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<th>Title</th>
<th>Determinant representation, Jacobi sum and de Rham discriminant (Algebraic Number Theory: Recent Developments and Their Backgrounds)</th>
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<tr>
<td>Author(s)</td>
<td>SAIOT, TAKESHI</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1993), 844: 79-83</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1993-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/83601">http://hdl.handle.net/2433/83601</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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<td>Institution</td>
<td>Kyoto University</td>
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Determinant representation, Jacobi sum and de Rham discriminant

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We give a description of the Galois action on the determinant of cohomologies of $\ell$-adic sheaves on varieties in terms of Jacobi sum Hecke characters and of the de Rham discriminant for general base fields. Let $k$ be an arbitrary base field, $U$ be a smooth scheme over $k$ and $\mathcal{F}$ be a smooth $\ell$-adic sheaf for $\ell \neq \text{ch}k$. We consider one-dimensional $\ell$-adic representation

$$\det R\Gamma_c(U_k, \mathcal{F}) = \bigotimes_i \det H^i_c(U_k, \mathcal{F})^\otimes(-1)^i$$

of $\text{Gal}(k^{sep}/k)$.

**Results.** 1. Constant coefficient.

First we consider the constant coefficient case. If we assume the resolution of singularity, the problem is reduced to the proper case. Let $X$ be a proper smooth variety over a field $k$ of dimension $n$ and $\chi$ be the Euler characteristic of $X_k$. Then it follows immediately from Poincare duality that

$$\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\frac{1}{2}n\chi) \otimes \begin{cases} 1, & n \text{ odd} \\ \kappa, & n \text{ even} \end{cases}$$

for some character $\kappa$ of order at most 2 of $\text{Gal}(k^{ab}/k)$.

**Theorem 1.** Assume $\text{ch}k \neq 2$, $X$ is projective and $n = 2m = \dim X$ is even. Let $\delta_X \in k^\times/(k^\times)^2$ be the discriminant of the cup product on the de Rham cohomology $H^n_{dR}(X/k)$ which is a non-degenerate symmetric bilinear form and let $b^- = \sum_{i<n} \dim H^i_{dR}(X/k)$. Then the character $\kappa$ corresponds to the quadratic extension $k(\sqrt{(-1)^m b^- \delta_X})/k$.

Remark. When $k$ is finite and $n = 2m$ even, Tate conjecture implies that $\kappa_{et}$ is trivial if and only if the rank of $\text{CH}^m(X_k')_{h}/\text{CH}^m(X_k)_{h}$ is even. Here the suffix $h$ denotes modulo the homological equivalence and $k'$ is the quadratic extension of $k$. In particular if $\kappa$ is not trivial, there is an algebraic cycle of $X_k$ not defined over $k$.

2. General coefficient.

We proceed to a general coefficient. Let $k, U$ and $\mathcal{F}$ be as above.
Theorem 2. We assume

1. There is a projective and smooth variety $X$ over $k$ containing $U$ such that the complement $D = X - U$ is a divisor with simple normal crossings.

2. The ramification of $\mathcal{F}$ along $D$ is tame.

3. The sheaf $\mathcal{F}$ is defined on a model of $U$ defined over a ring of finite type over $\mathbb{Z}$.

Then we have

$$\det R\Gamma_c(U_k, \mathcal{F}) \otimes \det R\Gamma_c(U_k, \mathbb{Q}_\ell)^{\otimes -\text{rank} \mathcal{F}} = c_{X, U/k}^*(\det \mathcal{F}) \otimes J_{D,F}^{\otimes -1}$$

as one-dimensional $\ell$-adic representations of $\text{Gal}(k^{ab}/k)$.

