Tunnel numbers of knots and Jones-Witten invariants

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1. Introduction and main results

The purpose of this paper is to discuss some topological consequences derived from Jones-Witten invariants for links in 3-manifolds. In [Jo], Jones obtained upper bounds for the values of the Jones polynomial at roots of unity in terms of the braid index. In the present paper we focus on classical invariants called the tunnel numbers of knots. Let K be a knot in S^3. We are going to describe a lower bound for the tunnel number $t(K)$ by means of special values of the Jones polynomial and its generalizations. The tunnel number is the minimal number of arcs $\gamma_1, \cdots, \gamma_t$ so that the closure of the complementary space of the regular neighborhood of $K \cup \gamma_1 \cup \cdots \cup \gamma_t$ is homeomorphic to a handlebody. In Figure 1 we indicate that the tunnel number of a trefoil knot is equal to one. It is clear that the tunnel number is bounded from above the minimal number of crossings of the knot diagram and that the minimal number of generators of the fundamental group of the knot complement is at most $t(K) + 1$. This definition of the tunnel number is also generalized in a natural way to a knot in a closed oriented 3-manifold. In the preceding article [Ko3], we developed a general framework of giving an estimate for the tunnel number in terms of the Jones-Witten invariants, based on the monodromy of conformal field theory. The scope of the present paper is to give a refinement of the inequality obtained in [Ko3] and to present a more combinatorial description using quantum $6j$-symbols at roots of unity.

To state our main results let us recall some basic construction for invariants of framed links associated with the quantum group $U_q(sl(2, \mathbf{C}))$ in the sense of Drinfel'd [D] and Jimbo [J]. We fix a positive integer l called a level and we put $r = l + 2$. In this paper we discuss the invariants at $q = e^{2\pi\sqrt{-1}/r}$. Let $P_+(l)$ be a set of half integers defined by

$$(1\text{-}1) \qquad\qquad P_+(l) = \{0, 1/2, 1, \cdots, l/2\}.$$

Here $j \in P_+(l)$ corresponds to an irreducible $U_q(sl(2, \mathbf{C}))$ module of dimension $2j + 1$. Let L be a framed link in S^3 with m components. As was constructed in Kirillov-Reshetikhin [KR], for $\lambda : \{1, \cdots, m\} \to P_+(l)$, we can associate an invariant of

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oriented framed link denoted by $J_r(L, \lambda)$. In this paper, we follow a normalization due to Kirby-Melvin [KM]. If $\lambda \equiv 1/2$, then $J_r(L, 1/2)$ is a variant of the Jones polynomial given by

$$(1\text{-}2) \qquad J_r(L, 1/2) = \left(2\cos\frac{\pi}{r}\right) t^{3L \cdot L} V_L(e^{-\frac{2\pi\sqrt{-1}}{r}})$$

with $t = e^{\pi\sqrt{-1}/2r}$ where V_L is the original Jones polynomial normalized as in [Jo]. In the above expression the symbol $L \cdot L$ is the sum of the sum of the entries of the linking matrix $\sum_{ij} lk(L_i, L_j)$, where $lk(L_i, L_i)$ is the framing on the component L_i. In the case of a knot K assigned with a color $j \in P_+(l)$, we have the following description. Putting for an integer $n = 0, 1, \cdots, l$, $J_{K,n} = J_r(K, n/2)$, we have a recursive formula

$$(1\text{-}3) \qquad J_{K,n} = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} J_{K^{n-2i},1}$$

where K^γ stands for the γ parallel push offs defined by means of the framing. Let us notice that $|J_r(K, j)|$ does not depend on the framing on K. For $i, j \in P_+(l)$ we put

$$S_{ij} = \sqrt{\frac{2}{r}} \sin \frac{(2i+1)(2j+1)\pi}{r}.$$

With this notation we have the following.

Theorem A. *Let K be a knot in S^3. Then, the tunnel number $t(K)$ satisfies*

$$\sum_{j \in P_+(l)} |J_r(K, j)|^2 \le 2^{-t(K)} S_{00}^{-2(t(K)+1)},$$

which gives a lower estimate for the tunnel number.

Let us recall that we have a symmetry

$$(1\text{-}4) \qquad |J_r(K, j)| = |J_r(K, l/2 - j)|$$

obtained in [KM]. Based on this, we can deduce from Theorem A that

$$(1\text{-}5) \qquad |J_r(K, j)|^2 \le \left(\frac{1}{2 S_{00}^2}\right)^{t(K)+1} - 1$$

for any $j \in P_+(l)$ such that $j \ne 0, l/2$, $j \ne l/2 - j$.

In [W], Witten introduced a series of invariants for a link in 3-manifold based on quantum field theory with Chern-Simons action. A Dehn surgery formula of the Witten invariant for $sl(2, \mathbb{C})$ was studied extensively by Reshetikhin-Turaev [RT] and

Kirby-Melvin [KM]. Let K be a knot in a closed oriented 3-manifold M assigned with $j \in P_+(l)$. We suppose that there is a framed link L in S^3 disjoint from K and (M, K) is obtained by the Dehn surgery on the framed link L. Let us denote by m the number of components of L. The invariant we are going to investigate is

$$(1\text{-}6) \qquad Z_r(M, K; j) = C^{sign(L)} \sum_\lambda S_{0\lambda(1)} \cdots S_{0\lambda(m)} J_r(K \cup L, j \cup \lambda)$$

where the sum is for any $\lambda : \{1, \cdots, m\} \to P_+(l)$, $C = \exp(-3(r-2)\pi\sqrt{-1}/4r)$, and $sign(L)$ denotes the signature of the linking matrix for L. In particular, in the case the knot K is empty, we obtain an invariant of 3-manifold $Z_r(M)$. Let us note that we normalize the invariant so that $Z_r(S^3) = 1$, in contrast to the Witten's original normalization. Our result for the tunnel numbers of knots in 3-manifolds is the following.

