \( u, v, w \), put

\[ P_{ijt}(u, v, w) = |\Gamma_i(u) \cap \Gamma_j(v) \cap \Gamma_t(w)|. \]

More precise descriptions about distance-regular graphs will be found in [3], [4].

The following Proposition is obtained directly from the definition and remarks in the previous section.
Proposition 1 Let $\Gamma$ be a distance-regular graph of diameter $d$ with the vertex set $X$. Put $|X| = n$ and let $D$ be one of the square roots of $n$. Let $t_0, t_1, \ldots, t_d$ be non-zero complex numbers and let $w$ be the complex valued function on $X \times X$ defined by $w(u, v) = t_i$ where $i = \partial(u, v)$. Then $(X, w)$ becomes a spin model if and only if the following conditions hold:

\begin{align*}
(C1) \quad \sum_{i=0}^{d} k_i t_i &= D t_0^{-1}, \\
(C2) \quad \sum_{i=0}^{d} k_i t_i^{-1} &= D t_0, \\
(C3) \quad \sum_{i=0}^{d} \sum_{j=0}^{d} p_{ij}^\alpha t_i t_j^{-1} &= 0 \quad (\alpha = 1, 2, \ldots, d), \\
(C4) \quad \text{For all vertices } u, v, w \text{ in } X, \\
\quad \sum_{\ell=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} P_{ij\ell}(u,v,w) t_i t_j t_\ell^{-1} &= D t_\alpha t_\beta^{-1} t_\gamma^{-1},
\end{align*}

where $\alpha = \partial(u, v)$, $\beta = \partial(u, w)$, $\gamma = \partial(v, w)$.

Remark 5 Though conditions (C1) and (C2) can be removed in the above, these are useful to find solutions of the equations.

3 Result

A distance-regular graph having the intersection array

$$\{4m, 4m-1, 2m, 1; 1, 2m, 4m-1, 4m\}$$

is called a Hadamard graph of order $4m$. There is a natural one-to-one correspondence between Hadamard graphs of order $4m$ and Hadamard matrices of order $4m$ (see [4] 1.8). Now our main result follows:

Theorem 2 Let $\Gamma$ be a Hadamard graph of order $4m$. Let $s, t_0, t_1$ be complex numbers such that

$$s^2 + 2(2m - 1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{m}}{(4m - 1)s + 1}, \quad t_1^4 = 1.$$

Put $t_2 = st_0$, $t_3 = -t_1$ and $t_4 = t_0$. Then $t_0, \ldots, t_4$ satisfy the conditions in Proposition 1 with $D = 4\sqrt{m}$.

Theorem 2 can be described without using distance-regular graphs as follows:
Theorem 3 Let $H$ be a Hadamard matrix of order $n$, $n \equiv 0$ (mod 4), and let $M$ be the weight matrix of the Potts model of size $n$. Let $\omega$ be one of the 4-th roots of 1, $\omega^4 = 1$. Define a $4n \times 4n$-matrix $W$ as:

$$W = \begin{pmatrix}
M & M & \omega H & -\omega H \\
M & M & -\omega H & \omega H \\
\omega H^t & -\omega H^t & M & M \\
-\omega H^t & \omega H^t & M & M
\end{pmatrix}$$

Then $W$ becomes the weight matrix of a spin model having $4n$ spins.

4 Proof of Theorem 2

Let $H$ be a Hadamard graph of order $4m$ and let $s, t_0, \ldots, t_4$ be complex numbers such that

$$s^2 + 2(2m - 1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{m}}{(4m - 1)s + 1},$$

$$t_1^4 = 1, \quad t_2 = st_0, \quad t_3 = -t_1, \quad t_4 = t_0.$$ 

By $k_{i-1}b_{i-1} = k_i c_i$, we get

$$k_0 = 1, \quad k_1 = 4m, \quad k_2 = 8m - 2, \quad k_3 = 4m, \quad k_4 = 1.$$ 

So (C1) becomes

$$t_0 + 4mt_1 + (8m - 2)t_2 + 4mt_3 + t_4 = 4\sqrt{m} t_0^{-1}.$$ 

By $t_3 = -t_1$, $t_0 = t_4$ and $t_2 = st_0$, this becomes

$$2t_0 + (8m - 2)st_0 = 4\sqrt{m} t_0^{-1}.$$ 

Clearly this holds by the assumption $t_0^2 = 2\sqrt{m}((4m - 1)s + 1)^{-1}$.

