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Horizontal divisors on arithmetic surfaces associated with Belyi uniformizations*)

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For a finite surjective morphism $f : Y \to X$ between some *arithmetic* surfaces and a *horizontal* prime divisor D on X, we consider questions related to connectedness of $f^{-1}(D)$. The results will then be applied to fundamental groups of related surfaces. This article owes much to Harbater's work [Hb], and contains an appendix on some proof by T. Saito.

By an arithmetic surface, we mean any two dimensional integral scheme of finite type having structure of a flat $\mathfrak{O}$-scheme, where $\mathfrak{O}$ is the ring of integers of a number field k (the dimension relative to $\mathfrak{O}$ is 1). Horizontal divisors are those finite over $\mathfrak{O}$. Let us begin by describing some special examples. First, if $\mathbf{P}^1_{\mathbf{Z}}$ is the projective line over $\mathbf{Z}, f : \mathbf{P}^1_{\mathbf{Z}} \to \mathbf{P}^1_{\mathbf{Z}}$ is defined by $y \to y^N = x$ $(N \geq 1)$, and D is defined by $x = 1$, then $f^{-1}(D) \simeq \mathrm{Spec}\ (\mathbf{Z}[y]/(y^N - 1))$ is connected, being the spectrum of the ring of virtual characters of a finite group ($\simeq \mathbf{Z}/N$ in this case; cf [S] 11·4). Each irreducible component of $f^{-1}(D)$ meets some other components on the special fibers $\mathbf{P}^1_{\mathbf{Z}} \otimes \mathbf{F}_p$ at $p|N$, to make $f^{-1}(D)$ connected. This remains valid if $\mathbf{Z}$ is replaced by any $\mathfrak{O}$. Secondly, if $f : Y \to X$ is *everywhere etale* and D is normal, then distinct irreducible components of $f^{-1}(D)$ cannot meet each other (cf. e.g. [G] Cor 9·11). As these examples show, when $f^{-1}(D)$ splits into the union of several irreducible components, the connectedness of $f^{-1}(D)$ is closely related to ramifications of f at special fibers (vertical prime divisors) of Y. In a sense, it gives a "horizontally patched" information on such ramifications.

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*) Interium report

The main results proved in this note are as follows. Let $X = \mathbf{P}^1_{\mathfrak{O}}$ be the projective t-line over $\mathfrak{O}$ ($\mathfrak{O}, k$ being as above), $L/k(t)$ be a finite extension *unramified outside* $t = 0, 1, \infty$ (the "Belyi uniformization"), and $f : Y \to X$ be the integral closure of X in L. For $a \in k^{\cup}(\infty)$, denote by D_a the prime divisor on X defined by $t = a$. Then

Theorem A (Th 2, Prop 1 of §2). *(i) If* $a = 0, 1, \infty, f^{-1}(D_a)$ *is connected; (ii) if* $a \in \mathbf{Q}, f^{-1}(D_a)$ *is again connected; (iii) there exists* $\mathfrak{O}$ *and* $a \in \mathfrak{O}$*, such that* $a, 1-a$ *are both units of* $\mathfrak{O}$ *(so that* D_a *does not meet* $D_0^{\cup}D_1^{\cup}D_\infty$*), and that* $f^{-1}(D_a)$ *is connected for any* f*.*

As direct applications, we obtain, for example:

Theorem B (i) (T. Saito). $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - D_0^{\cup}D_1^{\cup}D_\infty) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$; *(ii) if one of* $t = 0, 1, \infty$ *is totally ramified in* $L/k(t)$*, then* $\pi_1(Y) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$.

See §3 for more details (Proposition 2, Cor 1,2,3). Saito's original proof of (i) is quite different (see §3, and Appendix).

As for (ii), according to Belyi [B](Th 4 and its proof), every algebraic function field of one variable L over a number field k contains such an element t that $L/k(t)$ is unramified outside $t = 0, 1, \infty$ and, in fact, moreover, totally ramified at $t = \infty$ [when L has a prime divisor of degree 1 over k]. So, (ii) implies that every arithmetic surface over $\mathfrak{O}$ [(having a section over $\mathfrak{O}$)] has a normal model Y such that $\pi_1(Y) \simeq \pi_1(\mathrm{Spec}\,\mathfrak{O})$.

