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Determinant representation, Jacobi sum and de Rham discriminant

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We give a description of the Galois action on the determinant of cohomologies of ℓ-adic sheaves on varieties in terms of Jacobi sum Hecke characters and of the de Rham discriminant for general base fields. Let k be an arbitrary base field, U be a smooth scheme over k and $\mathcal{F}$ be a smooth ℓ-adic sheaf for $\ell \neq \mathrm{ch} k$. We consider one-dimensional ℓ-adic representation

$$\det R\Gamma_c(U_{\bar{k}}, \mathcal{F}) = \bigotimes_i \det H^i_c(U_{\bar{k}}, \mathcal{F})^{\otimes(-1)^i}$$

of $\mathrm{Gal}(k^{sep}/k)$.

Results. 1. Constant coefficient.

First we consider the constant coefficient case. If we assume the resolution of singularity, the problem is reduced to the proper case. Let X be a proper smooth variety over a field k of dimension n and χ be the Euler characteristic of $X_{\bar{k}}$. Then it follows immediately from Poincare duality that

$$\det R\Gamma(X_{\bar{k}}, \mathbf{Q}_\ell) = \mathbf{Q}_\ell(-\frac{1}{2}n\chi) \otimes \begin{cases} 1, & n \quad \text{odd}, \\ \kappa, & n \quad \text{even}. \end{cases}$$

for some character κ of order at most 2 of $\mathrm{Gal}(k^{\mathrm{ab}}/k)$.

THEOREM 1. *Assume* $\mathrm{ch} k \neq 2$, X *is projective and* $n = 2m = \dim X$ *is even. Let* $\delta_X \in k^\times/(k^\times)^2$ *be the discriminant of the cup product on the de Rham cohomology* $H^n_{dR}(X/k)$ *which is a non-degenerate symmetric bilinear form and let* $b^- = \sum_{i<n} \dim H^i_{dR}(X/k)$. *Then the character* κ *corresponds to the quadratic extension* $k(\sqrt{(-1)^{m\chi+b^-}\delta_X})/k$.

Remark. When k is finite and $n = 2m$ even, Tate conjecture implies that κ_{et} is trivial if and only if the rank of $CH^m(X_{k'})_h/CH^m(X_k)_h$ is even. Here the suffix h denotes modulo the homological equivalence and k' is the quadratic extension of k. In particular if κ is not trivial, there is an algebraic cycle of $X_{\bar{k}}$ not defined over k.

2. General coefficient.

We proceed to a general coefficient. Let k, U and $\mathcal{F}$ be as above.

THEOREM 2. *We assume*

(1) *There is a projective and smooth variety X over k containing U such that the complement $D = X - U$ is a divisor with simple normal classings.*
(2) *The ramification of $\mathcal{F}$ along D is tame.*
(3) *The sheaf $\mathcal{F}$ is defined on a model of U defined over a ring of finite type over $\mathbb{Z}$.*

Then we have

$$\det R\Gamma_c(U_{\bar{k}}, \mathcal{F}) \otimes \det R\Gamma_c(U_{\bar{k}}, \mathbb{Q}_\ell)^{\otimes -\mathrm{rank}\ \mathcal{F}} = c^*_{X,U/k}(\det \mathcal{F}) \otimes J_{D,\mathcal{F}}^{\otimes -1}$$

as one-dimensional ℓ-adic representations of $\mathrm{Gal}(k^{ab}/k)$.

The precise definition of the right hand side will be given later. A rough idea is as follows. The first term is the pull-back of the determinant character $\det \mathcal{F}$ of $\pi_1(U)^{\mathrm{ab,tame}}$ to $\mathrm{Gal}(k^{ab}/k)$ by the pairing with the relative canonical class $c_{X,U/k} \in CH_0(X, D)$. The second term $J_{D,\mathcal{F}}$ denotes a Jacobi sum Hecke character, which is determined by the ramification datum of ρ along D.

COROLLARY. *If k is an algebraic number field, the ℓ-adic representation $J_{D,\mathcal{F}}$ is defined by an algebraic Hecke character.*

Theorem 1 solves the conjecture (3.11) of [O] affirmatively. By Theorem 2 together with a formula for period integral (joint work with T.Terasoma), we verify a part of a conjecture of Deligne [D2] Conjecture 8.1 (iii): A motive of rank 1 is defined by an algebraic Hecke character, in certain cases.

Definitions. 1. Canonical cycle.

