数理解析研究所講究録
第845巻 1993年 129-138

A Generalization of Bochner's Tube Theorem for Elliptic Boundary Value Problems

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The classical Bochner's tube theorem states that every holomorphic function defined on a connected tube domain T, $T = \mathbf{R}^n + i\Omega$, in $\mathbf{C}^n$ can be extended holomorphically to the convex hull $\widetilde{T}$, $\widetilde{T} = \mathbf{R}^n + i\widetilde{\Omega}$, of T. As is well-known, this property of holomorphic functions in several variables can be microlocalized along a totally real manifold M in a complex manifold X and is called a local version of Bochner's tube theorem (*cf.* [SKK, chap.I, prop.1.5.4] and also [H, lem.2.5.10; Ko] for a more precise statement). This kind of (microlocal) analytic continuation theorem is also proved for a generic CR-submanifold M of a complex manifold X (*cf.* [AT2, BT]).

In this note, we announce that a local version of Bochner's tube theorem holds good for boundary value problems for elliptic systems of differential equations on a real manifold X (Theorem 1). Our method also gives a tempered version of Theorem 1 by using the recent result [AT1] of Andronikof and Tose, reported in this conference (*cf.* the exposition of Tose in this volume). As a related subject, in the last section, we note that one can prove quite easily Epstein's edge-of-the-wedge theorem for elliptic boundary value problems.

Proceedings of the conference "Microlocal Geometry", the Research Institute of Mathematical Sciences, Kyoto, August 1992.

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Contents.

1. Main Theorem

Let X be a real analytic manifold, with $\mathcal{A}_X$ being the sheaf of analytic functions on X, M a submanifold of X of codimension $d \geq 1$. Let $\mathcal{D}_X$ denote the sheaf of differential operators with analytic coefficients on X, and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on X. Throughout this section we assume the following conditions on $\mathcal{M}$:

(a.1) $\mathcal{M}$ is elliptic :

$$T^*_X\widetilde{X} \cap \mathrm{Char}(\mathcal{M}) \subset T^*_{\widetilde{X}}\widetilde{X},$$

where $\widetilde{X}$ is a complex neighborhood of X on which $\mathcal{M}$ is defined as coherent $\mathcal{D}_{\widetilde{X}}$-module, and $\mathrm{Char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$.

(a.2) The complexification Z of M in $\widetilde{X}$ is noncharacteristic for $\mathcal{M}$:

$$T^*_Z\widetilde{X} \cap \mathrm{Char}(\mathcal{M}) \subset T^*_{\widetilde{X}}\widetilde{X}.$$

We set : $\mathcal{A}_X{}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_X)$.

Let $\tau : T_MX \to M$ be the normal bundle of M in X. Recalling the specialization functor [KS]

$$\nu_M : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b_{\mathbf{R}^+}}(T^*_MX),$$

we have :

Theorem 1. *Let U be an open conic subset of T_MX with connected fibres, $\widetilde{U}$ the convex hull of U in each fibre. Then*

$$\Gamma(\widetilde{U}, H^0\nu_M(\mathcal{A}_X{}^\bullet)) \longrightarrow \Gamma(U, H^0\nu_M(\mathcal{A}_X{}^\bullet)) \tag{1.1}$$

is an isomorphism.

EXAMPLE. Let $(X^{\mathbf{C}}, \mathcal{O}_{X^{\mathbf{C}}})$ be a complex manifold, X the underlying real manifold of $X^{\mathbf{C}}$, M a generic CR-submanifold of $X^{\mathbf{C}}$. Let $\mathcal{M}$ be the Cauchy-Riemann system of differential equations on X. Then $(X, M, \mathcal{M})$ satisfies conditions (a.1) and (a.2). Hence the theorem above holds for

$$\mathcal{A}_X{}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_X) \cong \mathcal{O}_{X^{\mathbf{C}}};$$

this is nothing but the microlocal version of Bochner's tube theorem for a generic CR-submanifold M, proved by Aoki and Tajima [AT2] (*cf.* also [BT, sect.3] for a related, but different problem).

