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ON SIEGEL WAVE FORM ON THE COVERING GROUP OF $Sp(2,\mathbf{R})$

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In this paper, we shall show that the complete analogue of the shimura correspondence of modular forms of half integral weight holds for Siegel wave forms of degree 2.

§1. Commutation relation of differential operators

As usual, we consider an element in the center of the universal enveloping algebra of Lie algebra of Lie groups G as a differential operator on G. Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2,\mathbf{R})$ are given in [13].

Put

$$H_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\qquad H_2=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$

$$X_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix},\qquad X_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$X_3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad X_4=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$X_{-1}={}^tX_i,\qquad (i=1,2,3,4).$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2,\mathbf{R})$ in [6] are

$$\begin{aligned}L_1=&H_1H_1+H_2H_2+6H_1\\&+2H_2+4X_{-1}X_1+8X_{-4}X_4+4X_{-3}X_3+8X_{-2}X_2,\end{aligned}$$

$$\begin{aligned}L_2=&16X_{-4}X_{-4}X_4X_4+16X_{-4}X_{-3}X_3X_4\\&-32X_{-4}X_{-2}X_2X_4+16X_{-4}X_{-2}X_3X_3\\&+16X_{-4}X_{-1}X_1X_4+8X_{-4}H_1H_2X_4\\&+8X_{-4}(H_1-H_2)X_1X_3-16X_{-4}X_1X_1X_2\\&+16X_{-3}X_{-3}X_2X_4+16X_{-3}X_{-2}X_2X_3\\&+8X_{-3}X_{-1}(H_1-H_2)X_4+4X_{-3}H_2H_2X_3\\&+8X_{-3}(H_1+H_2)X_1X_2+16X_{-2}X_{-2}X_2X_2\end{aligned}$$

$$
\begin{aligned}
&- 16X_{-2}X_{-1}X_{-1}X_4 + 8X_{-2}X_{-1}(H_1 + H_2)X_3 \\
&+ 16X_{-2}X_{-1}X_1X_2 - 8X_{-2}H_1H_2X_2 \\
&+ 4X_{-1}H_1H_1X_1 + H_1H_1H_2H_2 \\
&- 16X_{-4}H_1X_4 + 32X_{-4}H_2X_4 + 32X_{-4}X_1X_3 \\
&+ 32X_{-3}X_{-1}X_4 - 8X_{-3}H_1X_3 + 16X_{-3}X_1X_2 \\
&+ 16X_{-2}X_{-1}X_3 - 16X_{-2}(H_1 + H_2)X_2 \\
&+ 24X_{-1}H_1X_1 + 2H_1H_1H_2 \\
&+ 6H_1H_2H_2 \\
&- 48X_{-4}X_4 - 24X_{-3}X_3 - 48X_{-2}X_2 \\
&+ 24X_{-1}X_1 - 2H_1H_1 + 12H_1H_2 \\
&+ 6H_2H_2 - 12H_1 + 12H_2.
\end{aligned}
$$

We define the Weil representation r of $G = Sp_2(\mathbf{R})$ on $V = M_{5,1}(\mathbf{R})$ by putting

$$
r\begin{pmatrix} E & S \\ 0 & E \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} = \exp(2\pi \mathrm{tr}(S({}^tX_1X_2 + {}^txx)))f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix},
$$

$$
r\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} = \gamma(\det A)^{5/2} f \begin{pmatrix} X_1A \\ xA \\ X_2A \end{pmatrix},
$$

$$
r\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f \begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} = -2i \int_V \int_V
$$

$$
\exp(2\pi i \mathrm{tr}({}^tY_1X_2 + {}^tY_2X_1 + 2{}^tyx)) f \begin{pmatrix} Y_1 \\ y \\ Y_2 \end{pmatrix} dY_1dY_2dy
$$

for $f \in \mathcal{S}(V \times V), S = {}^tS \in M_{2,2}(\mathbf{R}), A \in M_{2,2}(\mathbf{R})$ where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\gamma = 1$ or $-i$ according as $\det A > 0$ or < 0. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ put $J(g, Z) = \det(CZ + D)$. This definition of r for the generators determines r for general $g \in G$ up to a sign. We determine a branch of $\sqrt{J(g,Z)}$ by $-\pi < \arg(\sqrt{J(g,iE)}) \leq \pi$. For $X \in V$ put $\varphi_Z(X) = \exp(2\pi i Z[X])$ and determine the sign of $r(g)$ by

