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On q-analogues of multiple sine functions

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Basic back-ground problem: To calculate the sine function

$S_A(x) = \prod_{a \in A} (a-x)$ of a ring (or integral domain) A.

$$\left(\prod_{a\in A}(a-x) \text{ ``}=\text{'' } \exp\left(-\frac{\partial}{\partial s} \sum_{a \in A} (a-x)^{-s} \Big|_{s=0}\right) : \text{regularized product} \right)$$

Example 1. $S_{\mathbb{Z}}(x) = \prod_{m=-\infty}^{\infty} (m-x) \qquad (\operatorname{Im} x > 0)$

$$\underset{\text{def.}}{=} \exp\left(-\frac{\partial}{\partial s} \sum_{m=-\infty}^{\infty} (m-x)^{-s} \Big|_{s=0}\right)$$

$$= 1 - e^{2\pi i x}$$

$$\sim 2\sin(\pi x).$$

Example 2. Let τ be an imaginary quadratic integer, $\operatorname{Im}\tau > 0$, then

$$S_{\mathbb{Z}[\tau]}(x) = \prod_{m,n=-\infty}^{\infty} (m+n\tau+x)$$

$$= \exp\left(-\frac{\partial}{\partial s} \sum_{m,n} (m+n\tau+x)^{-s} \Big|_{s=0}\right)$$

$$= (1-q_x) \prod_{n=1}^{\infty} (1-q_\tau^n q_x)(1-q_\tau^n q_x^{-1})$$

for $0 < \operatorname{Im} x < \operatorname{Im}\tau$, where $q_x = e^{2\pi i x}$ and $q_\tau = e^{2\pi i \tau}$.

There are two proofs

$$\begin{cases} 1) \text{ Fourier expansion of "non-absolute Eisenstein series"} \\ 2) \text{ double gamma function.} \end{cases}$$

It turns out that $S_{\mathbb{Z}[\tau]}(x) = \sin_{q_\tau}(\pi_{q_\tau} x)$.

<u>Problem</u> $A = \mathcal{O}_K$ integer ring of a number field K. Then, what is $S_A(x)$? Moreover, $K^{ab} = K(S_A(K))$?

1° K : totally real $\Rightarrow$ Shintani's approximation to $S_A(x)$ via multiple sine functions.

2° K : not totally real $\Rightarrow$ q-analogues of multiple sine functions.

<u>generalizations</u>

sine function

$$\begin{tikzcd}
 & & S(x) = 2\sin(\pi x) \arrow[dl, Rightarrow, "\text{multiple}"'] \arrow[dr, squiggly, "q\text{-analogue}"] & \\
\text{Hölder type } [r=2] & \mathcal{S}_r(x) \arrow[d, Rightarrow] \arrow[drr, squiggly] & & S^q(x) \arrow[d, Rightarrow] \\
\text{Shintani-type } [r=2] & S_r(x) \arrow[d, Rightarrow] \arrow[drr, squiggly] & & \mathcal{S}_r^q(x) \arrow[d, Rightarrow] \\
 & S_r(x;\omega_1,\ldots,\omega_r) \arrow[drr, squiggly] & & S_r^q(x) \arrow[d, Rightarrow] \\
 & & & S_r^q(x;\omega_1,\ldots,\omega_r)
\end{tikzcd}$$

<u>basic properties</u>

⓪ periodicity

① distribution property (multiplication formula)

② relation to special values of zeta and L-functions.
(Dirichlet's "class number formula", ...)

③ relation to gamma factors of $\left\{\begin{array}{l}\text{Selberg}\\ \text{arithmetic}\end{array}\right\}$ zeta functions.

$\left[\begin{array}{l}\text{④ addition formula}\\ \text{⑤ algebraicity of special values}\end{array}\right]$ ... "difficult" in general

§1. Survey of the sine function.

Let $\mathcal{S}(x) = 2\sin(\pi x)$. Then :

⓪ $\mathcal{S}(x+1) = \mathcal{S}(x)\cdot(-1)$ with $(-1) = \mathcal{S}_0(x)^{-1}$.

① $\mathcal{S}(Nx) = \prod_{k=0}^{N-1} \mathcal{S}(x+\frac{k}{N})$ for integers $N \geqslant 2$.

In particular $\prod_{k=1}^{N-1} \mathcal{S}(\frac{k}{N}) = N$.