2. Specialization and boundary value morphism

In this section and the next section, we fix a field k of characteristic zero and work with sheaves of k_X-modules on a topological manifold X. We denote by $\mathrm{D^b}(X)$ the derived category of k_X-modules.

Let X be a C^2-manifold, M a submanifold of X of codimension $d \geq 1$, $j : M \hookrightarrow X$ the embedding, $\tau : T_M X \to M$ the normal bundle of M in X,

$$\nu_M : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b_{\mathbf{R}^+}}(T_M^* X)$$

the specialization functor [KS]. For $F \in \mathrm{Ob}(\mathrm{D^b}(X))$, we have the canonical morphism

$$\nu_M(F) \longrightarrow \tau^! \mathbf{R}\tau_! \nu_M(F) \cong \tau^{-1} j^! F \otimes \tau^! k_M. \tag{2.1}$$

Applying the functor $H^0(\bullet)$, we have a sheaf-homomorphism

$$b : H^0 \nu_M(F) \longrightarrow \tau^{-1} H^d_M(F) \otimes or_{M|X}, \tag{2.2}$$

with $or_{M|X}$ being the relative orientation sheaf for $M \to X$.

Let U be an open conic subset of $T_M X$. If $\tau|_U : U \to M$ has connected (non-empty) fibres on M, (2.2) gives

$$b_U : \Gamma(U, H^0 \nu_M(F)) \longrightarrow \Gamma(M, H^d_M(F) \otimes or_{M|X}). \tag{2.3}$$

This is nothing but the boundary value map to M for F. Note that we have a canonical map

$$H^0(U, \nu_M(F)) \longrightarrow \Gamma(U, H^0 \nu_M(F)),$$

and an isomorphism

$$H^0(U, \nu_M(F)) \cong \varinjlim_V H^0(V, F),$$

where V ranges through the family $\mathcal{V}_U$ of the open subsets of X satisfying $C_M(X \setminus V) \cap U = \varnothing$. Hence, from (2.3), we get a canonical map

$$(2.4) \qquad H^0(V, F) \longrightarrow \Gamma(M, H^d_M(F) \otimes or_{M|X}).$$

Remark. — The description of boundary value morphism given here is classical for $F = \mathcal{O}_X$ (*cf.* e.g. [SKK, chap.1]). On the other hand, Schapira [S] constructed the canonical boundary value morphism

$$\mathbf{R}\Gamma_V(F)|_M \longrightarrow \mathbf{R}\Gamma_M F \otimes or_{M|X}[d]$$

for an open subset V of X with $\overline{V} \supset M$, satisfying a weaker condition.

EXAMPLE. Let X, M be as in section 1. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on X, and assume the condition (a.2). Let $\mathcal{B}_X$ denote the sheaf of Sato's hyperfunctions on X and set : $F = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_X)$. Then the target of morphism (2.1) is isomorphic to $\tau^{-1}\mathbf{R}\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}_M, \mathcal{B}_M)$, with $\mathcal{M}_M$ being the induced coherent $\mathcal{D}_M$-module of $\mathcal{M}$ by $M \to X$. Thus we obtain a canonical boundary value morphism for hyperfunction solutions of $\mathcal{M}$:

$$(2.5) \qquad \mathcal{H}om_{\tau^{-1}(\mathcal{D}_X|_M)}(\tau^{-1}(\mathcal{M}|_M), H^0\nu_M(\mathcal{B}_X)) \longrightarrow \tau^{-1}\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}_M, \mathcal{B}_M).$$

Note that Ôaku [O] constructed the same homomorphism as (2.5) by using the notion of F-mild hyperfunctions, which is also proved by [O] to be injective.

3. A key lemma — Fourier-Sato transformation

In this section, since we work only in the derived category $\mathrm{D^b}(k_X)$, with k a fixed field, we denote simply by f_*, $f_!$ the right derived push-forward functors by a continuous map f.