$$
r(g)\varphi_Z(X) = \sqrt{J(g,Z)}^{-1} \varphi_{gZ}(X).
$$

Put

$$
a(h,g) = \sqrt{J(h,gZ)}\sqrt{J(g,Z)}/\sqrt{J(hg,Z)}.
$$

Define the double cover $\tilde{G} = \{(g,\epsilon) | g \in G, \epsilon = \pm 1\}$ of G by putting $(h,\epsilon)(g,\epsilon') = (hg, \epsilon\epsilon' a(h,g))$ and define a representation $\tilde{r}$ of $\tilde{G}$ by $\tilde{r}(g,\epsilon)f = \epsilon r(g)f$.

We can also define representation ρ of the "orthogonal group" G on $\mathcal{S}(V \times V)$ in the following manner. First, we define linear mapping σ by

$$\sigma(X) = \begin{pmatrix} 0 & a & c & -f \\ -a & 0 & -b & -c \\ -c & b & 0 & d \\ f & c & -d & 0 \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} \in M_{5,1}(\mathbf{R}).$$

Then $g \in G$ acts on $V \times V$ by

$$\begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix}^g = \left(\sigma^{-1}\left({}^t g \left(\sigma \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \end{pmatrix} \right) g \right), \sigma^{-1}\left({}^t g \left(\sigma \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{52} \end{pmatrix} \right) g \right) \right)$$

for

$$\begin{pmatrix} X_1 \\ x \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \end{pmatrix} \in M_{5,2}(\mathbf{R}) = V \times V.$$

Put

$$\rho(g)f(X) = f(X^g)$$

for $f \in \mathcal{S}(V \times V), g \in G$. Then the representations r, ρ induce the representations (differential representations) of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$ which we denote by the same letters r, ρ. With this notation, we obtain

THEOREM 1.

$$r(L_1) = \frac{1}{2}\rho(L_1) - 5,$$
$$r(L_2) = -\frac{1}{4}\rho(L_2) + \frac{1}{16}\rho(L_1)^2 - \rho(L_1) + 10.$$

§2. Standard and generaliged Whittaker functions

For a symmetric matrix T, we call functions $f(g) = h(Y)\exp(2\pi i \mathrm{tr}(TX))$ of $g \in G$ satisfying $L_1 f = \delta_1 f$ and $L_2 f = \delta_2 f$ with some functions h of Y and $X+iY = giE \in H_2$ generaliged Whittaker functions. These functions for definite T are investigated in [10],

[16]. We deal with Whittaker functions for indefinite T here. By the same reasoning as in [10], [16], we can probably show the followings which we assume hereafter. If $F(Z)$ is a Siegel wave form satisfying $F(Z+n_0N)=F(Z)$ for all $N={}^tN\in M_{2,2}(\mathbf{Z})$, $L_1F=\delta_1F$ and $L_2F=\delta_2F$, $F(Z)$ is expanded as follows:

$$F(Z)=\sum_{2n_0T=2n_0{}^tT\in M_{2,2}(\mathbf{Z})} A(T,Y)\exp(2\pi i\mathrm{tr}(TX))$$

where if T is indefinite

$$A(T,Y)=\int_{\Re s=\sigma} A_s(T)W_s(T,Y)ds$$

where

$$W_s(T,Y)=W_s(S,Y[{}^tN^{-1}]), S=\begin{pmatrix}0&1\\1&0\end{pmatrix}, T=S[N]={}^tNSN,$$

$$\begin{aligned}&W_s\left(S,t\begin{pmatrix}\sqrt{y}&0\\0&\frac{1}{\sqrt{y}}\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}1&0\\x&1\end{pmatrix}\begin{pmatrix}\sqrt{y}&0\\0&\frac{1}{\sqrt{y}}\end{pmatrix}\right)\\
&=y^st^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-2\pi xk_1k_2t)((k_1^2+1)(k_2^2+1)(x^2+1))^{s/2}\\
&\quad K_s(2\pi t\sqrt{(k_1^2+1)(k_2^2+1)(x^2+1)})\\
&\quad F((s+1/2+\nu_1)/2,(s+1/2-\nu_1)/2;1/2;-k_1^2)\\
&\quad F((s+1/2+\nu_2)/2,(s+1/2-\nu_2)/2;1/2;-k_2^2)dk_1dk_2,\end{aligned}$$

and

(1) $$\delta_1=(\lambda_1+\lambda_2-2)/8, \delta_2=((\lambda_1-\lambda_2)^2)/256-(\lambda_1+\lambda_2)/32+3/64$$

with

$$\nu_1=(-1+2\sqrt{\lambda_1})/2, \nu_2=(-1+2\sqrt{\lambda_2})/2.$$

$A_s(T)$ is a constant determined by the uniquness of the above expansinon. We call $A_s(T)$ a Fourier coeffiiciet of F.