In §1, we shall prove a criterion for connectedness of $f^{-1}(D)$ when $X = \mathbf{P}^1_{\mathfrak{O}}$ (Theorem 1). This is just a direct consequence of Harbater's criterion [Hb] for an algebraic function given as power series over $\mathfrak{O}$ to be rational (a modification of Dwork's criterion). Logically, this is just a simple remark. But the author could not find a reference with explicit statement on this connection, and so he thought it necessary to be presented. We note here that in the *geometric* cases (geometric surfaces, etc.), the connectedness of $f^{-1}(D)$ was established under some mild conditions (such as $(D^2) > 0$) in Hironaka-Matsumura

[H-M] cf. also [Ht]. There, the main point was the extendability of any formal-rational function on the completion of X along D to a global rational function on X. In our arithmetic case, one must also take care of neighborhoods of D above *archimedean* places of $\mathfrak{O}$ which is the role of archimedean radii of convergence appearing in the criterion.

In §2, we restrict ourselves to the case where only $t = 0, 1, \infty$ can be ramified in $f \otimes k$ ("Belyi uniformization"), and obtain Theorem 2, Proposition 1.

In §3, we prove Proposition 2 and its corollaries as direct applications of §2.

The next problem would be to find out whether Theorem 1 extends to more general arithmetic surfaces and a full arithmetic analogue of Hironaka-Matsumura criterion can be described using an appropriate Arakelov type theory. We hope to be able to discuss this problem more concretely in the near future.

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§1. In what follows, k will denote an algebraic number field, $\mathfrak{O}$ the ring of integers of k, and Σ the set of all distinct embeddings $\sigma : k \hookrightarrow \mathbb{C}$. We denote by $K = k(t)$ the rational function field of one variable, and by L/K a finite extension which may contain constant field extensions. Let $X = \mathbf{P}^1_{\mathfrak{O}} = \operatorname{Spec} \mathfrak{O}[t] \cup \operatorname{Spec} \mathfrak{O}[t^{-1}]$, and $f : Y \to X$ be the integral closure of X in L. For each $\sigma \in \Sigma$, let $f_\sigma : Y_\sigma \to X_\sigma$ denote the base change $\otimes_{k,\sigma}\mathbb{C}$ of f. Each f_σ defines a finite branched covering $Y_\sigma(\mathbb{C}) \to X_\sigma(\mathbb{C}) = \mathbf{P}^1(\mathbb{C})$ between (not necessarily connected) compact Riemann surfaces. For $r > 0$, put $B(r) = \{z \in \mathbb{C}; |\, z \,| < r\} \subset \mathbf{P}^1(\mathbb{C})$.

Theorem 1. *Let D_0 be the prime divisor of $X = \mathbf{P}^1_{\mathfrak{O}}$ defined by the equation $t = 0$. Assume that there exists $r_\sigma > 0$ for each $\sigma \in \Sigma$ such that f_σ is unramified above $B(r_\sigma)$ and $\Pi_\sigma r_\sigma \geq 1$. Then the $\mathfrak{O}$-scheme $f^{-1}(D_0) = Y \times_X D_0$ is connected.*

(Note that if L/K is a constant field extension, then $f^{-1}(D_0)$ is the spectrum of the corresponding ring of integers.)

This theorem is a direct consequence of the following result of Harbater ([Hb] Prop 2.1 and the preceding remarks).

Lemma (Harbater). *Let k be a number field with normalized absolute values $|\ |_v$ (so that $\Pi_v \mid a \mid_v = 1$ for all $a \in k^\times$). Suppose that $F(t) \in k[[t]]$ is algebraic over $k(t)$. Then one can choose $r_v > 0$ for each place v of k, with $r_v = 1$ for almost all v, such that $F(t)$ is v-adically convergent on the open disc of radius r_v (w.r.t. $|\ |_v$). If, moreover, one can choose r_v's such that $\Pi_v r_v \geq 1$, then $F(t)$ is rational, i.e. $F(t) \in k(t)$.*

Remark 1. For a complex archimedean place v corresponding to $\sigma, \bar{\sigma} \in \Sigma, r_v$ in this lemma corresponds to $r_\sigma r_{\bar{\sigma}} = r_\sigma^2$ in Theorem 1.