Condition (C2) becomes

$$t_0^{-1} + 4mt_1^{-1} + (8m - 2)t_2^{-1} + 4mt_3^{-1} + t_4^{-1} = 4\sqrt{m} t_0,$$

and it becomes

$$2t_0^{-1} + (8m - 2)t_2^{-1} = 4\sqrt{m} t_0,$$

$$1 + (4m - 1)s^{-1} = 2\sqrt{m} t_0^2.$$ 

By the assumption $t_0^2 = 2\sqrt{m}((4m - 1)s + 1)^{-1}$, it is equivalent to

$$1 + (4m - 1)s^{-1} = 2\sqrt{m} \cdot 2\sqrt{m}((4m - 1)s + 1)^{-1}.$$
This is implied by the assumption $s^2 + 2(2m - 1)s + 1 = 0$.

Next consider condition (C3). The values of $p_{ij}^\alpha$ are easily computed by the following formula ([4] 4.1.7).

$$p_{j+1,i}^\alpha = \frac{1}{c_{j+1}}(p_{j+1,\ell}^\alpha b_{\ell-1} + p_{j+1,\ell+1}^\alpha c_{\ell+1} - p_{j-1,\ell}^\alpha b_{j-1}).$$

Case $\alpha = 1$:

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>$p_{ij}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 1), (1, 0), (3, 4), (4, 3)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2), (2, 1), (2, 3), (3, 2)</td>
<td>$4m - 1$</td>
</tr>
</tbody>
</table>

Condition (C3) becomes

$$t_0t_1^{-1} + t_1t_0^{-1} + t_3t_4^{-1} + t_4t_3^{-1} + (4m - 1)(t_1t_2^{-1} + t_2t_1^{-1} + t_2t_3^{-1} + t_3t_2^{-1}) = 0.$$%

This holds by $t_3 = -t_1$ and $t_0 = t_4$.

Case $\alpha = 2$:

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>$p_{ij}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2), (2, 0), (2, 4), (4, 2)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 1), (1, 3), (3, 1), (3, 3)</td>
<td>$2m$</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$8m - 4$</td>
</tr>
</tbody>
</table>

(C3) becomes

$$t_0t_2^{-1} + t_2t_0^{-1} + t_4t_2^{-1} + t_4t_2^{-1} + 2m(t_1t_1^{-1} + t_1t_3^{-1} + t_3t_1^{-1} + t_3t_3^{-1}) + (8m - 4) = 0.$$%

This is implied by $t_3 = -t_1$, $t_0 = t_4$, $t_2 = st_0$ and $s^2 + 2(2m - 1)s + 1 = 0$.

Case $\alpha = 3$:

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>$p_{ij}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 3), (3, 0), (1, 4), (4, 1)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2), (2, 1), (2, 3), (3, 2)</td>
<td>$4m - 1$</td>
</tr>
</tbody>
</table>

$$t_0t_3^{-1} + t_3t_0^{-1} + t_1t_4^{-1} + t_4t_1^{-1} + (4m - 1)(t_1t_2^{-1} + t_2t_1^{-1} + t_2t_3^{-1} + t_3t_2^{-1}) = 0.$$%

This holds by $t_3 = -t_1$ and $t_0 = t_4$.

Case $\alpha = 4$:

<table>
<thead>
<tr>
<th>$(i, j)$</th>
<th>$p_{ij}^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 4), (4, 0)</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3), (3, 1)</td>
<td>$4m$</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$8m - 2$</td>
</tr>
</tbody>
</table>
\[ t_0t_4^{-1} + t_4t_0^{-1} + 4m(t_1t_3^{-1} + t_3t_1^{-1}) + (8m - 2)t_2t_2^{-1} = 0. \]

Clearly this holds.