The precise definition of the right hand side will be given later. A rough idea is as follows. The first term is the pull-back of the determinant character $\det \mathcal{F}$ of $\pi_1(U)^{ab, \text{tame}}$ to $\text{Gal}(k^{ab}/k)$ by the pairing with the relative canonical class $c_{X, U/k} \in CH_0(X, D)$. The second term $J_{D,F}$ denotes a Jacobi sum Hecke character, which is determined by the ramification datum of $\rho$ along $D$.

Corollary. If $k$ is an algebraic number field, the $\ell$-adic representation $J_{D,F}$ is defined by an algebraic Hecke character.

Theorem 2 solves the conjecture (3.11) of [O] affirmatively. By Theorem 2 together with a formula for period integral (joint work with T. Terasoma), we verify a part of a conjecture of Deligne [D2] Conjecture 8.1 (iii): A motive of rank 1 is defined by an algebraic Hecke character, in certain cases.

Definitions. 1. Canonical cycle.

First, we define the relative Chow group $CH_0(X, D)$ of dimension 0 and the relative canonical cycle $c_{X, U/k} \in CH_0(X, D)$. Let $X$ be a smooth scheme over a field $k$ of dimension $n$ and $D = \bigcup_{i \in I} D_i$ be a divisor with simple normal crossings. Let $\mathcal{K}_n(X)$ denotes the sheaf of Quillen's $K$-theory on $X_{\text{Zar}}$. Namely the Zariski sheafification of the presheaf $U \mapsto K_n(U)$. Let $\mathcal{K}_n(X, D)$ be the complex $[\mathcal{K}_n(X) \to \oplus_i \mathcal{K}_n(D_i)]$. Here $\mathcal{K}_n(X)$ is put on degree 0 and $\mathcal{K}_n(D_i)$ denotes their direct image on $X$. We call the hypercohomology $H^n(X, \mathcal{K}_n(X, D))$ the relative Chow group of dimension 0 and write

$$CH_0(X, D) = H^n(X, \mathcal{K}_n(X, D)).$$

We define the relative canonical class

$$c_{X, U/k} = (-1)^n c_n(\Omega^1_{X/k}(\log D), \text{res}) \in CH_0(X, D).$$
Let $V$ be the covariant vector bundle associated to the locally free $\mathcal{O}_X$-module $\Omega_{X/k}^1(\log D)$ of rank $n$. For each irreducible component $D_i$, let $\Delta_i = \pi_i^{-1}(1)$. Here $r_i : V|_{D_i} \to \mathcal{O}_{D_i}$ is induced by the Poincare residue $\text{res}_i : \Omega_{X/k}^1(\log D)|_{D_i} \to \mathcal{O}_{D_i}$ and $1 \subset \mathcal{A}^1$ is the 1-section. Let $\mathcal{K}_n(V, \Delta)$ be the complex $[\mathcal{K}_n(V) \to \bigoplus_i \mathcal{K}_n(\Delta_i)]$ defined similarly as above and $\{0\} \subset V$ be the zero section. Then we have

$$H^n_{\{0\}}(V, \mathcal{K}_n(V, \Delta)) \simeq H^n_{\{0\}}(V, \mathcal{K}_n(V)) \simeq H^0(X, \mathcal{K}_n(V, \Delta)) \simeq H^0(X, Z)$$

by the purity and homotopy property of K-cohomology. The relative top chern class $c_n(\Omega_{X/k}^1(\log D), \text{res}) \in CH_0(X, D)$ is defined as the image of $1 \in H^0(X, Z)$.

Next we consider the canonical pairing

$$CH_0(X, D) \times \text{Gal}(k^{ab}/k) \to \pi_1(U)^{ab, \text{tame}}.$$

For its definition require an adelic description of the group $CH_0(X, D)$, we only give a definition of a quotient

$$CH_0(X) \times \text{Gal}(k^{ab}/k) \to \pi_1(X)^{ab}.$$

It is characterized by the following property. For a closed point $x \in X$, the pairing with the class $[x]$ coincides with the inseparable degree times the Galois transfer followed by $i_{x*}$ for $i_x : x \to X$

$$\text{Gal}(k^{ab}/k) \xrightarrow{\text{tr}_x(x)/k \times [\kappa(x)/k]} \text{Gal}(\kappa(x)^{ab}/\kappa(x)) \xrightarrow{i_{x*}} \pi_1(X)^{ab}.$$

The required reciprocity law follows from the fact that $P^1$ is simply connected.