Remark 2. We shall only need the case where $F(t)$ belongs to $\mathfrak{O}[[t]]$ and is integral over $\mathfrak{O}[t]$. In this case, since we may choose $r_v = 1$ for all non-archimedean v, the assumption is $\Pi_\sigma r_\sigma \geq 1$. (It is not easy to make use of non-archimedean v with $r_v > 1$; see Remark 4 at the end of §1.) In this case, the proof in [Hb] is easy enough to be sketched. For each $\sigma, F_\sigma \in \mathbf{C}[[t]]$ is not only holomorphic in the open disc of radius r_σ, but extends to a continuous function on its closure, because F_σ is integral over $\mathbf{C}[t]$. Therefore, by the Riemann-Lebesgue lemma, one obtains $\mid a_n^\sigma \mid r_\sigma^n \to 0 (n \to \infty)$. Therefore, $\Pi_\sigma \mid a_n^\sigma \mid = N(a_n) \to 0$. But since $a_n \in \mathfrak{O}$, and hence $N(a_n) \in \mathbf{Z}$, this implies $N(a_n) = 0$ for $n \gg 0$, hence $F(t) \in \mathfrak{O}[t]$. For more details, and for comparison with classical Dwork criterion, see [Hb] §2.

Proof of Theorem 1. Choose any geometric point η of $Y_k = Y \otimes_{\mathfrak{O}} k$ above $t = 0$, and use the completion of L at η to embed L into $\bar{k}((t))$ ($\bar{k}$: an algebraic closure of k).

Claim 1A. $L \cap \mathfrak{O}[[t]] \subset k(t)$.

Proof. Take any $F = F(t) = \Sigma a_n t^n \in L \cap \mathfrak{O}[[t]]$, and by multiplying a suitable element $\neq 0$ of $\mathfrak{O}[t]$, we assume F to be integral over $\mathfrak{O}[t]$. Let $\sigma \in \Sigma$. Then $F_\sigma(t) = \Sigma a_n^\sigma t^n \in \mathbf{C}[[t]]$ extends to a holomorphic function on $B(r_\sigma)$ (and hence converges on $B(r_\sigma)$), because F_σ is integral over $\mathbf{C}[t]$ and f_σ is unramified above $B(r_\sigma)$. Since $\Pi_\sigma r_\sigma \geq 1$, the above lemma gives $F(t) \in k[t]$.

Claim 1B. *Let E be the quotient field of $\mathfrak{O}[[t]]$ ($k(t) \subset E \subset k((t))$). Then $L \cap E = k(t)$.*

Proof. Since $L \cap E$ is finite over $k(t)$, every element of $L \cap E$ is a $k(t)^\times$- multiple of some $g \in L \cap E$ which is integral over $\mathfrak{O}[t]$. Since $g \in E$ and integral over $\mathfrak{O}[[t]]$, $g \in \mathfrak{O}[[t]]$. Hence $g \in L \cap \mathfrak{O}[[t]] \subset k(t)$ by Claim 1A.

Claim 1C. *L and E are linearly disjoint over $k(t)$.*

Proof. Apply Claim 1B to the Galois closure of L over $k(t)$ (which does not change r_σ's).

Claim 1D. *Let B be the integrel closure of $\mathfrak{O}[t]$ in L. Then $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \simeq \varprojlim(B/t^N B)$ is an integral domain.*

Proof. Since $B \to L$ is injective and $\mathfrak{O}[[t]]/\mathfrak{O}[t]$ is flat, $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \to L \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]]$ is also injective. On the other hand, $\mathfrak{O}[[t]] \to E$ is injective and $L/\mathfrak{O}[t]$ is flat; hence $L \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]] \to L \otimes_{\mathfrak{O}[t]} E = L \otimes_{k(t)} E$ is also injective. By Claim 1C, $L \otimes_{k(t)} E$ is a field. Therefore, $B \otimes_{\mathfrak{O}[t]} \mathfrak{O}[[t]]$ is a domain.