Now we consider condition (C4). Since (C4) is symmetric in \( u, v \), we may assume \( \partial(u, w) \leq \partial(v, w) \). Fix three vertices \( u, v, w \). Put \( \partial(u, v) = \alpha, \partial(u, w) = \beta, \partial(v, w) = \gamma \) and \( P_{ij\ell} = P_{ij\ell}(u, v, w) \). If \( \beta = 0 \), we have \( u = w, \alpha = \gamma \), and \( P_{ij\ell} = 0 \) for \( i \neq \ell \). Therefore

\[ \sum_{i,j,\ell} P_{ij\ell}t_it_jt_\ell^{-1} = \sum_{j} \sum_{i} P_{iji}t_j = \sum_{j} k_jt_j, \]

and (C4) is equivalent to (C1) in the case \( \beta = 0 \). So we must verify (C4) in each of the following cases of \( (\alpha, \beta, \gamma) \):

\[
\begin{align*}
(0, 1, 1) & \quad (0, 2, 2) & \quad (0, 3, 3) & \quad (0, 4, 4) \\
(1, 1, 2) & \quad (1, 2, 3) & \quad (1, 3, 4) \\
(2, 1, 1) & \quad (2, 1, 3) & \quad (2, 2, 2) & \quad (2, 2, 4) & \quad (2, 3, 3) \\
(3, 1, 2) & \quad (3, 1, 4) & \quad (3, 2, 3) \\
(4, 1, 3) & \quad (4, 2, 2)
\end{align*}
\]

In the case \( (\alpha, \beta, \gamma) \neq (2, 2, 2) \), the values of \( P_{ij\ell} \) are easily computed. We need the following Lemma for the case \( (\alpha, \beta, \gamma) = (2, 2, 2) \).

**Lemma 4** If \( \partial(u, v) = \partial(u, w) = \partial(v, w) = 2 \), then \( w \) has precisely \( m \) neighbours in \( \Gamma_1(u) \cap \Gamma_1(v) \).

**Proof.** Put \( D_j^i = \Gamma_i(u) \cap \Gamma_j(v) \). We have \( w \in D_2^2 \). Put \( e(w, D_1^1) = r, \quad e(w, D_3^3) = s, \quad e(w, D_2^1) = s', \quad e(w, D_3^3) = r' \). Notice that every vertex \( x \in X \) has the unique opposite vertex \( x' \) such that \( \partial(x, x') = 4 \), since we have \( k_4 = 1 \). Since the opposite vertex \( x' \) of \( x \in D_1^1 \cap \Gamma_1(w) \) is in \( D_3^3 \), we get \( r' \leq |D_3^3| - r = 2m - r \). Similarly we get \( s' \leq 2m - s \).

On the other hand, we have \( r + s = 2m \) since \( w \) has precisely \( 2m \) neighbours in \( \Gamma_1(u) \). We have also \( s + r' = 2m \) since \( w \) has \( 2m \) neighbours in \( \Gamma_3(v) \). These imply \( r = r' \). By the same reason, we get \( s = s' \). Therefore we must have \( r = s = r' = s' = m \).

**Case** \( (\alpha, \beta, \gamma) = (0, 1, 1) \):

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>( P_{ij\ell} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 1), (1, 1, 0), (3, 3, 4), (4, 4, 3))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 2), (2, 2, 1), (2, 2, 3), (3, 3, 2))</td>
<td>(4m - 1)</td>
</tr>
</tbody>
</table>
So, condition (C4) becomes
\[
\begin{align*}
t_0^2t_1^{-1} + t_1^2t_0^{-1} + t_3^2t_4^{-1} + t_4^2t_3^{-1} + (4m - 1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) &= Dt_0t_1^{-2}, \\
2t_1^2t_0^{-1} + (8m - 2)t_1^2t_2^{-1} &= Dt_0t_1^{-2}.
\end{align*}
\]

By \(t_1^4 = 1\), this is equivalent to (C2).