Remark. If $k$ is finite, the pairing $CH_0(X) \times \hat{Z} \to \pi_1^{ab}(X)$ coincides with the reciprocity map of higher dimensional unramified class field theory.

For a smooth $\ell$-adic sheaf $\mathcal{F}$ on $U$ tamely ramified along $D$, the determinant $\det \mathcal{F}$ determines an $\ell$-adic character of $\pi_1(U)^{ab, \text{tame}}$. Therefore by pulling it back by the pairing with $c_{X,U/k}$, we obtain the first term $c_{X,U/k}^* (\det \mathcal{F})$.

2. Jacobi sum.

We call a Jacobi datum on $k$ a triple $(T, \chi, n)$ as follows

1. $T = (k_i)_{i \in I}$ is a finite family of finite separable extensions of $k$.
2. $\chi = (\chi_i)_{i \in I}$ is a family of characters $\chi_i : \mu_{d_i}(k_i)$ of the group of $d_i$-th roots of unity for some integer $d_i$ invertible in $k$ such that $\mu_{d_i} \simeq \mathbb{Z}/d_i$ on $k_i$.
3. $n = (n_i)_{i \in I}$ is a family of integers.
satisfying the condition
\[ \prod_{i \in I} N_{k_i/k}(\chi_i)^{n_i} = 1. \]

Here the norm \( N_{k_i/k}(\chi_i) : \zeta \in \mu_{d_i}(\overline{k}) \rightarrow \chi_i(N_{k_i/k}(\zeta)) \) is the product of the conjugates. It is easy to see that each \( N_{k_i/k}(\chi_i) \) factors some \( \mu_{d_i'} \) such that \( d_i'|d_i \) and \( \mu_{d_i'} \simeq \mathbb{Z}/d_i' \) on \( S \). The product is taken as a character of \( \mu_{d,k} \) for some common multiple of \( d_i' \)'s which is invertible on \( k \).

If \( k \) is finite of order \( q \), we define the Jacobi sum \( j_X = j_{T,\chi, n} \) for each Jacobi datum \((T, \chi, n)\) on \( k \) by
\[ j_X = \prod_{i \in I}(\tau_{k_i}(\overline{\chi}_i, \psi_0 \circ \text{Tr}_{k_i/k}))^{n_i}. \]

Here if \( k_i \) is of order \( q_i \), \( \overline{\chi}_i \) is a multiplicative character of \( k_i \) defined by \( \overline{\chi}_i(a) = \chi_i(a^{q_i-1}/d_i) \) for \( a \in k_i^x \), \( \psi_0 \) is a non-trivial additive character of \( k \) and \( \tau \) denotes the Gauss sum \( \tau_E(\chi, \psi) = -\sum_{\alpha \in E^x} \chi^{-1}(\alpha)\psi(\alpha) \).

The Jacobi sum \( j_X \) is independent of choice of \( \psi_0 \) by the condition \( \prod_{i \in I} N_{k_i/k}(\chi_i)^{n_i} = 1 \). In fact the product of the restrictions \( \prod_j \overline{\chi}_{ij}|_{k^x} \) coincides with \( N_{k_i/k}(\chi_i) \) regarded as a character of \( \mu_{q-1}(k) = k^x \).