The last isomorphism follows from a general fact; if A is a noetherian ring, M is a (not necessarily free) finite A-module, and I is an ideal of A, then $M \otimes \varprojlim(A/I^n) \simeq \varprojlim(M/I^n M)$ (cf [A-M] p108).

Claim 2. *If J, J' are ideals of B such that (i) $J + J' = (1)$, (ii) $J, J' \supset (t)$, (iii) $(JJ')^n \subset (t)$ for some $n \geq 1$, then either $J = (1)$ or $J' = (1)$.*

Proof. By these conditions,

$$\varprojlim(B/t^N B) \simeq \varprojlim(B/J^N) \oplus \varprojlim(B/J'^N)$$

which reduces the Claim to Claim 1D.

Completing the proof of Theorem 1. If $f^{-1}(D_0) = \mathrm{Spec}(B/tB)$ were not connected, it must be a disjoint union of two non-empty subsets S, S'. Let J (resp. J') be the intersection of all (minimal) primes of B belonging to S (resp. S'). Then J, J' satisfies the conditions of Claim 2. Therefore, J or $J' = (1)$, a contradiction. □

Remark 3. Perhaps we should show some example where $f^{-1}(D)$ is disconnected. This is the case when $L = \mathbf{Q}(t, y)$, with $y^2 - y = t$ and D is defined by $t = 0$. In fact, then $f^{-1}(D) \simeq \mathrm{Spec}(\mathbf{Z}[y]/y(y-1)) \cong \mathrm{Spec}\,\mathbf{Z} \sqcup \mathrm{Spec}\,\mathbf{Z}$. Note that the branch point $t = -\frac{1}{4}$ is "archimedean close" to $t = 0$.

Remark 4. At non-archimedean primes $\mathfrak{p}$, the radius of convergence can be strictly smaller than the distance from the center of the nearest branch point (cf. [Hb] §3 Remark 2, [D-R]). For this reason, we could not use non-archimedean primes to loosen the assumption of Theorem 1.

§2. Let $k, \mathfrak{O}$, L/K, $f : Y \to X$ ($X = \mathbf{P}^1_{\mathfrak{O}}$) be as at the beginning of §1, and now we assume that $f_k; Y_k \to X_k$ is unramified outside $t = 0, 1, \infty$. A prime divisor of X defined by $t = 0, 1$, or ∞ will be called *cuspidal.*

Theorem 2. *If f_k is unramified outside $t = 0, 1, \infty$, and D is a cuspidal prime divisor of $X = \mathbf{P}^1_{\mathfrak{O}}$, then $f^{-1}(D)$ is connected.*

Proof. We may assume that D is the cusp defined by $t = 0$. Replacing t by $t^{1/N}$ with a suitable N, we are reduced to the situation where f_k is unramified outside $t \in \mu_N$ (the group of N-th roots of unity). But then the connectedness of $f^{-1}(D)$ is an immediate consequence of Theorem 1. $\square$

For the closure D_a in $\mathbf{P}^1_{\mathfrak{O}}$ of other rational points $t = a \in k$ $(a \neq 0, 1)$ of $\mathbf{P}^1_k$, we can only prove:

Proposition 1. *If f_k is unramified outside $t = 0, 1, \infty$, and $a \in k$ $(a \neq 0, 1), f^{-1}(D_a)$ is connected at least in the following cases; (i) $a \in \mathbf{Q}$; (ii) $a = (1-\zeta)^{-1}$, where ζ is a root of unity whose order is not a prime power; (ii)′ $a = (1-\zeta')(\zeta - \zeta')^{-1}$, where ζ, ζ' are roots of unity such that none of the orders of $\zeta, \zeta', \zeta'\zeta^{-1}$ are prime powers.*

Remark 5. In cases (ii)(ii)′, a is a *special unit*, i.e., a and $1-a$ are both units. This means that D_a does not meet any cuspidal prime divisor. An example of (ii): $a = (1+\omega)^{-1} = -\omega$, where ω is a cubic root of unity.

By Theorem 1, $f^{-1}(D_a)$ is connected if there exists $\gamma \in GL_2(\mathfrak{O})$ (acting on $\mathbf{P}^1_{\mathfrak{O}}$ by linear fractional transformations) such that $\gamma(a) = 0$ and

$$\prod_{\sigma \in \Sigma} \mathrm{Min}(|\gamma(0)^\sigma|, |\gamma(1)^\sigma|, |\gamma(\infty)^\sigma|) \geq 1.$$

We shall show, in each of the cases (i)(ii)(ii)′, that such an element γ exists.