**Case (\(\alpha, \beta, \gamma\)) = (0, 2, 2):**

<table>
<thead>
<tr>
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<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 2), (2, 2, 0), (2, 2, 4), (4, 4, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 1), (1, 1, 3), (3, 3, 1), (3, 3, 3))</td>
<td>2m</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>8m - 4</td>
</tr>
</tbody>
</table>

Then condition (C4) becomes
\[
2(t_0^2t_2^{-1} + t_2^2t_0^{-1}) + (8m - 4)t_2 = D t_0 t_2^{-2},
\]
\[
s^{-1} + s^2 + (4m - 2)s = 2\sqrt{m} s^{-2} t_0^{-2}.
\]

By the assumption \(t_0^2 = 2\sqrt{m}((4m - 1)s + 1)^{-1}\), this becomes
\[
s^{-1} + s^2 + (4m - 2)s = (4m - 1)s^{-1} + s^{-2}.
\]

This is implied by the assumption \(s^2 + 2(2m - 1)s + 1 = 0\).

**Case (\(\alpha, \beta, \gamma\)) = (0, 3, 3):**

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 3), (1, 1, 4), (3, 3, 0), (4, 4, 1))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 2), (2, 2, 1), (2, 2, 3), (3, 3, 2))</td>
<td>4m - 1</td>
</tr>
</tbody>
</table>

(C4) becomes
\[
\begin{align*}
t_0^2t_3^{-1} + t_1^2t_4^{-1} + t_3^2t_0^{-1} + t_4^2t_1^{-1} + (4m - 1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) &= Dt_0t_3^{-2}.
\end{align*}
\]

This is equivalent to Case (\(\alpha, \beta, \gamma\)) = (0, 1, 1).

**Case (\(\alpha, \beta, \gamma\)) = (0, 4, 4):**

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 4), (4, 4, 0))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 1, 3), (3, 3, 1))</td>
<td>4m</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>8m - 2</td>
</tr>
</tbody>
</table>
\[ t_0^2 t_4^{-1} - t_4^2 t_0^{-1} + 4m(t_1^2 t_3^{-1} + t_3^2 t_1^{-1}) + (8m - 2)t_2^2 t_2^{-1} = Dt_0 t_4^{-1}. \]

**Case** \((\alpha, \beta, \gamma) = (1, 1, 2):\)

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 1, 1), (1, 0, 2), (1, 2, 0), (3, 2, 4), (3, 4, 2), (4, 3, 3))</td>
<td>1</td>
</tr>
<tr>
<td>((2, 1, 1), (2, 3, 3))</td>
<td>2m - 1</td>
</tr>
<tr>
<td>((2, 1, 3), (2, 3, 1))</td>
<td>2m</td>
</tr>
<tr>
<td>((1, 2, 2), (3, 2, 2))</td>
<td>4m - 2</td>
</tr>
</tbody>
</table>

\[ t_0 + t_4 + t_0 t_1 t_2^{-1} + t_1 t_2 t_0^{-1} + t_2 t_3 t_4^{-1} + t_3 t_4 t_2^{-1} + 2m(t_1 t_2 t_3^{-1} + t_2 t_3 t_1^{-1}) + (4m - 2)(t_1 + t_2 + t_3) = D t_2^{-1}. \]

**Case** \((\alpha, \beta, \gamma) = (1, 2, 3):\)

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 1, 2), (1, 0, 3), (2, 1, 4), (2, 3, 0), (3, 4, 1), (4, 3, 2))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 2, 3), (3, 2, 1))</td>
<td>2m - 1</td>
</tr>
<tr>
<td>((1, 2, 1), (3, 2, 3))</td>
<td>2m</td>
</tr>
<tr>
<td>((2, 1, 2), (2, 3, 2))</td>
<td>4m - 2</td>
</tr>
</tbody>
</table>

\[ t_0 t_1 t_2^{-1} + t_0 t_1 t_3^{-1} + t_1 t_2 t_0^{-1} + t_2 t_3 t_4^{-1} + t_3 t_4 t_2^{-1} + 2m(t_1 t_2 t_3^{-1} + t_2 t_3 t_1^{-1}) + (2m - 1)(t_1 t_2 t_3^{-1} + t_2 t_3 t_1^{-1}) + (4m - 2)(t_1 + t_3) + 4mt_2 = D t_1 t_2^{-1} t_3^{-1}. \]