Following [20],[21], we define an embedding of $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ into $Sp(2,\mathbf{R})$ by

$$\iota\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix},\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\right)=\begin{pmatrix}a_1&0&b_1&0\\0&a_2&0&b_2\\c_1&0&d_1&0\\0&c_2&0&d_2\end{pmatrix}$$

which is consistent to an embedding $(z_1, z_2) \mapsto \kappa(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ of $H_1 \times H_1$ into H_2. We describe the "Mellin transformation" of F which is analogous to [11] and gives rise to the spinor L function of F. The "Mellin transformation" of F is

$$L(n_0, s_1, s_2, F) = \int_0^\infty \int_0^\infty \int_0^{n_0} \int_0^{n_0} F(\kappa(x_1+iy_1, x_2+iy_2)) y_1^{s_1-1} y_2^{s_2-1} dx_1 dy_1 dx_2 dy_2$$
$$= 2n_0^{2s_1} \Gamma(s_1, s_2) \sum_{m \neq 0} A_{s_1-s_2} \begin{pmatrix} 0 & m/n_0 \\ m/n_0 & 0 \end{pmatrix} |m|^{-2s_1}$$

where

$$\Gamma(s_1, s_2) = \int_0^\infty W_{s_2-s_1}(S, tE) t^{s_2+s_1+1} dt$$
$$= 2^{-2} \pi^{-s_1-s_2-1} \Gamma((s_1 + s_2 - 1/2 + \nu_1)/2 + 1)$$
$$\Gamma((s_1 + s_2 - 1/2 - \nu_1)/2 + 1) \Gamma((s_1 + s_2 - 1/2 + \nu_2)/2 + 1)$$
$$\Gamma((s_1 + s_2 - 1/2 - \nu_2)/2 + 1) / \Gamma(s_1) \Gamma(s_2).$$

We note that when $s_1 = s_2$, the series in $L(n_0, s_1, s_2, F)$ essentially equals the spinor L function of F. If a function W on $\tilde{G}$ satisfies

$$L_1 W = \delta_1' W, L_2 W = \delta_2' W,$$
$$W\left(\left(\begin{pmatrix} 1 & x_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ 0 & 1 & x_2 & x_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 1\right)(g, \epsilon)\right)$$
$$= \exp(2\pi i(x_0 + x_3)) W((g, \epsilon)),$$
$$W((g, \epsilon)(k, \epsilon')) = W((g, \epsilon)) \epsilon' / \sqrt{J(k, iE)}$$

for all $k \in K = G \cap SO(4)$, we call it a Whittaker function. It is known that such a function is unique up to a constant multiple if it exists. As stated in [20], we can construct a Whittaker function using the above generalized Whittaker function. We describe an explicit formula of a Whittaker function W. Obviously, W is determined by the values

$$\tilde{W}(t, y) = W\left(\left(\begin{pmatrix} \sqrt{ty} & 0 & 0 & 0 \\ 0 & \sqrt{t/y} & 0 & 0 \\ 0 & 0 & 1/\sqrt{yt} & 0 \\ 0 & 0 & 0 & 1/\sqrt{t/y} \end{pmatrix}, 1\right)\right)$$

The values are

$$\tilde{W}(t, y) = t^2 \sqrt{y/t}/2 \int_{-\infty}^\infty \int_0^\infty w(s, t')$$
$$\exp(-\pi(yt'^2/(4t) + 4ty/t'^2 + 2t/y + 4s^2 t/y)) t'^{-3} ds dt$$

where

$$w(s,t') = t'^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-2\pi s k_1 k_2 t')$$
$$K_0(2\pi t'\sqrt{(k_1^2+1)(k_2^2+1)(s^2+1)})$$
$$F((1/2+\nu_1)/2,(1/2-\nu_1)/2;1/2;-k_1^2)$$
$$F((1/2+\nu_2)/2,(1/2-\nu_2)/2;1/2;-k_2^2)dk_1dk_2,$$
$$\delta_1' = \delta_1/2-5, \delta_2' = -\delta_2/4+\delta_1^2/16-\delta_1+10$$

with δ_1, δ_2 in (1).