Theorem B. *Let M be a closed oriented 3-manifold and let K be a knot in M. Then, by means of the Witten invariant, the tunnel number $t(K)$ has a lower estimate*

$$\sum_{j \in P_+(l)} |Z_r(M, K; j)|^2 \leq S_{00}^{-2(t(K)+1)}.$$

Let Σ be a closed oriented surface of genus g and we denote by $\mathcal{M}_g$ its mapping class group. To prove our main result we describe a projective unitary representation

$$\rho : \mathcal{M}_g \to GL(\mathcal{H}_\Sigma)$$

constructed using an idea due to Moore-Seiberg [MS] and Reshetikhin-Turaev [RT]. By means of this representation we show

$$(1\text{-}6) \qquad Z_r(M, K; j) = S_{00}^{-g} < v_j^*, \rho(h)v_0 >$$

with some $v_0, v_j \in \mathcal{H}_\Sigma$ if the knot exterior admits a splitting

$$E(K) = cl(V - N(K)) \cup_h (-V)$$

with a handlebody V of genus g. Here $N(K)$ is a tubular neighborhood of K, cl stands for the closure, and $h : \partial V \to \partial V$ is a glueing map. The equality holds when the glueing map is identity. Theorem B is a consequence of the unitarity of ρ, and Theorem A is shown as a special case $M = S^3$ with an improvement using the symmetry derived from the Dynkin diagram automorphism.

Let us observe that in the case K is empty the unitarity implies

$$(1\text{-}8) \qquad |Z_r(M)| \leq S_{00}^{-g}$$

4

which gives a lower bound for the Heegaard genus of M. The inequality 1.8 has been obtained in [Wa], [Ko2] and [G].

The above interpretation of the Jones-Witten invariants in terms of the mapping class group was studied in [Ko1] and independently in [C] and [CLM]. Even in the case of knots in S^3 the above 3-dimensional point of view plays an essential role to show our inequality, which reveals a close relationship between the special values Jones polynomial and the quantum field theory.

Let us also recall that the topological meaning of the values of the Jones polynomial at certain roots of unity is well studied (see [Jo]). Let M_L be the 2-fold branched covering of S^3 branched along a link L. In the case $r = 6$, it is known by Lipson [Lp] and Lickorish-Millet [LM] that

$$(1\text{-}9) \qquad |V_L(e^{2\pi\sqrt{-1}/6})| = 3^{d/2}$$

where $d = \dim H_1(M_L, \mathbb{Z}/3\mathbb{Z})$. Hence the inequality 1.5 implies in this case

$$(1\text{-}10) \qquad 2 \cdot 3^{d(K)+1} + |J_r(K,1)|^2 \leq 2 \cdot 6^{t(K)+1} - 2$$

with $d(K) = \dim H_1(M_K, \mathbb{Z}/3\mathbb{Z})$ for $r = 6$. A numerical computation in the case $r = 5$ shows that if a knot K satisfies

$$|V_K(e^{2\pi\sqrt{-1}/5})| > 2.1489...,$$

then we have $t(K) \geq 2$. Although we could not get an inequality sharper than this, we can derive several interesting consequences (see Remark 4.10).

The paper is organized in the following way. In section 2, we recall basic facts on quantum $6j$ symbols at roots of unity and the associated invariants of framed links following Kirillov and Reshetikhin [KR]. In section 3, we construct projective unitary representations of the mapping class groups and we prove our main results. In section 4, we discuss a generalization of our construction for a complex simple Lie algebra based on the monodromy of the Wess-Zumino-Witten model.
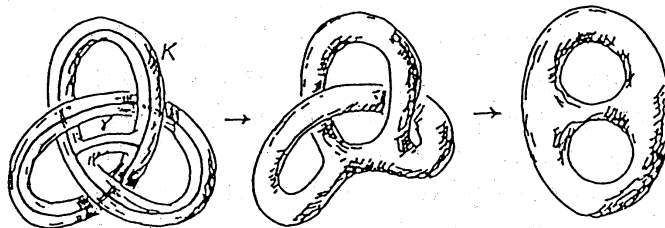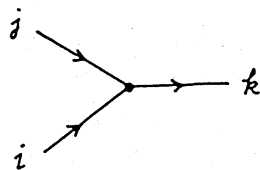
Figure 1

Figure 2

2. Quantum $6j$ symbols and tangle operators

Let us recall an explicit form of quantum $6j$-symbols at $q = e^{2\pi\sqrt{-1}/r}$ following Killirov and Reshetikhin [KR] (see also [TV]).

We say that a triple $i, j, k \in P_+(l)$ is admissible if and only if it satisfies the conditions

$$
\begin{aligned}
|i - j| &\leq k \leq i + j \\
i + j + k &\in \mathbf{Z} \\
i + j + k &\leq l.
\end{aligned}
$$

(2-1)

We adopt the following convention of a q-integer.

$$
[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin n\pi/r}{\sin \pi/r}.
$$

For an admissible triple (a, b, c) we put

$$
\Delta(abc) = \left(\frac{[a + b - c]![a + c - b]![b + c - a]!}{[a + b + c + 1]!} \right)^{1/2}.
$$

An ordered 6-tuple $a, b, c, d, e, f \in P_+(l)$ is said to be admissible if the triples

$$
(a, b, c), (c, d, e), (a, e, f), (b, d, f)
$$

are admissible in the above sense. For an admissible 6-tuple the Rakah-Wigner symbol is given by

$$
\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}^{RW} = \Delta(abc)\Delta(aef)\Delta(bdf)\Delta(cde) \sum_z (-1)^z [z + 1]! \times
$$

$$
\{[z - a - b - c]![z - a - e - f]![z - b - d - f]![z - c - d - e]!
$$

$$
\times [a + b + d + e - z]![a + c + d + f - z]![b + c + e + f - z]!\}^{-1}
$$

where the sum is for any non-negative integer z such that all expressions in the brackets are non-negative. Normalizing the Rakah-Wigner symbol, the quantm $6j$-symbol is given by

$$
\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = (-1)^{d+c+2c-a-b} \sqrt{\frac{S_{0c}}{S_{00}} \frac{S_{0f}}{S_{00}}} \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}^{RW}.
$$

For an admissible triple (i, j, k) and a trivalent graph depicted as in Figure 2, we associate a one dimensional complex vector space $\mathcal{H}_{ij}^k$ with basis denoted by e_{ij}^k. If (i, j, k) is not admissible, we define $\mathcal{H}_{ij}^k$ to be zero. When we reverse the orientation of the edge corresponding to i we use the convention that the same space is denoted by the symbol $\mathcal{H}_{i\dagger j}^k$. We adopt a similar convention for j and k.

