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Kyoto University
Analogy between Drinfeld modules and $\mathcal{D}$-modules

Yuichiro Taguchi

Introduction

The Trinity has long been an attractive motif in the creation of art, especially in the Western world. One of the most marvelous constructions which employ this principle extensively is perhaps "Messe in h-moll" by J. S. Bach [2]. More generally, one has various kinds of trinities, such as ζ(0), etc., which are well worth our contemplation. Recently, there has emerged a new trinity consisting of three D's: Drinfeld modules, $\mathcal{D}$-modules, and Dieudonné modules.

The formal analogy of Drinfeld modules to $\mathcal{D}$-modules has already been realized by Drinfeld since the very beginning of the history of Drinfeld modules [5], and was explained in [6] and [11]. The relation of Drinfeld modules and Dieudonné modules has been formulated, for example, by Drinfeld ([7] etc.) as the relation of Drinfeld modules and shtukas (or $F$-sheaves), and by Anderson ([1]) as the relation of abelian t-modules and t-motives. Finally, Dieudonné modules are originally the positive characteristic analogue of the Lie algebras of Lie groups — the Galois groups of $\mathcal{D}$-modules. Furthermore, over a more general base scheme than a perfect field, Dieudonné modules themselves must be furnished with connections.

Number theory is the most interesting when differentials appear. How cannot our Trinity be in Gloria? In this note, we concentrate on $\varphi$-modules, which arise naturally in the theory of Drinfeld modules and have a similar formalism to $\mathcal{D}$-modules — similarity summarized as follows:

<table>
<thead>
<tr>
<th>Drinfeld modules</th>
<th>$\mathcal{D}$-modules</th>
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<tbody>
<tr>
<td>$\phi_1 = a_0 + a_1 \sigma + \cdots + a_n \sigma^n$</td>
<td>$P = a_0 + a_1 \partial + \cdots + a_n \partial^n$</td>
</tr>
<tr>
<td>$a_i \in K = \mathbb{F}_q(t)$</td>
<td>$a_i \in K = \mathbb{C}(t)$</td>
</tr>
<tr>
<td>$\sigma \colon x \mapsto x^q$</td>
<td>$\partial = \frac{d}{dt}$</td>
</tr>
<tr>
<td>$K[\sigma] = \text{End}_{\text{linear}}(G_a/K)$</td>
<td>$K[\partial] = \text{Det}_{\text{linear}}(K)$</td>
</tr>
<tr>
<td>$\sigma x = x^q \sigma$ ($x \in K$)</td>
<td>$\partial x = x \partial + \frac{dx}{dt}$ ($x \in K$)</td>
</tr>
<tr>
<td>$\varphi$-module structures $\varphi : D(t) \to D$</td>
<td>connections $\nabla : D \to D \otimes \Omega^1_{K/C}$</td>
</tr>
<tr>
<td>Galois representations</td>
<td>local systems of horizontal sections</td>
</tr>
<tr>
<td>Galois extensions by $\varphi$-modules</td>
<td>Picard-Vessiot extensions of differential fields</td>
</tr>
<tr>
<td>solvable extensions</td>
<td>Liouville extensions</td>
</tr>
<tr>
<td>(Artin-Schreier, Kummer)</td>
<td>$(f, \exp(f))$</td>
</tr>
<tr>
<td>tame ramification</td>
<td>regular singularity</td>
</tr>
<tr>
<td>wild ramification</td>
<td>irregular singularity</td>
</tr>
<tr>
<td>???.</td>
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<td>!!!</td>
<td>Sato's theory</td>
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In §1, we give a generality on \( \varphi \)-modules. §2 is devoted to an exposition of results on regular singularity of \( \varphi \)-modules over a local field.

For a field \( K \), let \( K^{\text{sep}} \) denote a fixed separable closure of \( K \), and \( G_{K} \) the absolute Galois group \( \text{Gal}(K^{\text{sep}}/K) \).