Actually, we can also show that when a is a special unit, (ii)(ii)′ *are the only cases* where there exists some field $k \ni a$ and some $\gamma \in GL_2(\mathfrak{O})$ satisfying these conditions. Thus, in particular, when a is (a special unit which is) non-abelian over $\mathbf{Q}$, or when (for example) $a = \frac{1}{2}(1+\sqrt{5})$, there does not exist any such γ. We do not know whether $f^{-1}(D_a)$ is connected in such cases.

(i) *The case* $a \in \mathbf{Q}$ $(a \neq 0, 1)$. Write $a = -q/p$ $(p, q \in \mathbf{Z},\ (p,q) = 1,\ q > 0)$. It suffices to find an element $\gamma \in SL_2(\mathbf{Z})$ satisfying $\gamma(a) = 0,\ |\gamma(i)| \geq 1$ $(i = 0, 1, \infty)$. Define $q' \in \mathbf{Z}$

by $0 \leq q' < q$, $pq' \equiv 1 (\mathrm{mod}\, q)$, and $p' \in \mathbf{Z}$ by $p' = (pq'-1)/q$. Then

$$\gamma = \begin{pmatrix} p & q \\ p' & q' \end{pmatrix} \in SL_2(\mathbf{Z}),$$

$\gamma(a) = 0$, and $\gamma(0) = q/q'$, $\gamma(\infty) = p/p'$, $\gamma(1) = (p+q)/(p'+q')$. But $|q'/q| < 1$ and $|p'/p| = |q'/q - 1/pq| \leq 1$; hence $|\gamma(0)|, |\gamma(\infty)| \geq 1$. Moreover,

$$(p'+q')/(p+q) = q'/q - 1/q(p+q);$$

hence

$$-1 \leq q'/q - 1/q \leq (p'+q')/(p+q) \leq q'/q + 1/q \leq 1;$$

hence $|\gamma(1)| \geq 1$. Therefore, γ satisfies the desired properties.

(ii) In this case, it is enough to take $\gamma(t) = 1 - a^{-1}t$. In fact, then $\gamma(a) = 0$, $\gamma(0) = 1$, $\gamma(1) = \zeta$, $\gamma(\infty) = \infty$.

(ii)′ In this case, it is enough to take

$$\gamma = \begin{pmatrix} \zeta - \zeta' & \zeta' - 1 \\ \zeta - \zeta' & \zeta(\zeta' - 1) \end{pmatrix}.$$

In fact, then $\det \gamma = (\zeta - 1)(\zeta' - 1)(\zeta - \zeta') \in \mathfrak{O}^\times$, $\gamma(a) = 0$, $\gamma(0) = \zeta^{-1}$, $\gamma(1) = \zeta'^{-1}$, $\gamma(\infty) = 1$. □

§3. In general, let Y, Z be connected locally noetherian schemes, $f : Z \to Y$ be a morphism and $f_* : \pi_1(Z, \zeta) \to \pi_1(Y, \eta)$ be the induced homomorphism between their fundamental groups, where ζ is any geometric point of Z and $\eta = f(\zeta)$. Then by their definitions [G], f_* is *surjective* if and only if $Z' = Z \times_Y Y'$ is *connected* for any finite etale connected covering Y'/Y of Y. We apply this to the determination of $\pi_1(Y)$ for some special arithmetic surfaces Y, by using horizontal prime divisors $Z \hookrightarrow Y$ and the results of §2.

The following is a direct application.