First, we define the relative Chow group $CH_0(X, D)$ of dimension 0 and the relative canonical cycle $c_{X,U/k} \in CH_0(X, D)$. Let X be a smooth scheme over a field k of dimension n and $D = \cup_{i \in I} D_i$ be a divisor with simple normal crossings. Let $\mathcal{K}_n(X)$ denotes the sheaf of Quillen's K-theory on X_{Zar}. Namely the Zariski sheafification of the presheaf $U \mapsto K_n(U)$. Let $\mathcal{K}_n(X, D)$ be the complex $[\mathcal{K}_n(X) \to \oplus_i \mathcal{K}_n(D_i)]$. Here $\mathcal{K}_n(X)$ is put on degree 0 and $\mathcal{K}_n(D_i)$ denotes their direct image on X. We call the hypercohomology $H^n(X, \mathcal{K}_n(X, D))$ the relative Chow group of dimension 0 and write

$$CH_0(X, D) = H^n(X, \mathcal{K}_n(X, D)).$$

We define the relative canonical class

$$c_{X,U/k} = (-1)^n c_n(\Omega^1_{X/k}(\log D), res) \in CH_0(X, D).$$

Let V be the covariant vector bundle associated to the locally free $\mathcal{O}_X$-module $\Omega^1_{X/k}(\log D)$ of rank n. For each irreducible component D_i, let $\Delta_i = r_i^{-1}(1)$. Here $r_i : V|_{D_i} \to \mathbb{A}^1_{D_i}$ is induced by the Poincare residue $res_i : \Omega^1_{X/k}(\log D)|_{D_i} \to \mathcal{O}_{D_i}$ and $1 \subset \mathbb{A}^1$ is the 1-section. Let $\mathcal{K}_n(V, \Delta)$ be the complex $[\mathcal{K}_n(V) \to \oplus_i \mathcal{K}_n(\Delta_i)]$ defined similarly as above and $\{0\} \subset V$ be the zero section. Then we have

$$\begin{array}{c} H^n_{\{0\}}(V,\mathcal{K}_n(V,\Delta)) \simeq H^n_{\{0\}}(V,\mathcal{K}_n(V)) \simeq H^0(X,\mathbf{Z}) \\ \downarrow \\ H^n(V,\mathcal{K}_n(V,\Delta)) \simeq H^n(X,\mathcal{K}_n(X,D)) = CH_0(X,D) \end{array}$$

by the purity and homotopy property of K-cohomology. The relative top chern class $c_n(\Omega^1_{X/k}(\log D), res) \in CH_0(X, D)$ is defined as the image of $1 \in H^0(X, \mathbf{Z})$.

Next we consider the canonical pairing

$$CH_0(X,D) \times \mathrm{Gal}(k^{ab}/k) \to \pi_1(U)^{\mathrm{ab,tame}}.$$

For its definition require an adelic description of the group $CH_0(X, D)$, we only give a definition of a quotient

$$CH_0(X) \times \mathrm{Gal}(k^{ab}/k) \to \pi_1(X)^{\mathrm{ab}}.$$

It is characterized by the following property. For a closed point $x \in X$, the pairing with the class $[x]$ coincides with the inseparable degree times the Galois transfer followed by i_{x*} for $i_x : x \to X$

$$Gal(k^{ab}/k) \overset{tr_{\kappa(x)/k} \times [\kappa(x):k]_i}{\longrightarrow} Gal(\kappa(x)^{ab}/\kappa(x)) \xrightarrow{i_{x*}} \pi_1(X)^{\mathrm{ab}}.$$

The required reciprocity law follows from the fact that $\mathbb{P}^1$ is simply connected

Remark. If k is finite, the pairing $CH_0(X) \times \hat{\mathbf{Z}} \to \pi_1^{ab}(X)$ coincides with the reciprocity map of higher dimensional unramified class field theory.

For a smooth ℓ-adic sheaf $\mathcal{F}$ on U tamely ramified along D, the determinant $\det \mathcal{F}$ determines an ℓ-adic character of $\pi_1(U)^{\mathrm{ab,tame}}$. Therefore by pulling it back by the pairing with $c_{X,U/k}$, we obtain the first term $c^*_{X,U/k}(\det \mathcal{F})$.

2. Jacobi sum.

We call a Jacobi datum on k a triple (T, χ, n) as follows

(1) $T = (k_i)_{i \in I}$ is a finite family of finite separable extensions of k.

(2) $\chi = (\chi_i)_{i \in I}$ is a family of characters $\chi_i : \mu_{d_i}(k_i)$ of the group of d_i-th roots of unity for some integer d_i invertible in k such that $\mu_{d_i} \simeq \mathbf{Z}/d_i$ on k_i.

(3) $n = (n_i)_{i \in I}$ is a family of integers.

satisfying the condition

$$\prod_{i\in I} N_{k_i/k}(\chi_i)^{n_i} = 1.$$

Here the norm $N_{k_i/k}(\chi_i) : \zeta \in \mu_{d_i}(\bar{k}) \to \chi_i(N_{k_i/k}(\zeta))$ is the product of the conjugates. It is easy to see that each $N_{k_i/k}(\chi_i)$ factors some $\mu_{d'_i}$ such that $d'_i|d_i$ and $\mu_{d'_i} \simeq \mathbf{Z}/d'_i$ on S. The product is taken as a character of $\mu_{d,k}$ for some common multiple of d'_i's which is invertible on k.