1. \( \varphi \)-modules and Galois representations

Let \((S, \sigma)\) be a couple of a commutative ring \( S \) and an endomorphism \( \sigma \) of \( S \); \( \lambda \mapsto \lambda^{\sigma} \).

**Definition (1.1).** A \( \varphi \)-module \((D, \varphi)\) (or simply, \( D \)) over \((S, \sigma)\) (or simply, \( S \)) is an \( S \)-module which is endowed with a \( \sigma \)-semi-linear map \( \varphi : D \to D \) (i.e. \( \varphi \) is additive and \( \varphi(\lambda x) = \lambda^{\sigma} \varphi(x) \) for all \( \lambda \in S \) and \( x \in D \)). A morphism of \( \varphi \)-modules over \((S, \sigma)\) is an \( S \)-module homomorphism which commutes with the \( \varphi \)'s.

The \( \sigma \)-semi-linear map \( \varphi \) can be viewed as an \( S \)-linear map \( \varphi : D^{(\sigma)} \to D \), where \( D^{(\sigma)} \) is the base extension of \( D \) by \( \sigma : S \to S \).

A \( \varphi \)-module \( D \) is said to be étale if \( D \) is of finite type over \( S \) and \( \varphi : D^{(\sigma)} \to D \) is an isomorphism.

\( \varphi \)-modules arise, for example, as follows. Let \( R \) be an \( \mathbb{F}_{q} \)-algebra endowed with an \( \mathbb{F}_{q} \)-algebra homomorphism \( \alpha : \mathbb{F}_{q}[[\pi]] \to R \), where \( \pi \) is an indeterminate. Let \( S \) be the formal power series ring \( R[[\pi]] \), with an endomorphism \( \sigma : \sum r_{i} \pi^{i} \mapsto \sum r_{i}^{q} \pi^{i} \). Then the \( \pi \)-adic \( \pi \)-sheaf \( E_{G} \) associated with a \( \pi \)-divisible group \( G \) over \( R \) ([12], §6) can be regarded as a \( \varphi \)-module over \( S \), where the map \( \varphi \) is induced by the Frobenius morphism of \( G \). This kind of \( \varphi \)-modules \( E_{G} \) are free over \( S \) if \( R \) is local, and \( \varphi \) acts on \( \text{det}_{S} E_{G} \) by multiplication by \( (\text{power of } (\pi - \alpha(\pi))) \times \text{(unit)} \).

In the following, we assume all \( \varphi \)-modules are étale over a field \( S = K \) containing the finite field \( \mathbb{F}_{q} \) of \( q \) elements, and \( \sigma \) is the \( q \)-th power Frobenius. Thus \( D \) is a finite dimensional \( K \)-vector space, whose dimension is called the rank of \( D \), and \( D^{(\sigma)} \) is written as \( D^{(\sigma)} \).

We define the tensor product \((D, \varphi) = (D_{1}, \varphi_{1}) \otimes (D_{2}, \varphi_{2})\) of two \( \varphi \)-modules \((D_{1}, \varphi_{1})\) and \((D_{2}, \varphi_{2})\) by setting \( D := D_{1} \otimes_{K} D_{2} \) and defining \( \varphi : D \to D \) to be the map \( \varphi_{1} \otimes \varphi_{2} \). With this tensor product, \( \mathcal{F}_{K} \) becomes a \( \otimes \)-category.

For any \( \varphi \)-module \((D, \varphi)\) over \( K \) and any field extension \( L/K \), we make \( D_{L} := L \otimes_{K} D \) a \( \varphi \)-module over \( L \) by defining \( \varphi : D_{L} \to D_{L} \) to be the map \( \sum a \otimes x \mapsto \sum a^{q} \otimes \varphi(x) \).

If the extension is Galois, then the Galois group acts on \( D_{L} \) via the first factor.