By means of the quantum $6j$-symbols we first define the fusing operators

$$
\text{(2-2)} \qquad
\begin{aligned}
F_+ &: \oplus_j \mathcal{H}_{ab}^j \otimes \mathcal{H}_{jc}^d \to \oplus_i \mathcal{H}_{bc}^i \otimes \mathcal{H}_{ai}^d \\
F_- &: \oplus_j \mathcal{H}_{bc}^j \otimes \mathcal{H}_{aj}^d \to \oplus_i \mathcal{H}_{ab}^i \otimes \mathcal{H}_{ic}^d
\end{aligned}
$$

in the following way (see Figure 3)

$$
F_+(e_{ab}^j \otimes e_{jc}^d) = \sum_i \left\{ \begin{matrix} a & b & j \\ c & d & i \end{matrix} \right\} e_{bc}^i \otimes e_{ai}^d
$$

$$
F_-(e_{bc}^j \otimes e_{aj}^d) = \sum_i \left\{ \begin{matrix} c & b & j \\ a & d & i \end{matrix} \right\} e_{ab}^i \otimes e_{ic}^d .
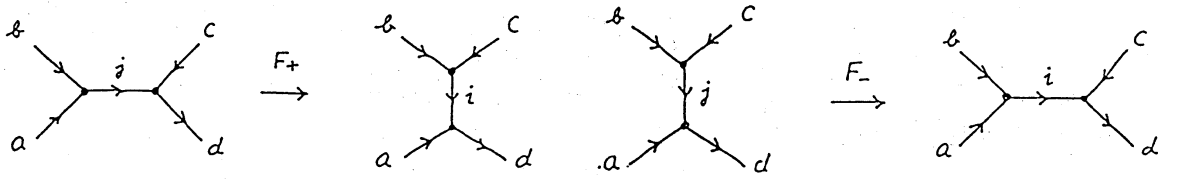$$

Figure 3

Let us introduce linear maps Ω_+ and Ω_-

$$
\Omega_\pm : \oplus_j \mathcal{H}_{j_1 j_2}^j \to \oplus_j \mathcal{H}_{j_2 j_1}^j
$$

given by

$$
\text{(2-3)} \qquad \Omega_\pm(e_{j_1 j_2}^j) = (-1)^{j_1 + j_2 - j} \exp\left(\pm \pi \sqrt{-1}(\Delta_j - \Delta_{j_1} - \Delta_{j_2}) \right) e_{j_2 j_1}^j .
$$

We define the braiding operators

$$
B_\pm : \oplus_j \mathcal{H}_{ab}^j \otimes \mathcal{H}_{jc}^d \to \oplus_j \mathcal{H}_{ac}^j \otimes \mathcal{H}_{jb}^d
$$

by

$$
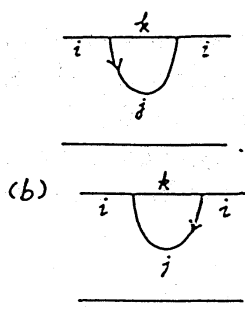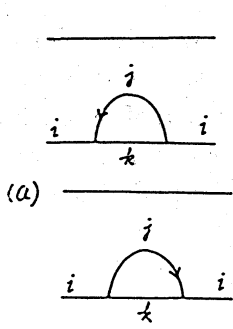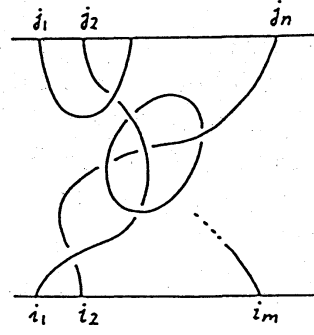\text{(2-4)} \qquad B_\pm = F_-(\Omega_\pm \otimes id)F_+ .
$$

Figure 4

Figure 5

Thus we obtain the following commutative diagram.

$$
\begin{array}{ccc}
\oplus_j \mathcal{H}^j_{ab} \otimes \mathcal{H}^d_{jc} & \xrightarrow{\ F_+\ } & \oplus_i \mathcal{H}^i_{bc} \otimes \mathcal{H}^d_{ai} \\
\Big\downarrow{\scriptstyle B_\pm} & & \Big\downarrow{\scriptstyle \Omega_\pm \otimes id} \\
\oplus_j \mathcal{H}^j_{ac} \otimes \mathcal{H}^d_{jb} & \xleftarrow[\ F_-\ ]{} & \oplus_i \mathcal{H}^j_{cb} \otimes \mathcal{H}^d_{ai}
\end{array}
$$

Associated with the diagram (a) in Figure 4, the fusing operators give the linear maps

$$
Y^0_{jj^\dagger} : \oplus_k \mathcal{H}^k_{ij} \otimes \mathcal{H}^i_{kj^\dagger} \to \mathcal{H}^0_{jj^\dagger} \otimes \mathcal{H}^i_{i0} \cong \mathbb{C}
$$

$$
Y^0_{j^\dagger j} : \oplus_k \mathcal{H}^k_{ij^\dagger} \otimes \mathcal{H}^i_{kj} \to \mathcal{H}^0_{j^\dagger j} \otimes \mathcal{H}^i_{i0} \cong \mathbb{C}.
$$