② Let χ be a primitive even Dirichlet character modulo $N \geqslant 2$, and $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ the Dirichlet L-function. Then $L(0,\chi) = 0$ and

$$L'(0,\chi) = -\frac{1}{2}\sum_{k=1}^{N-1} \chi(k) \log \mathcal{S}(\tfrac{k}{N}).$$

(Dirichlet's "class number formula" : usually written for $L(1,\chi)$ via functional equation.)

③ Euler (discovery) : $\zeta(1-s) = \zeta(s)(2\pi)^{-s}\Gamma(s)\mathcal{S}(\frac{s+1}{2})$

$\Downarrow$ symmetrize

§2. Survey of multiple sine functions (Kurokawa 1990)

Let $$S_r(x;\underline{\omega}) = \Gamma_r(x,\underline{\omega})^{-1}\,\Gamma_r(|\underline{\omega}|-x,\underline{\omega})^{(-1)^r},$$

where $\underline{\omega} = (\omega_1,\ldots,\omega_r)$, $|\underline{\omega}| = \omega_1+\cdots+\omega_r$

and
$$\Gamma_r(x,\underline{\omega})^{-1} = \prod_{\underline{n}\geq \underline{0}} (\underline{n}\cdot\underline{\omega}+x) \underset{\text{def.}}{=} \exp\left(-\frac{\partial}{\partial s}\zeta_r(0,x,\underline{\omega})\right)$$

with the multiple Hurwitz zeta function
$$\zeta_r(s,x,\underline{\omega}) = \sum_{\underline{n}\geqslant\underline{0}} (\underline{n}\cdot\underline{\omega}+x)^{-s}.$$

We put
$$S_r(x) = S_r(x;(1,\ldots,1))$$

and
$$\mathcal{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{n=-\infty}^{\infty}{}' P_r\left(\frac{x}{n}\right)^{n^{r-1}}$$
$$= e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty}\left(P_r\left(\frac{x}{n}\right)P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}}$$

for $r \geqslant 2$ and $\mathcal{S}_1(x) = 2\sin(\pi x)$; these are meromorphic of order r.

Theorem $$\mathcal{S}_r(x) = C_r\prod_{k=1}^{r} S_k(x)^{c(r,k)}$$

with
$$C_r = \begin{cases} 1 & \cdots\cdots\ r:\text{even} \\ e^{2\zeta'(1-r)} & \cdots\ r:\text{odd}\end{cases}$$

and $$c(r,k) = \frac{1}{k}\sum_{l=1}^{k}(-1)^{l-1}\binom{k}{l} l^r.$$

$(c(r,1)=1,\ \ldots,\ c(r,r)=(-1)^{r-1}(r-1)!\ )$

⓪ $S_r(x+\omega_i, \underline{\omega}) = S_r(x, \underline{\omega})\, S_{r-1}(x, \underline{\omega}(i))^{-1}$

where $\underline{\omega}(i) = (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_r)$.

(similar for $S_r(x)$, $\mathcal{S}_r(x)$ also)

① $$\begin{cases} S_r(Nx, \underline{\omega}) = \prod\limits_{\substack{0\le k_i \le N-1 \\ i=1,\cdots,r}} S_r\left(x+\frac{k\cdot\omega}{N}, \underline{\omega}\right), \\ \prod\limits_{\substack{0\le k_i \le N-1 \\ i=1,\cdots,r}}' S_r\left(\frac{k\cdot\omega}{N}, \underline{\omega}\right) = N. \end{cases}$$

[and homogenuity: $S_r(cx, c\underline{\omega}) = S_r(x, \underline{\omega})$ for $c>0$].

② a) Let χ be a primitive character modulo $N \geq 2$ satisfying $\chi(-1) = (-1)^{r-1}$ for $r \geq 1$. Then

$$L'(1-r, \chi) = \sum_{\substack{1\le j\le r \\ 1\le k\le N-1}} c_{kj}^N\, \chi(k) \log S_j\left(\frac{k}{N}\right).$$

(similar for $\mathcal{S}_j\left(\frac{k}{N}\right)$ instead of $S_j\left(\frac{k}{N}\right)$.)