For a \( \varphi \)-module \((D, \varphi)\) over \( K \), put \( V(D) := (K^{\text{sep}} \otimes_{K} D)^{\varphi} \), the set of fixed points of \( D_{K^{\text{sep}}} \) by \( \varphi \). It is clear that \( V(D) \) is an \( \mathbb{F}_{q} \)-vector space which is stable under the action of \( G_{K} \) on \( D_{K^{\text{sep}}} \). We have thus an \( \mathbb{F}_{q} \)-linear representation \( V(D) \) of \( G_{K} \).
Conversely, if $V$ is a finite dimensional $\mathbb{F}_q$-linear representation of $G_K$, put

$$D(V) := (K^{\text{sep}} \otimes_{\mathbb{F}_q} V)^{G_K},$$

the set of points of $K^{\text{sep}} \otimes_{\mathbb{F}_q} V$ which are fixed by the diagonal action of $G_K$. Clearly $D(V)$ is a $K$-vector space, which we make a $\varphi$-module by defining $\varphi : D(V) \to D(V)$ to be the map

$$\sum a \otimes x \mapsto \sum a^q \otimes x.$$

The following lemma holds in fact in much greater generality; both the base scheme Spec $K$ and the coefficient $\mathbb{F}_q$ can be generalized (cf. [13], §0; [8], §A.1; [7], Proposition 2.1; [12], Proposition (1.7)).

**Lemma (1.2).** Let $\mathcal{F}_K$ (resp. $\mathcal{G}_K$) be the category of $\varphi$-modules over $K$ (resp. the category of finite dimensional $\mathbb{F}_q$-linear representations of $G_K$). Then by the construction explained above, we have a $\otimes$-equivalence of $\otimes$-categories $V : \mathcal{F}_K \to \mathcal{G}_K$, with a quasi-inverse $D : \mathcal{G}_K \to \mathcal{F}_K$.

This correspondence is the most primitive version of the $\varphi$-module analogue of the correspondence of $D$-modules and local systems of horizontal sections.

In what follows, $n$ denotes the rank of the $\varphi$-module under consideration.

A vector $x$ of a $\varphi$-module $D$ is said to be cyclic if the $n$ vectors $x, \varphi(x), \ldots, \varphi^{n-1}(x)$ form a $K$-base of $D$. As in Lemme 1.3, Chapitre II of [3], we have

**Lemma (1.3).** If the base field $K$ is infinite, there exists a cyclic vector for $(D, \varphi)$.

From now on, we assume the $\varphi$-module $(D, \varphi)$ has a cyclic vector $x$. With $x$ is associated a polynomial $P_x(X) \in K[X]$ as follows: if

$$a_0 x + a_1 \varphi(x) + \cdots + a_{n-1} \varphi^{n-1}(x) + \varphi^n(x) = 0 \text{ with } a_i \in K,$$

then put

$$P_x(X) := a_0 X + a_1 X^q + \cdots + a_{n-1} X^{q^{n-1}} + X^{q^n}.$$

Thus we have recovered a presentation of “classical” appearance. This polynomial is determined by $x$ uniquely. Multiplying $x$ by a scalar $a \in K^\times$ yields:

$$P_{ax}(X) = a^n P_x(a^{-1} X)$$

$$= a_0 a^{q^n-1} X + a_1 a^{q^n-q} X^q + \cdots + a_{n-1} a^{q^{n-1}-q^{n-1}} X^{q^{n-1}} + X^{q^n}.$$

We also define

$$\check{V}_x(D) := \{ \alpha \in K^{\text{sep}} ; P_x(\alpha) = 0 \}.$$

This is clearly an $\mathbb{F}_q$-vector space on which $G_K$ acts.