Normalizing these, we define the annihilation operators by

(2-5)
$$
\sqrt{S_{0j}/S_{00}}\, Y^0_{jj^\dagger} \quad \text{and} \quad (-1)^{2j}\sqrt{S_{0j}/S_{00}}\, Y^0_{j^\dagger j}
$$

respectively. In a similar way, the fusion operators induce

$$
Y^{jj^\dagger}_0 : \mathbb{C} \cong \mathcal{H}^0_{jj^\dagger} \otimes \mathcal{H}^i_{i0} \to \oplus_k \mathcal{H}^k_{ij} \otimes \mathcal{H}^i_{kj^\dagger}
$$

$$
Y^{j^\dagger j}_0 : \mathbb{C} \cong \mathcal{H}^0_{j^\dagger j} \otimes \mathcal{H}^i_{i0} \to \oplus_k \mathcal{H}^k_{ij^\dagger} \otimes \mathcal{H}^i_{kj}
$$

associated with the diagram (b) in Figure 4. Using this, we define the creation operators by

(2-6)
$$
\sqrt{S_{0j}/S_{00}}\, Y^{jj^\dagger}_0 \quad \text{and} \quad (-1)^{2j}\sqrt{S_{0j}/S_{00}}\, Y^{j^\dagger j}_0
$$

respectively.

We put

$$
\mathcal{H}_{j_1\cdots j_n} = \oplus_{k_1,\cdots,k_{n-1}\in P_+(l)} \mathcal{H}^{k_1}_{0j_1} \otimes \mathcal{H}^{k_2}_{k_1 j_2} \otimes \cdots \mathcal{H}^0_{k_{n-1}j_n}.
$$

Let T be an oriented framed (m, n) tangle, by which we mean a union of oriented arcs in $\mathbf{R}^2 \times [0, 1]$ with boundary points contained in $\mathbf{R}^2 \times \{0\} \cup \mathbf{R}^2 \times \{1\}$ as shown in Figure 5. Here the framing is represented by the blackboard framing in the sense of [KM]. By decomposing the tangle diagram into elementary diagrams and by associating to each elementary diagram the braiding, creation or annihilation operator defined above, we obtain a linear map.

$$J_r(T) : \mathcal{H}_{j_1 \cdots j_n} \to \mathcal{H}_{i_1 \cdots i_m}.$$

In particular, for an oriented framed link L with m components assigned with color λ : $\{1, \cdots, m\} \to P_+(l)$ we obtain an invariant of colored oriented framed link $J_r(L, \lambda)$. It turns out that this invariant does not depend on the orientation on the link L. In the case $\lambda \equiv 1/2$, the invariant $J_L = J_r(L, 1/2)$ is characterized by the skein relation

(2-7)
$$t J_{L_+} - t^{-1} J_{L_-} = (t^2 - t^{-2}) J_{L_0}$$
$$J_{\bigcirc} = [2] = 2 \cos \frac{\pi}{r},$$

where we use the usual notation L_+, L_-, L_0 as in [Jo].

Let R be the representation ring of $sl(2, \mathbf{C})$. The above invariant defines a linear map

(2-8)
$$J_r(L, \cdot) : R^{\otimes m} \to \mathbf{C}.$$

Moreover, it factors through the fusion algebra R_r for the $SU(2)$ Wess-Zumino-Witten model at level $l = r + 2$ (see [V]). This linear map is compatible with the product structure as shown in Figure 6. Here R_r has generators v_j, $j \in P_+(l)$ with the product structure

$$v_i \cdot v_j = \sum_k N_{ij}^k v_k$$

where $N_{ij}^k = \dim \mathcal{H}_{ij}^k$ (see also [MoS1] and [MoS2]). As the notation indicates it is clear from the above consideration that the invariant $J_r(K, j)$ for a framed knot K coincides with the one defined in the introduction by means of the recursive formula 1-3.
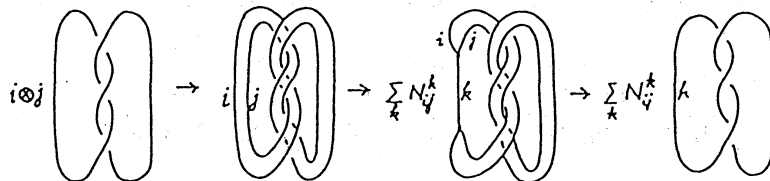

Figure 6

Let us recall that we have the following (see [KM]).

2.9 Proposition. *Let T be a colored oriented framed tangle. We write $J(T)$ for $J_r(T)$. Then, with respect to the following local modifications of the tangle we have*

$$J_{\bigcirc^j} = \frac{S_{0j}}{S_{00}} J_{\big)}$$

$$J_{\mathcal{O}\!\!\!\nwarrow}{}^j = \exp 2\pi \sqrt{-1} \Delta_j \, J_{(}{}^{j},$$

$$J_{\bigcup^i_j} = \frac{S_{ij}}{S_{0i}} J_{\big)}{}_{i\cdot}$$

Here Δ_j is given by

$$\Delta_j = \frac{j(j+1)}{r}.$$

2.10 *Remark.* Let us note that the sign convention adopted here is different from the one in [KR] and [MoS1], where the creation and annihilation operators are defined without the sign $(-1)^{2j}$ and consequently the effect of local modifications in Proposition 2.9 appear with signs.
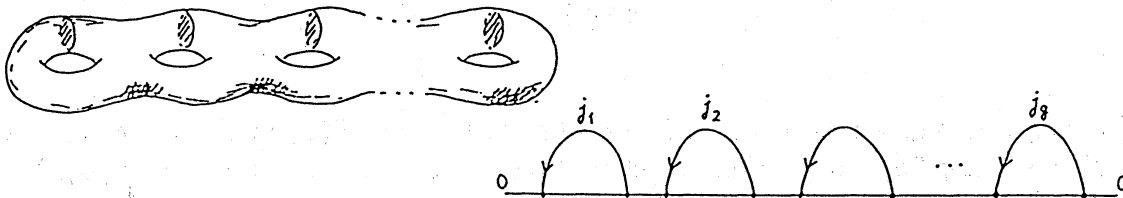
3. Proof of main results