§3. The Shimura correspondence

In this section, we define Shimura correspondence by a theta lifting and compute the "Mellin transform" of the lifted function, which is a sort of convolution of an original function and a theta function

$$\theta(Z) = \sum_{N\in M_{2,1}(\mathbf{Z})} \exp(2\pi i Z[N]).$$

For an odd integer N, put

$$\Gamma_0(4N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbf{Z}) \middle| C \equiv 0 \bmod 4N \right\},$$

$j(\gamma,Z) = \theta(\gamma Z)/\theta(Z)$ and let χ be a Dirichlet character. If a slowly increasing function $f(Z)$ on H_2 satisfies $L_1 f = \delta_1' f, L_2 f = \delta_2' f, f(\gamma Z) = j(\gamma,Z)\chi(\det D) f(Z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma = \Gamma_0(4N)$, we call f a Siegel wave form. If $\int_{U\cap\Gamma\backslash U} f(uZ)du = 0$ for all unipotent radicals U of maximal parabolic subgroups of G, we call f a cusp form. We assume f is a cusp form hereafter. Since $f(Z+N) = f(Z)$ for $N = {}^tN \in M_{2,2}(\mathbf{Z})$, $f(Z)$ is expanded as follows:

$$f(Z) = \sum_{2T=2{}^tT\in M_{2,2}(\mathbf{Z})} a(T,Y)\exp(2\pi i \operatorname{tr}(TX)).$$

Since $a(T,Y[N]) = a(T[{}^tN^{-1}],Y)$,

$$a\left(\begin{pmatrix} 0 & 0 \\ 0 & m^2 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right]\right)$$

is expanded as the Fourier series in z. The uniqueness of Whittaker functions implies

(2) $$a\left(\begin{pmatrix}0&0\\0&m^2\end{pmatrix},\begin{pmatrix}ty&0\\0&t/y\end{pmatrix}\left[\begin{pmatrix}1&0\\x&1\end{pmatrix}\right]\right)$$

$$=\sum_{n\neq 0} a_n^{m^2} W\left(\left(\begin{pmatrix}mn&0&0&0\\0&m&0&0\\0&0&1/(mn)&0\\0&0&0&1/m\end{pmatrix}\begin{pmatrix}1&x&0&0\\0&1&0&0\\0&0&1&0\\0&0&-x&1\end{pmatrix}\right.\right.$$

$$\left.\left.\begin{pmatrix}\sqrt{ty}&0&0&0\\0&\sqrt{t/y}&0&0\\0&0&1/\sqrt{yt}&0\\0&0&0&1/\sqrt{t/y}\end{pmatrix},1\right)\right)$$

$$=\sum_{n\neq 0} a_n^{m^2}\tilde{W}(m^2|n|t,|n|y)\exp(2\pi inx)\begin{cases}1, & \text{if } n>0,\\ -i, & \text{if } n<0.\end{cases}$$

Let S_0 be $\begin{pmatrix}E&0&0\\0&2&0\\0&0&E\end{pmatrix}$. Put $k(X)=\exp(\pi\mathrm{tr}S_0[X])$ for $X\in M_{5,2}(\mathbf{R})$,

$$\mathcal{L}_0=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\middle|\, a,b,c,d\in(1/\sqrt{4N})\mathbf{Z}\right\},$$

$$\mathcal{L}_\theta=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\middle|\, a,b,c,d\in\sqrt{N}\mathbf{Z}\right\},$$

$$\mathcal{L}_2=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\middle|\, a,b,c,d\in\sqrt{4N}\mathbf{Z}\right\}.$$

Define a character χ' by $\chi'(n)=\chi(n)\left(\frac{-N}{n}\right)$ and theta functions by

$$\Theta'(g,h)=\sum_{l_1\in\mathcal{L}_0,l_\theta\in\mathcal{L}_\theta,l_2\in\mathcal{L}_2}\chi'^{-1}(4N\det l_0)r(g)\rho(h)k\begin{pmatrix}l_0\\l_\theta\\l_2\end{pmatrix},$$

$$\Theta(Z,W)=\Theta'(g,h)\sqrt{J(g,iE)}$$

for $Z=giE, W=hiE\in H_2$. Put $h(Z)=f(-(4NZ)^{-1})/\sqrt{\det Z}$ and consider

$$F(W)=\int_{\Gamma\backslash H_2} h(Z)\Theta(Z,W)|Y|^{1/2}dXdY/|Y|^3\quad (Z=X+iY).$$