**Case** \((\alpha, \beta, \gamma) = (1, 3, 4):\)

<table>
<thead>
<tr>
<th>((i, j, \ell))</th>
<th>(P_{ij\ell})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 1, 3), (1, 0, 4), (3, 4, 0), (4, 3, 1))</td>
<td>1</td>
</tr>
<tr>
<td>((1, 2, 2), (2, 1, 3), (2, 3, 1), (3, 2, 2))</td>
<td>4m - 1</td>
</tr>
</tbody>
</table>

\[ t_0 t_1 t_3^{-1} + t_0 t_4 t_0^{-1} + t_3 t_4 t_0^{-1} + t_3 t_4 t_1^{-1} + (4m - 1)(t_1 + t_3 + t_1 t_2 t_3^{-1} + t_2 t_3 t_1^{-1}) = D t_1 t_3^{-1} t_4^{-1}. \]

**Case** \((\alpha, \beta, \gamma) = (2, 1, 1):\)
$P_{ij\ell}$

<table>
<thead>
<tr>
<th>$(i, j, \ell)$</th>
<th>$P_{ij\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 1), (2, 0, 1), (2, 4, 3), (4, 2, 3), (1, 1, 0), (3, 3, 4)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 1, 2), (3, 3, 2)$</td>
<td>$2m - 1$</td>
</tr>
<tr>
<td>$(1, 3, 2), (3, 1, 2)$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$(2, 2, 1), (2, 2, 3)$</td>
<td>$4m - 2$</td>
</tr>
</tbody>
</table>

$2m - 1$

$(2, 2, 1), (2, 2, 3)$

$4m - 2$

$t_{1}^{2}t_{0}^{-1} + t_{4}^{2}t_{4}^{-1} + 2(t_{0}t_{t_{1}}^{-1} + t_{2}t_{4}t_{3}^{-1}) + (2m - 1)(t_{1}^{2}t_{2}^{-1} + t_{3}^{2}t_{2}^{-1})$

$+(4m - 2)(t_{1}^{2}t_{1}^{-1} + t_{2}^{2}t_{3}^{-1}) + 4mt_{1}t_{3}t_{2}^{-1} = D_{t_{1}}t_{1}^{-2}$.

Case $(\alpha, \beta, \gamma) = (2, 1, 3)$:

$P_{ij\ell}$

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</tr>
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<td>$(0, 2, 1), (2, 0, 3), (2, 4, 1), (4, 2, 3), (1, 3, 0), (3, 1, 4)$</td>
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<td>$2m$</td>
</tr>
<tr>
<td>$(2, 2, 1), (2, 2, 3)$</td>
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</tr>
</tbody>
</table>

$2m - 1$

$(2, 2, 1), (2, 2, 3)$

$4m - 2$

$t_{t_{0}}t_{2}t_{1}^{-1} + t_{0}t_{2}t_{3}^{-1} + t_{2}t_{4}t_{3}^{-1} + t_{1}t_{3}t_{0}^{-1} + t_{1}t_{3}t_{t_{1}}^{-1} + 2m(t_{1}^{2}t_{3}^{-1} + t_{3}^{2}t_{1}^{-1})$

$+(4m - 2)(t_{1}t_{3}t_{2}^{-1} + t_{2}t_{t_{1}}^{-1} + t_{2}^{2}t_{t_{3}}^{-1}) = D_{t_{2}}t_{1}^{-1}t_{3}^{-1}$.

Case $(\alpha, \beta, \gamma) = (2, 2, 2)$:

$P_{ij\ell}$

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<tr>
<th>$(i, j, \ell)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 2), (2, 0, 2), (2, 2, 0), (2, 4, 2), (2, 4, 2), (4, 2, 2), (4, 2, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 1)$</td>
<td>$m$</td>
</tr>
<tr>
<td>$(3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3)$</td>
<td>$m$</td>
</tr>
<tr>
<td>$(2, 2, 2)$</td>
<td>$8m - 6$</td>
</tr>
</tbody>
</table>

$2m - 1$

$(2, 2, 1), (2, 2, 3)$

$4m - 2$

$t_{t_{0}}t_{2}t_{t_{1}}^{-1} + t_{0}t_{t_{2}}t_{3}^{-1} + 2(t_{0} + t_{t_{4}}) + m(t_{1}^{2}t_{3}^{-1} + t_{3}^{2}t_{1}^{-1}) + 3m(t_{1} + t_{3})$

$+(8m - 6)t_{2} = D_{t_{t_{2}}}t_{2}^{-1}$.