Figure 7

Let Σ be a closed oriented surface of genus g. The mapping class group of Σ denoted by $\mathcal{M}_g$ is by definition

$$\mathcal{M}_g = Diff^+(\Sigma)/\sim$$

where $Diff^+(\Sigma)$ denotes the group of orientation preserving diffeomorphisms of Σ and $\sim$ stands for the equivalence modulo isotopy. We are going to investigate an action of the mapping class group $\mathcal{M}_g$ on a vector space $\mathcal{H}_\Sigma$ defined by

(3-1)
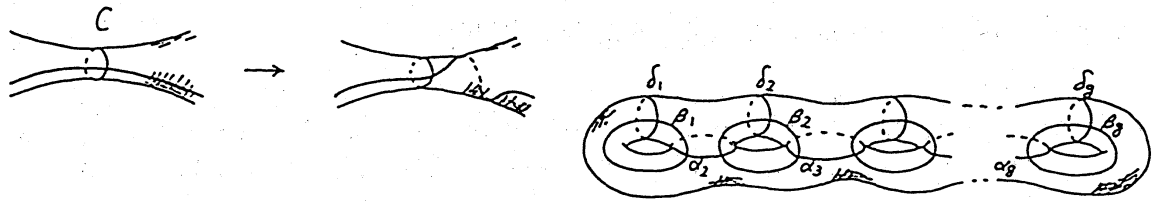$$\mathcal{H}_\Sigma = \oplus_{j_1,\cdots,j_g \in P_+(l)} \mathcal{H}_{j_1 j_1^! \cdots j_g j_g^!}$$

Figure 8

(see Figure 7).

Let us recall that the Dehn twist along a circle $C \subset \Sigma$ is a diffeomorphim of Σ defined by cutting the surface Σ along C and by reglueing after the rotation of 2π in the positive direction (see Figure 8). It is known by Lickorish [L] that $\mathcal{M}_g$ is generated by

$$\alpha_i, \quad 2 \leq i \leq g; \quad \beta_i, \quad 1 \leq i \leq g; \quad \delta_i, \quad 1 \leq i \leq g$$

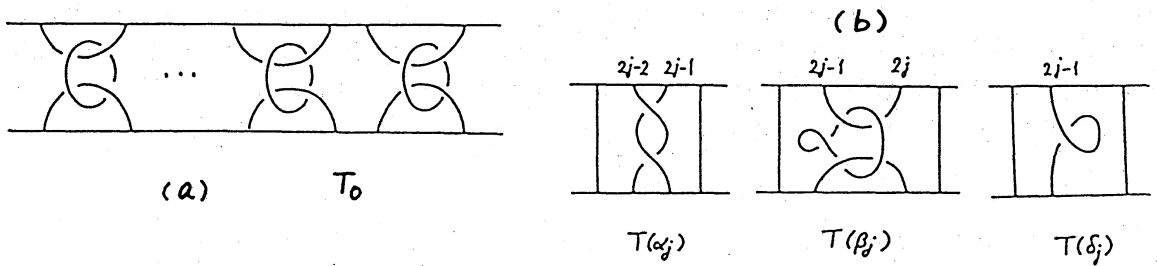which are Dehn twists along circles shown as in Figure 8.

Figure 9

To each $x \in \mathcal{M}_g$ we associate a framed $(2g, 2g)$ tangle $T(x)$ in the following way. Let T_0 be a $(2g, 2g)$ tangle illustrated in Figure 9a. To each x_i, $(x = \alpha, \beta, \delta)$ we associate a $(2g, 2g)$ tangle $T(x_i)$ depicted as in Figure 9b and for x_i^{-1} we associate its mirror image $T(x_i^{-1})$. Let T_0 be a $(2g, 2g)$ tangle shown in Figure 9. Given $x \in \mathcal{M}_g$ we express x using Lickorish generators as $x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$, $\epsilon_i = \pm 1$ and we define a tangle $T(x)$ by the composition

$$(3\text{-}2) \qquad T(x) = T_0 \circ T(x_{i_1}^{\epsilon_1}) \circ \cdots \circ T(x_{i_k}^{\epsilon_k}).$$

Let $T(x)$ be an oriented framed $(2g, 2g)$ tangle defined for $x \in \mathcal{M}_g$ in the previous paragraph by fixing a way of writing x in terms of Lickorish generators. We divide the set of arcs in $T(x)$ into $E^0(x)$, $E^1(x)$ and $L(x)$ defined by

$$E^0(x) = \{\gamma; \partial\gamma \subset \mathbf{R}^2 \times \{0\}\},$$
$$E^1(x) = \{\gamma; \partial\gamma \subset \mathbf{R}^2 \times \{1\}\},$$
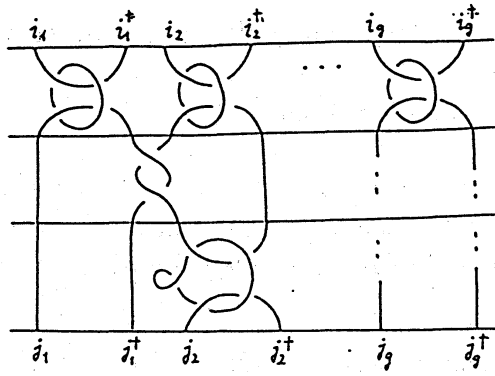$$L(x) = \{\gamma; \partial\gamma = \phi\}$$

Figure 10

(see Figure 10).

We define a linear map
$$\rho(x) : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$$

in the following way. For

$$u \in \mathcal{H}_{j_1 j_1^\dagger \cdots j_g j_g^\dagger}, \quad v^* \in \left(\mathcal{H}_{i_1 i_1^\dagger \cdots i_g i_g^\dagger} \right)^*$$

we associate the colors to the arcs in $E^0(x)$ and $E^1(x)$ as shown in Figure 10.