Then it is easy to see $F(\gamma Z)=\chi'^2(d)F(Z)$ for all $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ with $D=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ in $\Gamma_*^0(4N)$ where

$$\Gamma_*^0(4N)=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(2,\mathbf{Z})\middle|\, B\equiv 0 \bmod 4N,\ D\equiv\begin{pmatrix}*&0\\ *&*\end{pmatrix}\bmod 4N\right\}.$$

Put

$$L_{4N}(s_1,s_2)=\left(\sum_{n\neq 0,(n,4N)=1}\chi'(n)n^{-s_1-s_2-2}\right)\left(\sum_{m\neq 0,(m,4N)=1}\chi'(m)m^{s_1-s_2-1}\right).$$

Then we obtain

THEOREM 2.

$$
\begin{aligned}
&L(4N, s_1, s_2, F)\\
&= \int_0^\infty \int_0^\infty \int_0^{2N} \int_0^{2N} F(\kappa(x_1+iy_1, x_2+iy_2)) y_1^{s_1-1} y_2^{s_2-1} dx_1 dy_1 dx_2 dy_2\\
&= c(s_1,s_2)\Gamma'(s_1,s_2)L_{4N}(s_1,s_2)\\
&\left(\sum_{m\neq 0}\sum_{n>0} a_n^{m^2}(m^2n)^{-s_2+1/2} n^{s_1+1} \sum_{c>0} c^{-2s_1-2} \sum_{(c,d)=1, d \bmod c} \exp(2\pi i dn/c)\right)
\end{aligned}
$$

where $c(s_1, s_2)$ *is a product of powers of* $2^{s_1}, 2^{s_2}, N^{s_1}, N^{s_2}, \pi^{s_1}, \pi^{s_2}, 2, N, \pi$ *and*

$$
\begin{aligned}
&\Gamma'(s_1,s_2)\\
&=\pi^{s_1-s_2-1/2}\Gamma((s_2-s_1+1)/2)\Gamma((s_2+s_1+2)/2)/\Gamma(s_1+1)\\
&\quad\int_0^\infty\int_0^\infty \tilde{W}(t,y)\exp(-2\pi t/y)y^{-3/2}K_{s_1+1/2}(2\pi y)t^{s_2-3/2}dtdy\\
&=c'(s_1,s_2)\\
&\quad\Gamma((s_1+s_2+2-1/2+\nu_1)/2)\Gamma((s_1+s_2+2-1/2-\nu_1)/2)\\
&\quad\Gamma((-s_1+s_2+2-1/2+\nu_1)/2)\Gamma((-s_1+s_2+2-1/2-\nu_1)/2)\\
&\quad\Gamma((s_1+s_2+2-1/2+\nu_2)/2)\Gamma((s_1+s_2+2-1/2-\nu_2)/2)\\
&\quad\Gamma((-s_1+s_2+2-1/2+\nu_2)/2)\Gamma((-s_1+s_2+2-1/2-\nu_2)/2)\\
&\quad\Gamma(s_2+1)^{-1}\Gamma((-s_1+s_2+1)/2)^{-2}\Gamma((s_1+s_2+2)/2)^{-2}\\
&\quad\Gamma(s_2+1)\Gamma((-s_1+s_2+1)/2)\Gamma((s_1+s_2+2)/2)\Gamma(s_1+1)^{-1}
\end{aligned}
$$

with a product $c'(s_1, s_2)$ *of powers of* $2^{s_1}, 2^{s_2}, N^{s_1}, N^{s_2}, \pi^{s_1}, \pi^{s_2}, 2, N, \pi$.
$\Gamma'(s_1, s_2)$ *essentially equals* $\Gamma(s_1, s_2)$ *at least when* $s_2 = s_1$.