<u>examples</u>

$$L'(-1,\chi) \underset{\chi:\text{odd}}{=} \frac{1}{2}\log \prod_{k=1}^{N-1}\left(\frac{\mathcal{S}_2\left(\frac{k}{N}\right)^N}{\mathcal{S}_1\left(\frac{k}{N}\right)^k}\right)^{\chi(k)}$$

$$= -\frac{1}{2}\log \prod_{k=1}^{N-1}\left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k\right)^{\chi(k)},$$

$$L'(-2,\chi) \underset{\chi:\text{even}}{=} -\frac{1}{2}\log \prod_{k=1}^{N-1}\left(\frac{\mathcal{S}_3\left(\frac{k}{N}\right)^{N^2}\mathcal{S}_1\left(\frac{k}{N}\right)^{k^2}}{\mathcal{S}_2\left(\frac{k}{N}\right)^{2Nk}}\right)^{\chi(k)}$$

$$= -\frac{1}{2}\log \prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\chi(k)}.$$

two proofs
1) poly-logarithm $\underset{\text{relation}}{\overset{(*)}{\Longleftrightarrow}}$ $\mathcal{S}_r(x) \Rightarrow L(r,\chi) \Rightarrow$ ft. eq.
2) $\zeta(s-r+1, x) \underset{\text{relation}}{\Longleftrightarrow}$ $\zeta_r(s,x) \Rightarrow L'(1-r,\chi)$.

(*)
$$\mathcal{S}_r(x) = \exp\Big(-\frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(2\pi i)^k}{k!} x^k Li_{r-k}(e^{-2\pi i x}) + \frac{\pi i}{r} x^r + \frac{(r-1)!}{(2\pi i)^{r-1}} \zeta(r)\Big)$$

for $\mathrm{Im}\, x < 0$ (or $0 < x < 1$); "Debye polylogarithm" appears.

b) $K/\mathbb{Q}$ totally real, χ certain type Dirichlet character,

$$L'_K(0,\chi) \doteqdot \sum_{\substack{k \le [K:\mathbb{Q}] \\ j:\text{ finite index}}} c_{k,j} \log \mathcal{S}_k(\alpha_j, \underline{\omega}_j)$$

for some $\alpha_j \in K$; originally due to Shintani via $\Gamma_k(\alpha_j, \underline{\omega}_j)$.

3 gamma factors of Selberg zeta functions.

Let $M = \Gamma\backslash G/K$ be a compact locally symmetric space of rank one. Assume that $\dim M$ is even ($\Leftrightarrow$ the gamma factor is non-trivial $\Leftrightarrow$ $G \not\simeq SO(1, 2n-1)$).

Then, the Selberg zeta function $Z_M(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ of order $\dim M$ with the functional equation:

$$Z_M(2\rho_M - s) = Z_M(s)\exp\left(\mathrm{vol}(M)\int_0^{s-\rho_M} \mu_M(it)\,dt\right)$$

where $\mu_M(t)$ is the Plancherel measure. This is due to Selberg (1956 for $G = SL_2(\mathbb{R}) \sim SO(1,2)$) and Gangolli (1977, general G). The problem is to calculate $\exp(\cdots)$ explicitly and obtain the gamma factor $\Gamma_M(s)$ satisfying $\exp(\cdots) = \dfrac{\Gamma_M(s)}{\Gamma_M(2\rho_M - s)}$.

<u>Theorem</u>. Let

$$S_M(s) \underset{\mathrm{def}}{=} \left(\frac{\det\left(\sqrt{\Delta_{M'} + \rho_M^2} + (s-\rho_M)\right)}{\det\left(\sqrt{\Delta_{M'} + \rho_M^2} - (s-\rho_M)\right)}\right)^{\mathrm{vol}(M)(-1)^{\dim M/2}},$$

where $M' = G'/K$ is the compact dual symmetric space and $\Delta_{M'}$ is the Laplacian. Then

$$S_M(s) = \exp\left(\mathrm{vol}(M)\int_0^{s-\rho_M} \mu_M(it)\,dt\right)$$

$$= \begin{cases} \left(S_{2n}(s)\,S_{2n}(s+1)\right)^{\mathrm{vol}(M)(-1)^n} & \cdots\ G = SO(1,2n), \\ \left(\prod_{k=0}^{n} S_{2n}(s+k)^{\binom{n}{k}^2}\right)^{\mathrm{vol}(M)(-1)^n} & \cdots\ G = SU(1,n), \\ \left(\prod_{k=0}^{2n-1} S_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}}\right)^{\mathrm{vol}(M)} & \cdots\ G = Sp(1,n), \\ \left(S_{16}(s)\,S_{16}(s+1)^{10}\,S_{16}(s+2)^{28}\,S_{16}(s+3)^{28}\,S_{16}(s+4)^{10}\,S_{16}(s+5)\right)^{\mathrm{vol}(M)} & \cdots\ G = F_4. \end{cases}$$