As is explained above, each time we associate a color

$$\lambda : \{1, 2, \cdots, m\} \to P_+(l)$$

to the components of the oriented framed link $L(x) = L_1 \cup \cdots \cup L_m$ we have a linear map $J_r(T(x), \lambda) : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$. Let us denote by

$$< \cdot, \cdot >: \mathcal{H}_\Sigma^* \times \mathcal{H}_\Sigma \to \mathbf{C}$$

the canonical pairing with the dual space. We define $\rho(x)$ by

(3-3)
$$< v^*, \rho(x)u > = \sqrt{S_{0j_1} \cdots S_{0j_g}} \sqrt{S_{0i_1} \cdots S_{0i_g}}$$
$$\sum_\lambda C^{sign(L(x))} S_{0\lambda(1)} \cdots S_{0\lambda(m)} < v^*, J_r(T(x), \lambda)u >$$

where λ runs over all colors on the components of the link $L(x)$.

For $x, y \in \mathcal{M}_g$ we fix a way of writing x and y in terms of Lickorish generators and we define $(2g, 2g)$ tangles $T(x)$, $T(y)$ and $T(xy)$ in the above way. Corresponding to this we have oriented framed links $L(x)$, $L(y)$ and $L(xy)$. We put

(3-4)
$$\sigma(x, y) = sign(L(xy)) - sign(L(x)) - sign(L(y))$$

and we define $\xi(x, y)$ by
$$\xi(x, y) = C^{\sigma(x,y)}.$$

Let us recall that a closed oriented 3-manifold M is represented as

$$M = V \cup_h (-V)$$

with a handlebody V and a glueing map $h : \partial V \to \partial V$. This description is called a Heegaard splitting. The minimal g such that M admits a Heegaard splitting with a handlebody of genus g is called the Heegaard genus and is denoted by $g(M)$.

3.5 Proposition. *The above construction defines a projectively linear representation*

$$\rho : \mathcal{M}_g \to GL(\mathcal{H}_\Sigma)$$

with a 2-cocycle ξ. Namely, we have

$$\rho(xy) = \xi(x,y)\rho(x)\rho(y)$$

for any $x, y \in \mathcal{M}_g$. Moreover, with respect to a basis of $\mathcal{H}_\Sigma$

$$e^{k_1}_{0j_1} \otimes e^{k_2}_{k_1 j_1^\dagger} \otimes \cdots \otimes e^0_{k_{2g-1} j_g^\dagger}, \quad j_1, \cdots, j_g, k_1, \cdots k_{2g-1} \in P_+(l),$$

$\rho(x)$ is represented as a unitary matrix for any $x \in \mathcal{M}_g$.

Proof: To show the well-definedness of $\rho(h)$ for $h \in \mathcal{M}_g$ let us first recall that a key step to show the topological invariance of the expresstion 1.6 in [RT] is the following equality for the tangle operators corresponding to Kirby moves (see [K] and [FR]). For the tangle diagrams Γ and Γ' as shown in Figure 11, we have

$$(3\text{-}6) \qquad C^{-1} \sum_j S_{0j} J_r(\Gamma, i_1, \cdots, i_n, j) = J_r(\Gamma', i_1, \cdots, i_n).$$

In our situation this equality can be applied to any local modification of the above type, allowing also the arcs in $E^0(h)$ and $E^1(h)$ as strands passing through the circle in the tangle diagram Γ.
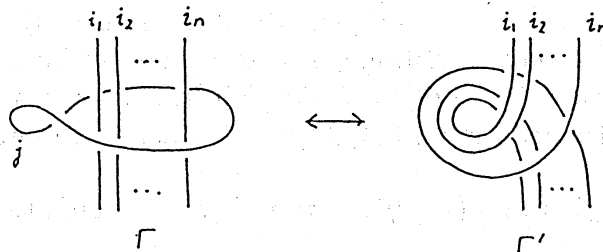

Figure 11

Let us also recall that associated with a decomposition of h into a product of Lickorish generators we can construct a link in S^3 so that the Dehn surgery on this link gives a 3-manifold obtained as a Heegaard splitting with a glueing map h (see [Ro]). In our case the Dehn surgery on the link $L(h)$ gives a 3-manifold $V_g \cup_h (-V_g)$. The arcs in $E^i(h)$, $i = 0, 1$ appear as a spine of the handlebody. Now we apply a relative version of Kirby's theorem (see [K] and [RT]) and we can conclude that if h and h' are isotopic the associated tangle diagrams $T(h)$ and $T(h')$ are obtainable one from the other by a finitlely many steps of Kirby moves shown in Figure 11. Here we allow any arcs in the tangle diagram as strands passing through the circle. Hence the equality 3.6 implies $\rho(h) = \rho(h')$.

Now we describe the relation between $\rho(xy)$ and $\rho(x)\rho(y)$. First, let us notice that the linear operator $\rho(T_0)$ associated with the tangle T_0 is the identity operator. This can be shown by considering the composition $T_0 \cdot T_0$ in the following way. Applying Kirby moves, one sees that $\rho(T_0)$ is idempotent. Since $\rho(T_0)$ is non-singular, this implies that $\rho(T_0)$ is identity. Combining with the definition 3.3, it is easy to deduce the assertion $\rho(xy) = \xi(x, y)\rho(x)\rho(y)$.

Noticing that the entries of the fusing matrices are real at $q = e^{2\pi\sqrt{-1}/r}$, it follows from the orthogonality of the quantum $6j$ symbol (see [KR] and [RT]) that the braiding operator is represented by a unitary matrix with respect to the above basis. The unitarity of $\rho(x_i)$ for the Lickorish generators follows immediately by looking at the above tangle representation. This completes the proof.