Recall that we have a canonical inclusion $D \subset D_{K^{\text{sep}}}$ (by definition) and a canonical identification $D_{K^{\text{sep}}} = K^{\text{sep}} \otimes_{\mathbb{F}_q} V(D)$ (by Lemma (1.2)).
Lemma (1.4). Suppose that $z$ is expressed by a column vector $(x_0, \ldots, x_{n-1})$, $x_i \in K^{sep}$, with respect to an $\mathbb{F}_q$-base $(e_i)_{0 \leq i \leq n-1}$ of $V(D)$. Then the $n$ elements $x_0, \ldots, x_{n-1}$ form an $\mathbb{F}_q$-base of $\tilde{V}(D)$, so $\tilde{V}(D)$ is an $n$-dimensional $\mathbb{F}_q$-linear representation of $G_K$. The two representations of $G_K$, $V(D)$ and $\tilde{V}(D)$, are contragredient to each other.

All this explained above reminds us of the theory of Picard-Vessiot on differential Galois extensions. A formulation à la Deligne ([4], §9) of our theory will be as follows: The functor $\omega : \mathcal{F}_K \rightarrow \text{Vect}(K)$ which associates with a $\varphi$-module over $K$ its underlying $K$-vector space is a fibre functor. So the category $\mathcal{F}_K$ is Tannakian. For a $\varphi$-module $X$, we set $X^\vee := \text{Hom}_K(X, K)$, which we make a $\varphi$-module by defining $\varphi^* : X^{\vee(q)} \rightarrow X^\vee$ to be the map $f \mapsto f \circ \varphi^{-1}$. Note that $X^{\vee(q)}$ is canonically isomorphic to $\text{Hom}_K(X^{(q)}, K)$ and that $\varphi : X^{(q)} \rightarrow X$ is an isomorphism since we are assuming $X$ is étale. Let $(X)_\otimes$ denote the full subcategory of $\mathcal{F}_K$ whose objects are subquotients of sums of $X^{\otimes l} \otimes X^{\otimes m}$. This is again a Tannakian category.

Suppose that $\omega_0 : (X)_\otimes \rightarrow \text{Vect}(\mathbb{F}_q)$ is a fibre functor (e.g. $\omega_0(D) = V(D) = (K^{sep} \otimes_K D)^\varphi$). Put $G := \text{Aut}^\otimes(\omega_0) \subset \text{GL}_n(\omega_0(X))$ (i.e. intuitively, the algebraic subgroup of automorphisms of the $\mathbb{F}_q$-vector space $\omega_0(X)$ which commute with $\varphi$-module endomorphisms of $X$), $G_K := G \otimes_{\mathbb{F}_q} K$, and $P$ the $G_K$-torsor $\text{Isom}^\otimes_{K}(\omega_0 \otimes_{\mathbb{F}_q} K, \omega_{(X)_\otimes})$. $P$ is a $K$-subscheme of the $\text{GL}_n(\omega_0(X))$-torsor $\text{Isom}_{K}^\otimes(\omega_0(X) \otimes_{\mathbb{F}_q} K, X)$. (It is the locus where "the $\varphi$-module structure is preserved"). The functor $\text{Rep}(G) \rightarrow (X)_\otimes$; $V \mapsto (V \otimes_{\mathbb{F}_q} K)^P$ is an equivalence of $\otimes$-categories (cf. Lemma (1.2)). Here the $\varphi$-module structure of $(V \otimes_{\mathbb{F}_q} K)^P$ is induced by the geometric ($q$-th power) Frobenius $P \rightarrow P^{(q)}$ over $K$.