If k is finite of order q, we define the Jacobi sum $j_\chi = j_{T,\chi,n}$ for each Jacobi datum (T, χ, n) on k by

$$j_\chi = \prod_{i\in I} (\tau_{k_i}(\bar{\chi}_i, \psi_0 \circ \mathrm{Tr}_{k_i/k}))^{n_i}.$$

Here if k_i is of order q_i, $\bar{\chi}_i$ is a multiplicative character of k_i defined by $\bar{\chi}_i(a) = \chi_i(a^{(q_i-1)/d_i})$ for $a \in k_i^\times$, ψ_0 is a non-trivial additive character of k and τ denotes the Gauss sum $\tau_E(\chi, \psi) = -\sum_{a\in E^\times} \chi^{-1}(a)\psi(a)$. The Jacobi sum j_χ is independent of choice of ψ_0 by the condition $\prod_{i\in I} N_{k_i/k}(\chi_i)^{n_i} = 1$. In fact the product of the restrictions $\prod_j \bar{\chi}_{ij}|_{k^\times}$ coincides with $N_{k_i/k}(\chi_i)$ regarded as a character of $\mu_{q-1}(k) = k^\times$.

To each Jacobi datum (T, χ, n) on a field k, we define an ℓ-adic representation J_χ of $\mathrm{Gal}(k^{sep}/k)$ as follows. A Jacobi datum on k is defined on a normal ring A of finite type over $\mathbf{Z}$. The representation J_χ is the pull-back of one of $\pi_1(\mathrm{Spec}\ A)^{\mathrm{ab}}$ characterized by the following condition: For each closed point s of Spec A, the action of the geometric Frobenius Fr_s at s is given by the multiplication by the Jacobi sum $J_\chi(s)$ defined by the reduction of the Jacobi datum (T, χ, n) at s. Uniqueness follows from the Cebotarev density and the existence is essentially shown in SGA $4\frac{1}{2}$.

Let U and $\mathcal{F}$ be as in Theorem 1 and we define a Jacobi datum on k associated to the ramification of $\mathcal{F}$ along D. Let $k_{i,i\in I}$ be the constant field of irreducible components D_i of D. Let ρ be the ℓ-adic representation of $\pi_1(U, \bar{x})^{\mathrm{tame}}$ corresponding to $\mathcal{F}$. The kernel $\pi_1(U, \bar{x})^{\mathrm{tame}} \to \pi_1(X, \bar{x})$ is the normal subgroup topologically generated by the local monodromy groups $\hat{\mathbf{Z}}'(1)_{D_i}$ along D_i's where $\hat{\mathbf{Z}}'(1) = \varprojlim \mu_d$ with d invertible in k. Let ρ_i be the restriction of ρ to $\hat{\mathbf{Z}}'(1)_{D_i} \simeq \hat{\mathbf{Z}}'(1)_{k_i}$. By the assumption of the existence of a model of finite type over $\mathbf{Z}$ and by the monodromy theorem of Grothendieck, the restrictions ρ_i's are quasi-unipotent. Namely the action of $\hat{\mathbf{Z}}'(1)$ on the semi-simplification ρ_i^{ss} factors a finite quotient. Hence we can decompose it in the form $\rho_i^{ss} \simeq \bigoplus_{j\in I_i} \mathrm{Tr}_{k_{ij}/k_i}(\chi_{ij})$. Here

k_{ij} is the finite extension of k_i obtained by adjoining the d_{ij}-th roots of unity, χ_{ij} is a character of $\mu_{d_{ij}}(k_{ij})$ of order d_{ij} and Tr denotes the direct sum of the conjugates. For $i \in I$, let $D_i^* = D_i - \bigcup_{j \neq i} D_j$ and c_i be the Euler number of $D_i^* \otimes_{k_i} \bar{k}_i$. Thus we obtain a triple (T, χ, n) by putting the index set $\bar{I} = \coprod_i I_i$, $T = (k_{ij}), \chi = (\chi_{ij})$ and $n = (n_{ij})$ with $n_{ij} = c_i$ for $i \in I$ and $j \in I_i$. The second term $J_{D,\mathcal{F}}$ is defined as the ℓ-adic representation determind by the Jacobi datum (T, χ, n).

Outline of proof.

For the detail of the proof, we refer to [S1] and [S2]. We give an outline of the proof of theorems. By a standard argument using Cebotarev density and specialization, we may assume k is finite. Then the determinant of the Frobenius is the constant of the functional equation of the L-function. We apply the product formula of Deligne-Laumon [D1],[L] for the constant by taking a Lefshetz pencil [SGA7]. For theorem 1, we show that the local terms are the Hessians at the singularities of the pencil and relate them to the de Rham discriminant using the Picard-Lefshetz formula (loc.cit).

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