REFERENCES

1. ***Automorphic forms on*** $GL(3, \mathbf{R})$, Lect. Notes in Math. **1083** (1984).
2. D. Bump , *Barnes second lemma and its application to Rankin-Selberg convolutions*, Amer. J. of Math. **109** (1987), 179-186.
3. ———, "The Rankin-Selberg method. In : Number theory, trace formulas and discrete groups: a symposium in honor of Atle Selberg," Academic Press, 1989.
4. D. Bump, S. Friedberg, J. Hoffstein , *Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives*, Ann. Math. **131** (1990), 53-127.
5. ———, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Invent. Math. **102** (1990), 543-618.
6. S. Friedberg and S. Wong, *On the Shimura correspondence for* $GSp(4)$, Math. Ann. **290** (1991), 183-207.
7. I. S. Gradshteyn and I. M. Ryzhik, "Tables of integrals, series, and products," Academic Press, 1980.
8. M. Hashizume , *Whittaker functions on semesimple Lie groups*, Hiroshima Math. J. **12** (1982), 259-293.

9. ______, *Whittaker functions on semesimple Lie group and their applications*, Seminar Note, RIMS **631** (1987), 123-137.
10. A. Hori, *Andrianov's L-function associated to Siegel wave forms of degree two*, preprint.
11. R. Howe and I. I. Piatetski-Shapiro, *Some examples of automorphic forms on Sp_4*, Duke Math. J. **50** (1983), 55-106.
12. T. Ibukiyama, *Construction of half integral weight Siegel modular forms of $Sp(2,\mathbf{R})$ from automorphic forms of the compact twist $Sp(2)$*, J. Reine Angew. Math. **359** (1985), 188-220.
13. S. Nakajima , *Invariant differential operators on $SO(2,q)/SO(2)\times SO(q)$ $(q \geq 3)$*, Master these, Univ. of Tokyo.
14. ______, *On invariant differential operators on bounded symmetric domains of type 4*, Proc. Japan Acad. **58**, Ser. A (1982), 235-238.
15. S. Niwa , *Modular forms of half integral weight and the integral of certain theta functions*, Nagoya Math. J. **56** (1974), 147-161.
16. ______, *On generalized Whittaker functions on Siegel's upper half space of degree 2*, Nagoya Math. J. **121** (1991), 171-184.
17. ______, *On Whittaker functions on groups of low rank*, Seminar Note, RIMS **752** (1991), 14-22.
16. M. E. Novodvolsky , *Fonctions J pour $GSp(4)$*, C. R. Acad. Sci. Paris Sér. A **280** (1975), 191-192.
17. ______, *Automorphic L-functions for symplectic group $GSp(4)$*, Proc. Symp. Pure Math. **33** (1979), 87-95.
18. T. Oda , *On Whittaker functions of class 1 on $Sp(2,\mathbf{R}) = Sp_4(\mathbf{R})$*, Seminar Note, RIMS **689** (1989), 148-164.
19. ______, *An explicit integral representation of Whittaker functions for the representations of the discrete series. - The case of $Sp(2,\mathbf{R})$ -*, preprint.
20. I. I. Piatetski-Shapiro and D. Soudry , *Automorphic forms on the symplectic group of order four*, Lecture Notes, I.H.E.S. (1983).
21. ______, *L and ϵ functions for $GSP(4)\times GL(2)$*, Proc. Natl. Acad. Sci. USA **81** (1984), 3924-3927.
22. G. Shimura , *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440-481.
23. ______, *Invariant differential operators on Hermitian symmetric spaces*, Ann. of Math. **132** (1990), 237–272.
23. T. Shintani, *On construction of holomorphic cusp form of half integral weight*, Nagoya Math. J **58** (1975), 83-126.
24. D. Soudry , *A uniqueness theorem for representations of $GSO(6)$ and the strong multiplicity one theorem for generic representations of $GSp(4)$*, Israel J. of Math. **58** (1987), 257-287.
25. V. G. Zhuravlev , *Hecke rings for a covering of the symplectic group*, Math. USSR Sbolnik **49** (1984), 379-399.
26. ______, *Euler expansions of theta transforms of Siegel modular forms of half-integral weight and their analytic properties*, Math. USSR Sbornik **51** (1985), 169-190.
27. T. Hina, *On Siegel modular forms of half integral weight*, preprint (1984).
28. H. Maass, *Dirichletshe Reihen und Modulformen zweiten Grades*, Acta Arith. **24** (1973), 225-238.
29. B. Diehl, *Die analytische Fortsetzung der Eisensteinreihe zur Siegelschen Modulgruppe*, J. Reine Angew. Math. **317** (1980), 40-73.
30. W. N. Bailey , *Some infinite integrals involving Bessel functions (1)*, Proc. London Math. Soc. (2) **40** (1936), 37-48.
31. ______, *Some infinite integrals involving Bessel functions (2)*, J. London Math. Soc. **11** (1936), 16-20.