Now we take $\omega_0(D)$ to be $V(D)$ as our fibre functor. Then $G(\mathbb{F}_q)$ is the bi-commutant of the image of $G_K$ in $\text{GL}_n(V(X))$, and $(V \otimes_{\mathbb{F}_q} K)^P = D(V)$. If $L$ is a subfield of $K^{sep}$ such that $G_L$ acts trivially on $V(X)$ (i.e. the $\varphi$-module $X$ is trivialized over $L$) then for any object $D$ of $(X)_\otimes$, one has an isomorphism of $\varphi$-modules over $L$:

$$(D \otimes_K L)\varphi \otimes_{\mathbb{F}_q} L \xrightarrow{\approx} D \otimes_K L.$$

This gives an $L$-valued point of $P$:

$$\text{Spec } L \rightarrow \text{Isom}^\otimes_{K}(\omega_0 \otimes_{\mathbb{F}_q} K, \omega_{(X)_\otimes}).$$

As in [4], §9, we have

Proposition (1.5). The $K$-scheme $P$ is étale and connected. The subfield $L$ of $K^{sep}$ which corresponds to the kernel of the action of $G_K$ on $V(X)$ is the function field of $P$. 

2. Regular singularity of $\varphi$-modules

In this section, we present some results on regular singularity of $\varphi$-modules, in analogy with the classical theory of ordinary differential equations (see e.g. [10], [3], [9]). Let $K$ be a complete discrete valuation field containing $\mathbb{F}_q$, with valuation $v$ and residue field $k$. Let $p$ be the characteristic of $K$.

**Definition (2.1).** A polynomial $f(X) = a_0 X + a_1 X^p + \cdots + a_n X^{p^n} \in K[X]$ is said to be regular (at $v$) if $a_0 \neq 0$, $a_n \neq 0$, and

\[(2.1.1) \quad v(a_i) - v(a_n) \geq \frac{p^n - p^i}{p^n - 1} (v(a_0) - v(a_n))\]

for all $i = 1, \ldots, n - 1$.

A regular polynomial $f(X)$ is separable because $f'(X) = a_0 \neq 0$. The condition (2.1.1) is saying that the Newton polygon of $f(X)$ is a straight line. This is equivalent to that all non-zero roots of $f(X)$ have the same valuation $(v(a_0) - v(a_n))/(p^n - 1)$.

Regularity of $f(X)$ is invariant by multiplying $f(X)$ by an element of $K^\times$. If $a_n$ is a unit (i.e. $v(a_n) = 0$), then (2.1.1) is simply

\[(2.1.2) \quad v(a_i) \geq \frac{p^n - p^i}{p^n - 1} v(a_0).\]

Regularity of $f(X)$ is invariant also under the change of variable $X \mapsto aX$ with $a \in K^\times$.

For any separable polynomial $f(X) \in K[X]$, we denote by $K_f$ the minimal splitting field of $f$ contained in $K^{sep}$.

**Proposition (2.2).** Let $f$ be a regular polynomial over $K$. Then the extension $K_f/K$ is tamely ramified at $v$.

We shall interpret the regularity (in the sense of (2.1)) of the polynomial $P_z$ of §1 in terms of lattices, Galois actions, and connections. For any algebraic extension $L/K$, the valuation $v$ of $K$ extend uniquely to $L$, which are again denoted $v$. We denote by $\mathcal{O}_L$ the valuation ring of $L$.

Let $(D, \varphi)$ be a $\varphi$-module over $K$. An $\mathcal{O}_K$-lattice $D^0$ of $D$ is said to be $\varphi$-stable if $\varphi(D^0) \subset D^0$ and $\mathcal{O}_K \cdot \varphi(D^0) = D^0$ (i.e. if $(D^0, \varphi)$ is an étale $\varphi$-module over $\mathcal{O}_K$). The existence of a $\varphi$-stable $\mathcal{O}_K$-lattice means that $V(D)$ is "finite étale" over $\mathcal{O}_K$;

**Lemma (2.3).** Let $(D, \varphi)$ be a $\varphi$-module over $K$.

(i) The following conditions are equivalent:

1. There exists a $\varphi$-stable $\mathcal{O}_K$-lattice $D^0$ in $D$.
2. The representation of $G_K$ on $V(D)$ is unramified.
(ii) If \( G_K \) acts on \( V(D) \) trivially, then the \( \varphi \)-stable \( \mathcal{O}_K \)-lattice is the \( \mathcal{O}_K \)-submodule of \( D \) spanned by \( V(D) \).