Now we are in a position to prove our main results. Let M be a closed oriented 3-manifold and let K be a knot in M. We denote by $E(K)$ the knot exterior. Namely, $E(K)$ is the closure of the complementary space of the tubular neighborhood $N(K)$. If the tunnel number of the knot K is $t(K)$, then we have a splitting of the knot exterior

$$E(K) = cl(V - N(K)) \cup_h (-V)$$

with a handlebody V of genus $g = t(K) + 1$ and a glueing map $h : \partial V \to \partial V$. Here the knot K is embedded in a simple position as shown in Figure 12. More precisely, there exists a separating disc D properly embedded in V with $\partial D \subset V$ such that K is embedded as a core of the solid torus obtained by cutting the handlebody V along D. Let us note that $cl(V - N(K))$ is a special case of a so-called compression body.
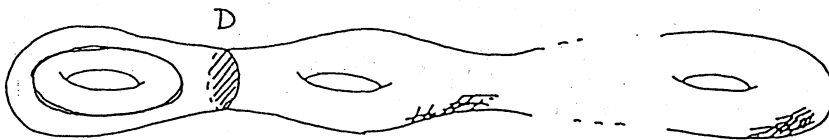

Figure 12

We put

$$v_j = e^j_{0j} \otimes e^0_{jj^\dagger} \otimes e^0_{00} \otimes \cdots \otimes e^0_{00}$$

which is a basis of

$$\mathcal{H}^j_{0j} \otimes \mathcal{H}^0_{jj^\dagger} \otimes \mathcal{H}^0_{00} \otimes \cdots \otimes \mathcal{H}^0_{00} \cong \mathbf{C}.$$

With this notation we obtain the following.

3.7 Proposition. *In the above situation we have*

$$Z_r(M, K; j) = S^{-g}_{00} < v^*_j, \rho(h)v_0 > .$$

Proof: Let us close the $(2g, 2g)$ tangle diagram $T(h)$ to get a plat diagram as in Figure 13. This is a link diagram expressed as $L(h) \cup L$ where L contains $2g$ components $L_1, \cdots, L_{2g}$. The color on L_1 is equal to j and the color on the other components is zero. Let us recall that by the Dehn surgery on the framed link $L(h)$ we get our 3-manifold M with link L in it.
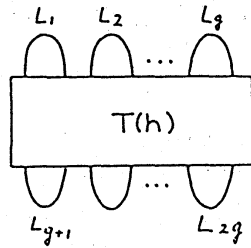
Figure 13

Let us consider the above situation with $K = L_1$. Noticing that the fusion operator

$$Y^0_{jj^\dagger} : \mathbf{C} \cong \mathcal{H}^j_{0j} \otimes \mathcal{H}^0_{jj^\dagger} \to \mathcal{H}^0_{jj^\dagger} \otimes \mathcal{H}^0_{00} \cong \mathbf{C}$$

is an identity map, we obtain the assertion from definition 1.5 and 3.3.

Now the proof of our main results is completed in the following way. We obtain from Proposition 3.5 that $< v^*_j, \rho(h)v_0 >$ is an entry of a unitary matrix. Hence it follows that

$$\sum_{j \in P_+(l)} | < v^*_j, \rho(h)v_0 > |^2 \leq 1,$$

which shows Theorem B. In the case $M = S^3$ we have

$$Z_r(S^3, K; j) = J_r(K, j).$$

Moreover, by means of the symmetry principle due to Kirby and Melvin [KM], we have

$$|J_r(K, j) = J_r(L_1 \cup L_2 \cup \cdots L_g, \lambda)|$$

for $\lambda(1) = j$ and

$$\lambda(k) = 0 \text{ or } l/2, \quad 2 \le k \le g.$$

Combining this symmetry with Theorem B, we obtain Theorem A. This completes the proof.

4. Comments on generalizations and discussions

We discuss briefly how our results can be generalized to an arbitrary complex simple Lie algebra $\mathcal{G}$. We refer the readers to [Ko3] for a more detailed description. We fix a positive integer l and we adopt as the set of colors

$$(4\text{-}1) \qquad P_+(l) = \{\lambda \in P_+; 0 \le (\lambda, \theta) \le l\}$$

where $(\cdot, \cdot)$ is the Cartan-Killing form normalized so that $(\theta, \theta) = 2$ for the longest root. Let L be an oriented framed link in S^3. By looking at the monodromy representation of the Knizhinik-Zamolodchikov equation in Wess-Zumino-Witten model at level l (see [TK]), for a color $\lambda : \{1, \cdots, m\} \to P_+(l)$, one can define an invariant of colored framed link $J_{\mathcal{G}, r}(L, \lambda)$ where we put $r = l + h^*$ with the dual Coxeter number h^*.

The phase factor derived from the change of framing in Proposition 2.9 is expressed by the conformal weight

$$\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(l + h^*)}$$

where ρ is the half sum of the positive roots.

Now in the situation of Theorem B, the Witten invariant for $\mathcal{G}$ at level l is expressed as

$$(4\text{-}2) \qquad Z_{\mathcal{G}, r}(M, K, \lambda) = C^{sign(L)} \sum_\mu S_{0\mu(1)} \cdots S_{0\mu(m)} J_{\dot{\mathcal{G}}, r}(K \cup L, \lambda \cup \mu)$$

where the sum is for any $\mu : \{1, \cdots, m\} \to P_+(l)$. In the above expression C and $S_{0\lambda}$ are given by

$$(4\text{-}3) \qquad \begin{aligned} C &= \exp\left(-\frac{l \dim \mathcal{G} \, \pi \sqrt{-1}}{4(l + h^*)}\right) \\ S_{0\lambda} &= \frac{1}{(l + h^*)^{rank(\mathcal{G})/2}} \left(\frac{vol \, \Lambda^W}{vol \, \Lambda^R}\right)^{1/2} \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi(\lambda + \rho, \alpha)}{l + h^*}. \end{aligned}$$

Here $vol \, \Lambda^W$ is the volume of the weight lattice, $vol \, \Lambda^R$ is the volume of the coroot lattice, and Δ_+ stands for the set of positive roots. Let us notice that C can be

written as $\left(\exp 2\pi\sqrt{-1}\frac{c}{24}\right)^{-3}$ with c the central charge of the Virasoro algebra acting on the integrable highest weight module of highest weight λ. We also recall that $S_{0\lambda}$ appears as an entry of the modular transformation S matrix on the characters of level l integrable highest weight modules determined in [KP] and [KW].