Case $(\alpha, \beta, \gamma) = (2, 2, 4)$:

$P_{ij\ell}$

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<thead>
<tr>
<th>$(i, j, \ell)$</th>
<th>$P_{ij\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 2, 2), (2, 0, 4), (2, 4, 0), (4, 2, 2)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 1, 3), (1, 3, 1), (3, 1, 3), (3, 3, 1)$</td>
<td>$2m$</td>
</tr>
<tr>
<td>$(2, 2, 2)$</td>
<td>$8m - 4$</td>
</tr>
</tbody>
</table>

$2m - 1$

$(2, 2, 1), (2, 2, 3)$

$4m - 2$

$t_{0} + t_{4} + t_{0}t_{t_{2}}t_{t_{4}}^{-1} + t_{2}t_{4}t_{t_{0}}^{-1} + 2m(t_{1} + t_{3} + t_{1}^{2}t_{3}^{-1} + t_{3}^{2}t_{1}^{-1}) + (8m - 4)t_{2} = D_{t_{4}}t_{2}^{-1}$.

Case $(\alpha, \beta, \gamma) = (2, 3, 3)$:
\[
t^{2}_{4}t^{-1} + t^{2}_{3}t_{0} + 2(t_{0}t_{2}t^{-1} + t_{2}t_{4}t_{1}^{-1}) + (2m-1)(t^{2}_{1}t_{2}^{-1} + t^{2}_{3}t_{2}^{-1}) \\
+ (4m-2)(t^{2}_{1}t_{1}^{-1} + t^{2}_{3}t_{1}^{-1}) + 4mt_{1}t_{3}t_{2}^{-1} = Dt_{2}t_{3}^{-2}.
\]

Case \((\alpha, \beta, \gamma) = (3,1,2)\):

\[
t_{0}t_{3}t_{1}^{-1} + t_{0}t_{3}t_{4}^{-1} + t_{1}t_{4}t_{0}^{-1} + t_{1}t_{4}t_{3}^{-1} + (4m-1)(t_{1} + t_{3}) \\
+ (4m-1)(t_{1}t_{2}t_{3}^{-1} + t_{2}t_{3}t_{1}^{-1}) = Dt_{3}t_{1}^{-1}t_{4}^{-1}.
\]

Case \((\alpha, \beta, \gamma) = (3,2,3)\):

\[
t + t_{4} + t_{0}t_{3}t_{2}^{-1} + t_{2}t_{3}t_{0}^{-1} + t_{1}t_{2}t_{4}^{-1} + t_{1}t_{4}t_{2}^{-1} + 2m(t_{1}t_{2}t_{3}^{-1} + t_{2}t_{3}t_{1}^{-1}) \\
+ (4m-2)(t_{1} + t_{2} + t_{3}) = Dt_{2}^{-1}.
\]

Case \((\alpha, \beta, \gamma) = (4,1,3)\):
\[ t_{0} t_{4} t_{1}^{-1} + t_{0} t_{4} t_{3}^{-1} + t_{1} t_{3} t_{4}^{-1} + t_{4}^{-1} t_{1}^{-1} t_{3}^{-1} + (4m - 1)(t_{2}^{-1} t_{1}^{-1} + t_{2}^{-1} t_{3}^{-1}) + (8m - 2)t_{1} t_{3} t_{2}^{-1} = Dt_{4} t_{1}^{-1} t_{3}^{-1}. \]

Case \((\alpha, \beta, \gamma) = (4, 2, 2)\):

\[ t_{2}^{-1} t_{0}^{-1} + t_{2}^{-1} t_{4}^{-1} + 2t_{0} t_{4} t_{2}^{-1} + 4m(t_{1} + t_{3}) + (8m - 4)t_{2} = Dt_{4} t_{2}^{-2}. \]

### Reference