(iii) There exists a finite separable extension \( L/K \) such that \( D_L \) has a \( \varphi \)-stable \( \mathcal{O}_L \)-lattice.

DEFINITION (2.4). A \( \varphi \)-module \((D, \varphi)\) over \( K \) is said to have regular singularity (at \( v \)) if it is the direct sum of \( \varphi \)-submodules \((D_i, \varphi_i)\) each of which has a cyclic vector \( z_i \) such that the associated polynomial \( P_{z_i} \) is regular in the sense of (2.1).

When the residue field \( k \) of \( K \) is separably closed, the \( D_i \) in the above definition can be taken to be irreducible if once \((D, \varphi)\) has regular singularity.

THEOREM (2.5). Assume the residue field \( k \) of \( K \) is separably closed. Let \((D, \varphi)\) be a \( \varphi \)-module over \( K \). Then the following conditions are equivalent:

1. The \( \varphi \)-module \((D, \varphi)\) is regular.
2. There exists a finite tamely ramified extension \( L/K \) such that \( D_L \) has a \( \varphi \)-stable \( \mathcal{O}_L \)-lattice \( D_L^{0} \).
3. The Galois representation \( V(D) \) is tamely ramified.

Now we turn our attention to the connection associated with a \( \varphi \)-module \( D \). Recall (e.g. [8], A.2.2) that there exists on \( D \) a unique connection \( \nabla : D \to D \otimes_K \Omega_{K/k}^{1} \) for which \( \varphi : D \to D \) is horizontal; \( \nabla \circ \varphi = (\varphi \otimes \text{id}) \circ \nabla \). If the Galois representation \( V(D) \) is trivial, then \( M^{\nabla} := \text{Ker}(\nabla) = k \otimes_{\mathbb{F}_q} V(D) \).

If \( x \in D \) is expressed by a column vector \( ^t(x_0, \ldots, x_{n-1}) \), \( x_j \in K \), with respect to an \( \mathbb{F}_q \)-base of \( V(D) \), then we have

\[
\nabla(x) = ^t(dx_0, \ldots, dx_{n-1}).
\]

The connection may also be regarded as a \( K \)-linear map

\[
\nabla : \text{Der}_k(K) \to \text{End}_k(D)
\]
such that, for all \( \partial \in \text{Der}_k(K) \simeq \text{Hom}_K(\Omega_{K/k}^{1}, K) \), one has \( \nabla(\partial) = (1 \otimes \partial) \circ \nabla \).

Let \( \| - \| \) be the norm on \( D_{K^{\text{sep}}} \) for which the unit ball is the \( \varphi \)-stable \( \mathcal{O}_{K^{\text{sep}}} \)-lattice \( D_{K^{\text{sep}}}^{0} = \mathcal{O}_{K^{\text{sep}}} \cdot V(D) \).

THEOREM (2.6). Assume the residue field \( k \) of \( K \) is separably closed. Let \( t \) be a uniformizer of \( K \). Then the following conditions are equivalent:

1. The \( \varphi \)-module \( D \) has regular singularity;
2. For any \( x \in D \), we have \( \| \nabla(t^i \frac{d}{dt})(x) \| \leq \| x \| \).

The condition (2) may be rephrased that the norm of \( \nabla \),

\[
\| \nabla \| := \sup_{x \in D} \frac{\| \nabla(t^i \frac{d}{dt})(x) \|}{\| x \|},
\]
equals 1 (We have always \( \| \nabla \| \geq 1 \)). Also, it may be rephrased that there exists in \( D \) a \( \nabla(t^{\frac{d}{dt}}) \)-stable \( \mathcal{O}_K \)-lattice which is a "proper ball" with respect to the norm \( \| - \| \).
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Tokyo Metropolitan University, Hachioji, Tokyo, 192–03 JAPAN
taguchi@math.metro-u.ac.jp