Several approaches have been developed to establish a more combinatorial description of such invariants for classical Lie algebras (see [M], [TW], [We] and [KT]). The associated representation of the mapping class group is related to the monodromy representation of the conformal field theory on Riemann surfaces (see [TUY]).

Let K be a knot in a closed oriented 3-manifold M. Generalizing Theorem B, we have a lower estimate for the tunnel number $t(K)$

$$
(4.4) \qquad \sum_{\lambda \in P_+(l)} |Z_{\mathcal{G},r}(M,K;\lambda)|^2 \le S_{00}^{-2(t(K)+1)}.
$$

We also notice that considering the case K is empty, we have the following lower bound for the Heegaard genus (see [Wa], [Ko2] and [G]).

$$
(4.5) \qquad |Z_{\mathcal{G},r}(M)| \le S_{00}^{-g(M)}
$$

Let us discuss the case $\mathcal{G} = sl(n, \mathbb{C})$. We denote by α_i, $1 \le i \le n-1$, a set of simple roots and s_{α_i} the reflection on the weight lattice defined by $s_{\alpha_i}(v) = v - (v, \alpha_i)\alpha_i$. The set of weights $P_+(l)$ is written as

$$
P_+(l) = \{ \sum_{i=1}^{n-1} a_i \Lambda_i ; a_i \in \mathbb{Z}, a_i \ge 0, \sum_{i=1}^{n-1} a_i \le l \}
$$

with the fundamental weights Λ_i, $1 \le i \le n-1$, which is naturally identified with the set of dominant integral weights for the corresponding affine Lie algebra $\widehat{\mathcal{G}}$ at level l. Hence we have a Dynkin diagram automorphism for the affine Lie algebra, which induces an automorphism on the set $P_+(l)$. This automorphism is isomorphic to a cyclic group of order n generated by σ defined by

$$
\sigma(\lambda) = C(\lambda) + l\Lambda_1
$$

where C is the Coxeter transformation $s_{\alpha_1} \cdots s_{\alpha_{n-1}}$. In [KT] we described the phase factor coming from this action, which implies that for a knot K in S^3 we have

$$
(4.6) \qquad |J_{\mathcal{G},r}(K, \sigma(\lambda))| = |J_{\mathcal{G},r}(K, \lambda)|.
$$

Let us notice that σ has a fixed point $l/n \sum_{i=1}^{n-1} \Lambda_i$ if $l \equiv 0 \mod n$ and is fixed point free otherwise. We suppose that $\lambda \in P_+(l)$ is not a fixed point of σ and that λ is not contained in the orbit of $0 \in P_+(l)$. Then, we have an inequality

$$
(4.7) \qquad \sum_{\lambda \in P_+(l)} |J_{\mathcal{G},r}(K,\lambda)|^2 \le n^{-t(K)} S_{00}^{-2(t(K)+1)}.
$$

We recall that the skein (or HOMFLY) polynomial $P_L(a,z)$ for an oriented link is a two-variable generalization of the Jones polynomial (see [Jo]) characterized by the skein relation:

(4-8)
$$P_{\bigcirc} = 1$$
$$a^{-1}P_{L_+} - aP_{L_-} = zP_{L_0}.$$

Let us suppose that $n \geq 3$ and $l \geq 2$. Considering the case when $\lambda \in P_+(l)$ corresponds to the vector representation, we have

$$P_K(s^n, s - s^{-1}) = \frac{\sin \pi/r}{\sin n\pi/r} J_{\mathcal{G},r}(K, \lambda)$$

for $s = e^{\pi\sqrt{-1}/r}$. Combining with 4.6 and 4.7, we obtain a upper bound for the above special values of the skein polynomial. A similar estimate can be formulated for special values of the Kauffman polynomial by examining the case $\mathcal{G} = o(n, \mathbf{C}), sp(n, \mathbf{C})$.

Our construction may also be generalized to certain links. Let $L = L_1 \cup L_2$ be a link with 2 components in a closed oriented 3-manifold M. We define its tunnel number $t(L)$ to be the minimal number $g+1$ such that the link exterior $E(L) = cl(M - N(L))$ admits a splitting
$$E(L) = (V^1 - N(L_1)) \cup (V^2 - N(L_2))$$

with a handlebody V^1 of genus g and its second copy V^2 with a reversed orientation. Here we assume that L_i, $i = 1, 2$, is embedded in V^i in a simple position as shown in Figure 12. In this situation we obtain for $\lambda_1 \in P_+(l)$

(4-9)
$$\sum_{\lambda_2 \in P_+(l)} |Z_{\mathcal{G},r}(M, L; \lambda_1, \lambda_2)|^2 \leq S_{00}^{-2(t(L)+1)}$$

where λ_i is a color on L_i, $i = 1, 2$.

4.10 Concluding Remarks.

(i) It was shown by Jones [Jo] that if an oriented link L is a closed n braid, then

(4-11)
$$|V_L(e^{2\pi\sqrt{-1}/r})| \leq \left(2\cos\frac{\pi}{r}\right)^{n-1}, \quad r = 3, 4, \cdots.$$

The unitarity of the monodromy representation enables us to generalize this inequality in the form
$$|J_{\mathcal{G},r}(L, \lambda)| \leq \left(\frac{S_{0\lambda}}{S_{00}}\right)^n$$

using the framed link invariant with color $\lambda \in P_+(l)$ for each component of L. Combining with our estimate of the tunnel number, we can deduce several consequences.

For example, if a knot K satisfies the equality for $r = 5$ and $n = 3$ in 4.11 (see [Jo]), then we have $t(K) \geq 2$.

(ii) For a connected sum of knots we have

$$(4\text{-}12) \qquad J_{\mathcal{G},r}(K_1 \natural K_2, \lambda) = J_{\mathcal{G},r}(K_1, \lambda) J_{\mathcal{G},r}(K_2, \lambda) \frac{S_{00}}{S_{0\lambda}}.$$

For a knot K, let us consider $K \natural \cdots \natural K$ (n times). Our result can be used to give a lower estimate for the tunnel number of such knot in terms of a function of n which is of linear order with respect to n.

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