To each Jacobi datum \((T, \chi, n)\) on a field \( k \), we define an \( \ell \)-adic representation \( J_\chi \) of \( \text{Gal}(k^{\text{sep}}/k) \) as follows. A Jacobi datum on \( k \) is defined on a normal ring \( A \) of finite type over \( \mathcal{O} \). The representation \( J_\chi \) is the pull-back of one of \( \pi_1(\text{Spec } A)^{ab} \) characterized by the following condition: For each closed point \( s \) of \( \text{Spec } A \), the action of the geometric Frobenius \( \mathcal{F}_s \) at \( s \) is given by the multiplication by the Jacobi sum \( J_\chi(s) \) defined by the reduction of the Jacobi datum \((T, \chi, n)\) at \( s \). Uniqueness follows from the Cebotarev density and the existence is essentially shown in SGA 4\( \frac{1}{2} \).

Let \( \mathcal{U} \) and \( \mathcal{F} \) be as in Theorem 1 and we define a Jacobi datum on \( k \) associated to the ramification of \( \mathcal{F} \) along \( D \). Let \( k_i \in I \) be the constant field of irreducible components \( D_i \) of \( D \). Let \( \rho \) be the \( \ell \)-adic representation of \( \pi_1(\mathcal{U}, \mathcal{E})^{\text{tame}} \) corresponding to \( \mathcal{F} \). The kernel \( \pi_1(\mathcal{U}, \mathcal{E})^{\text{tame}} \rightarrow \pi_1(\mathcal{U}, \mathcal{E}) \) is the normal subgroup topologically generated by the local monodromy groups \( \hat{\mathcal{U}}(1)_{D_i} \) along \( D_i \)'s where \( \hat{\mathcal{U}}(1) = \lim_{\rightarrow} \mu_d \) with \( d \) invertible in \( k \).

Let \( \rho_i \) be the restriction of \( \rho \) to \( \hat{\mathcal{U}}(1)_{D_i} \simeq \hat{\mathcal{U}}(1)_{k_i} \). By the assumption of the existence of a model of finite type over \( \mathcal{O} \) and by the monodromy theorem of Grothendieck, the restrictions \( \rho_i \)'s are quasi-unipotent. Namely the action of \( \hat{\mathcal{U}}(1) \) on the semi-simplification \( \rho_i^{ss} \) factors a finite quotient. Hence we can decompose it in the form \( \rho_i^{ss} \simeq \bigoplus_{j \in I_i} \text{Tr}_{k_{ij}/k_i}(\chi_{ij}) \). Here
$k_{ij}$ is the finite extension of $k_i$ obtained by adjoining the $d_{ij}$-th roots of unity, $\chi_{ij}$ is a character of $\mu_{d_{ij}}(k_{ij})$ of order $d_{ij}$ and $\text{Tr}$ denotes the direct sum of the conjugates. For $i \in I$, let $D^*_i = D_i - \bigcup_{j \neq i} D_j$ and $c_i$ be the Euler number of $D^*_i \otimes_{k_i} \overline{k_i}$. Thus we obtain a triple $(T, \chi, n)$ by putting the index set $\overline{I} = \coprod_{i} I_i$, $T = (k_{ij})$, $\chi = (\chi_{ij})$ and $n = (n_{ij})$ with $n_{ij} = c_i$ for $i \in I$ and $j \in I_i$. The second term $J_{D,F}$ is defined as the $p$-adic representation determined by the Jacobi datum $(T, \chi, n)$.

**Outline of proof.**

For the detail of the proof, we refer to [S1] and [S2]. We give an outline of the proof of theorems. By a standard argument using Cebotarev density and specialization, we may assume $k$ is finite. Then the determinant of the Frobenius is the constant of the functional equation of the $L$-function. We apply the product formula of Deligne-Laumon [D1],[L] for the constant by taking a Lefshetz pencil [SGA7]. For theorem 1, we show that the local terms are the Hessians at the singularities of the pencil and relate them to the de Rham discriminant using the Picard-Lefshetz formula (loc.cit).

**REFERENCES**


[S1] T.Saito, $\epsilon$-factor of a tamely ramified sheaf on a variety, to appear in Inventiones Math.