Proposition 2. *Let k be a number field, $\mathfrak{O}$ its ring of integers, and $X = \mathbf{P}^1_{\mathfrak{O}}$ (the projective t-line over $\mathfrak{O}$). Let $L/k(t)$ be a finite extension field, which is unramified outside $t = 0, 1, \infty$, and $f : Y \to X$ be the normalization of X in L. Let $a \in k^{\cup}(\infty)$ be either $a \in \mathbf{Q}^{\cup}(\infty)$ (including $0, 1, \infty$) or of the form* (ii) *or* (ii)$'$ *of Proposition* 1, *and D_a be the prime divisor on X defined by $t = a$. Let E be any closed subscheme of Y contained in (the support of) $f^{-1}(D_0 {}^{\cup} D_1 {}^{\cup} D_\infty)$, which does not meet $f^{-1}(D_a)$ (for example, $E = \emptyset$). Then the natural homomorphism*

$$\pi_1(f^{-1}(D_a)^{\mathrm{red}}) \longrightarrow \pi_1(Y - E)$$

is surjective. In particular, (i) *if $f^{-1}(D_a)^{\mathrm{red}} \xrightarrow{\sim} \operatorname{Spec}\mathfrak{O}$, then $\pi_1(Y - E) \xrightarrow{\sim} \pi_1(\operatorname{Spec}\mathfrak{O})$;* (ii) *if $f^{-1}(D_a)^{\mathrm{red}}$ is a tree-like union of* Spec $\mathfrak{O}$ *(see below) and $\pi_1(\operatorname{Spec}\mathfrak{O}) = (1)$, then $\pi_1(Y - E) = (1)$.*

Here, $f^{-1}(D_a)^{\mathrm{red}}$ (the reduced part of $f^{-1}(D_a)$) is called *tree-like* if its graph (edges = irreducible components, vertices on an edge = closed points on the corresponding irreducible component) is a tree.

Proof. The prime divisor $F = f^{-1}(D_a)^{\mathrm{red}}$ is a closed subscheme of $Y_1 = Y - E$. If Y_1'/Y_1 is any connected finite etale covering, $Y_1' \times_{Y_1} F \simeq Y' \times_Y F$, where Y' is the integral closure of Y (and also of $\mathbf{P}^1_{\mathfrak{O}}$) in the function field of Y_1'. By Proposition 1, $Y' \times_Y f^{-1}(D_a) = Y' \times_X D_a$ is connected; hence $Y' \times_Y F$ is also connected. Therefore, $\pi_1(F) \to \pi_1(Y_1)$ is surjective.

When $F \xrightarrow{\sim} \operatorname{Spec}\mathfrak{O}$, this defines a section $\operatorname{Spec}\mathfrak{O} \to Y_1$, and hence we have a surjection $\alpha : \pi_1(\operatorname{Spec}\mathfrak{O}) \to \pi_1(Y_1)$, and the structural homomorphism $\beta : \pi_1(Y_1) \to \pi_1(\operatorname{Spec}\mathfrak{O})$, with $\beta \circ \alpha = id$. Therefore, $\pi_1(Y_1) \xrightarrow{\sim} \pi_1(\operatorname{Spec}\mathfrak{O})$. In case (ii), F has no non-trivial connected finite etale coverings, because each irreducible component $\simeq \operatorname{Spec}\mathfrak{O}$ is simply connected, and there can be no non-trivial connected "mock coverings" (graph-theoretically produced finite connected etale coverings) because F is tree-like. □

Corollary 1 (T. Saito). $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - D_0 \cup D_1 \cup D_\infty) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$.

This fact may well have been known, but the author could not find any reference, except that Example 3.1 in [Hb] §3 is quite close. (It gives $\pi_1(\operatorname{Spec} \mathbf{Z}[t, (t^N - 1)^{-1}]) = (1)$, to which the case $\mathfrak{O} = \mathbf{Z}$ reduces directly, and [Hb] contains enough tools for treating the case of general $\mathfrak{O}$.) As far as the author knows, the first proof of this was provided by T. Saito. It is a direct application of generalized Abhyankar lemma (see Appendix). Our argument gives it an alternative proof which is more archimedean in nature.