Hence we obtain the gamma factor

$$\Gamma_M(s) = \det\left(\sqrt{\Delta_{M'} + \rho_M^2} + (s-\rho_M)\right)^{vol(M)(-1)^{\dim M/2}}$$

$$= \begin{cases} \left(\Gamma_{2n}(s)\Gamma_{2n}(s+1)\right)^{vol(M)(-1)^{n-1}} & \cdots G = SO(1,2n) \\ \left(\prod_{k=0}^{n} \Gamma_{2n}(s+k)^{\binom{n}{k}^2}\right)^{vol(M)(-1)^{n-1}} & \cdots G = SU(1,n) \\ \left(\prod_{k=0}^{2n-1} \Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}}\right)^{-Vol(M)} & \cdots G = Sp(1,n) \\ \left(\Gamma_{16}(s)\Gamma_{16}(s+1)^{10}\Gamma_{16}(s+2)^{28}\Gamma_{16}(s+3)^{28}\Gamma_{16}(s+4)^{10}\Gamma_{16}(s+5)\right)^{-vol(M)} & \cdots G = F_4 . \end{cases}$$

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§3. q-analogue of the sine function

We assume $q > 1$ for simplicity.

$$S^q(x) \underset{def}{=} \frac{\Gamma^q(\frac{1}{2})^2}{\Gamma^q(x)\,\Gamma^q(1-x)} = \bar{S}_q(x)\, q^{-(x-\frac{1}{2})^2} \prod_{n=1}^{\infty}\left(1 - q^{-(n+\frac{1}{2})}\right)^{-2}$$

where $\Gamma^q(x)$ is the Jackson's q-gamma function and

$$\bar{S}_q(x) = (1-q^{-x})\prod_{n=1}^{\infty}\left(1-q^{-(n+x)}\right)\left(1-q^{-(n-x)}\right)$$

$$\sim \sigma(-\tau, -\tau x) \text{ if } q = e^{2\pi i \tau} \text{ with } \tau \in H_- \ (\mathrm{Im}\,\tau < 0)$$

Let $L_q(s,\chi) \underset{def}{=} \sum_{n=1}^{\infty} \chi(n)\,[n]^{-s}$

where $[n] = [n]_q = \dfrac{q^n - 1}{q-1}$ is the q-integer.

⓪ periodicity, and ① distribution property are due to Jackson.

② **Theorem** Let χ be a primitive even Dirichlet character modulo $N \geqslant 2$. Then $L_q(s,\chi)$ is meromorphic on $\mathbb{C}$ and $L_q(0,\chi)=0$. Moreover

$$L_q'(0,\chi) = -\frac{1}{2}\sum_{k=1}^{N-1}\chi(k)\log S^{q^N}\left(\frac{k}{N}\right).$$

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§4. q-multiple sine functions $(q>1)$

We notice only the most basic case.

Let

$$\overline{S}_r^{\,q}(x,\underline{\omega}) \underset{\mathrm{def}}{=} \prod_{\underline{n}\geqslant 0}\left(1-q^{-(\underline{n}\cdot\underline{\omega}+x)}\right)\left(1-q^{-(\underline{n}\cdot\underline{\omega}+|\underline{\omega}|-x)}\right)^{(-1)^{r-1}}$$

for $\underline{\omega}=(\omega_1,\cdots,\omega_r)$. Then:

⓪ $\overline{S}_r^{\,q}(x+\omega_i,\underline{\omega}) = \overline{S}_r^{\,q}(x,\underline{\omega})\,\overline{S}_{r-1}^{\,q}(x,\underline{\omega}(i))^{-1}$.

① $\displaystyle \overline{S}_r^{\,q}(Nx,\underline{\omega}) = \prod_{\substack{0\leq k_i\leq N-1\\ i=1,\cdots,r}}\overline{S}_r^{\,q^N}\left(x+\frac{\underline{k}\cdot\underline{\omega}}{N},\underline{\omega}\right)$

and

$$\overline{S}_r^{\,q}(cx,c\underline{\omega}) = \overline{S}_r^{\,q^c}(x,\underline{\omega}) \quad \text{for } c>0.$$

We have results similar to §2. The above $\overline{S}_r^{\,q}(x,\underline{\omega})$ can be considered as multi-parametric q-analogue of the sine function by setting $(q_1,\cdots,q_r)=(q^{\omega_1},\cdots,q^{\omega_r})$.

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