Proof. First, take some a as in Prop. 1 (ii) or (ii)$'$, and choose k such that $k \ni a$. In Prop. 2, take $Y = X$, $E = D_0 \cup D_1 \cup D_\infty$. Since $D_a \cap E = \emptyset$, Prop. 2 (i) applies to this case, and we conclude that $\pi_1(\mathbf{P}^1_{\mathfrak{O}} - E) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$ for $\mathfrak{O}$: big enough. But then, for any $\mathfrak{O}$, $\mathbf{P}^1_{\mathfrak{O}} - E$ cannot have finite etale connected coverings other than constant ring extensions (which must be etale). Therefore, our assertion holds for any $\mathfrak{O}$. □

Corollary 2. *Let $f : Y \to X$ be as at the beginning of Prop. 2 (the first two sentences preserved). Suppose that one of the cusps, say $t = \infty$, is totally ramified in $f_k = f \otimes k$: $Y_k \to X_k$. Then $\pi_1(Y) \xrightarrow{\sim} \pi_1(\operatorname{Spec} \mathfrak{O})$, or more strongly,*

$$\pi_1(Y - D_0 \cup D_1) \cong \pi_1(\operatorname{Spec} \mathfrak{O}).$$

Proof. In fact, in this case $f^{-1}(D_\infty)^{\mathrm{red}} \simeq \operatorname{Spec} \mathfrak{O}$.

In particular,

Corollary 3. *Let p be a prime, $a, b, c \in \mathbf{Z}$, $a + b + c = 0$, $abc \not\equiv 0 \pmod p$, and $L = \mathbf{Q}(t, y)$, where*

$$y^p = (-1)^c t^a (1 - t)^b$$

(a "primitive Fermat curve"). Let $f : Y \to \mathbf{P}^1_{\mathbf{Z}}$ be the normalization of $\mathbf{P}^1_{\mathbf{Z}}$ (the t-line) in

L. Then for $i, j \in \{0, 1, \infty\}$, $i \neq j$,

$$\pi_1(Y - f^{-1}(D_i \cup D_j)) = (1).$$

[Appendix] *T. Saito's original proof of Cor.* 1 *of Prop.* 2

It proceeds as follows. Let $L/k(t)$, $f : Y \to X = \mathbf{P}^1_{\mathfrak{O}}$ be as at the beginning of Proposition 2. Suppose that $f : Y \to X$ is etale outside $D_0 \cup D_1 \cup D_\infty$. Let $\mathfrak{p}$ be any prime ideal of $\mathfrak{O}$, and put $X_\mathfrak{p} = X \otimes_{\mathfrak{O}} (\mathfrak{O}/\mathfrak{p})$. Choose any cuspidal prime divisor D_i $(i = 0, 1, \infty)$ on X, and let P be the intersection of D_i with $X_\mathfrak{p}$, which is a closed point on $X_\mathfrak{p}$. Then the only prime divisor on X passing through P, along which f is possibly ramified, is D_i. From this follows, by the generalized Abhyankar lemma ([G] Exp. XIII §5), that the ramification indices of $f_k = f \otimes k$ above $t = i$ cannot be divisible by the residue characteristic of $\mathfrak{p}$. Since $\mathfrak{p}$ and i are arbitrary, f must be etale also above D_0, D_1, D_∞; hence $\pi_1(X - D_0 \cup D_1 \cup D_\infty) \simeq \pi_1(X) \simeq \pi_1(\operatorname{Spec} \mathfrak{O})$, as desired.

Saito has also noted that the same argument holds for a somewhat more general case; $\mathbf{P}^1_{\mathfrak{O}} - \bigcup_{a \in A} D_a$ where A is a finite set of elements of $k^{\cup}(\infty)$ satisfying the following conditions. For each pair of $\mathfrak{p}$ and $a \in A$, put $P(a, \mathfrak{p}) = D_a \cap X_\mathfrak{p}$ (a closed point on $X_\mathfrak{p}$). Then for each pair $(a, \mathfrak{p})$, either $P(a, \mathfrak{p}) \neq P(a', \mathfrak{p})$ for all $a' \neq a$ $(a' \in A)$, or there exists exactly one $a' \in A$, $a' \neq a$ with $P(a', \mathfrak{p}) = P(a, \mathfrak{p})$, *and* in this case the maximal ideal of the local ring of X at $P(a, \mathfrak{p})$ is generated by two elements defining D_a and $D_{a'}$ at $P(a, \mathfrak{p})$. (Roughly speaking, the conditions require that the only singularities of $\bigcup D_a$ are "ordinary double points".)

An example: $\mathfrak{O} = \mathbf{Z}$, $A = \{0, 1, 2, 3, \infty\}$.

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