Let M be a C^1-manifold, $\tau : E \to M$ a C^1 vector bundle on M, $\pi : E^* \to M$ the dual bundle of E. Consider the diagram

$$\begin{array}{ccc} E \times_M E^* & \xrightarrow{p_2} & E^* \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\tau} & M \end{array}$$

and set :

$$P' = \{(x,y) \in E \times_M E^* \mid \langle x,\, y\rangle \leq 0\}.$$

Recall the Fourier-Sato transformation [KS, *cf.* also BMV]

$$\Phi : \mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(E^*), \qquad \Phi(G) = p_{2!}(p_1^{-1}G)_{P'}$$

for $G \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(E))$. Then we have

Theorem [KS, BMV]. *There is a canonical isomorphism :*

$$G \xrightarrow{\sim} p_{1*}\mathbf{R}\Gamma_{P'}(p_2{}^!\Phi(G)).$$

Moreover we have the following result :

Lemma 3.1. *There is a canonical commutative diagram :*

$$\begin{array}{ccc} G & \longrightarrow & p_{1*}\mathbf{R}\Gamma_{P'}(p_2{}^!\Phi(G)) \\ \downarrow & & \downarrow \\ \tau^!\tau_!G & \longrightarrow & p_{1*}p_2{}^!\Phi(G), \end{array}$$

where the vertical arrows are natural ones. In this diagram, every horizontal arrow is an isomorphism.

This lemma is proved by direct, but careful calculation. It is not very difficult to obtain an isomorphism from $\tau^!\tau_!G$ to $p_{1*}p_2{}^!\Phi(G)$, but we have to be more careful in proving commutativity of the diagram.

As a corollary of 3.1, we have :

Corollary 3.2. *There is a canonical distinguished triangle in* $\mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(E)$:

$$G \longrightarrow \tau^! \tau_! G \longrightarrow p^+_{1\,*} p^{+\,!}_2 \Phi(G) \xrightarrow{+1},$$

where $p^+_1 = p_1|_{P^+}$ *and* $p^+_2 = p_2|_{P^+}$, *with*

$$P^+ = \{(x,y) \in E \times_M E^* \mid \langle x,\, y\rangle > 0\}.$$

Remark. — In my talk at the conference, I reported the result of Corollary 3.2 by working on the sphere bundle $S(E \setminus M)$ and its dual $S(E^* \setminus M)$. In this case, the calculation is more complicated.

4. Elliptic boundary value problems

Let M, X, $\mathcal{M}$ be as in section 1. In particular, $\mathcal{M}$ is an elliptic system of differential equations on X.

Let $\pi : T^*_M X \to M$ be the conormal bundle of M in X. Recalling the Sato microlocalization functor [KS]

$$\mu_M : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(T^*_M X),$$

we have :

Theorem 4.1 [KK]. *For* $j < d$, $H^j \mu_M(\mathcal{A}_X{}^\bullet) = 0$.

This is a conclusion of the isomorphism obtained in [KK].

5. Proof of Theorem 1

Let M, X, $\mathcal{M}$ be as in section 1, and set : $G = \nu_M(\mathcal{A}_X{}^\bullet)$; then G is an object of $\mathrm{D}^{\mathrm{b}}(T_M X)$ and by definition $\Phi(G) = \mu_M(\mathcal{A}_X{}^\bullet)$. Therefore, by Theorem 4.1, we have $H^j(\Phi(G)) = 0$ for $j < d$. Hence, from Lemma 3.2, we have an exact sequence of sheaf-homomorphisms on $T_M X$:

$$0 \to H^0(G) \to \tau^{-1}\mathbf{R}^d \tau_! G \otimes or_{T_M X|M} \to p^+_{1\,*} p^{+\,-1}_2 (H^d \Phi(G) \otimes or_{T^*_M X|M}).$$

We note here that $\mathbf{R}^d \tau_! G \cong H^d_M(\mathcal{A}_X{}^\bullet)|_M$ and the second arrow of this sequence is nothing but morphism (2.2) for $F = \mathcal{A}_X{}^\bullet$. Using this exact sequence, and following the argument of [SKK, chap.1, prop.1.5.4], we can easily prove Theorem 1. The details are left to the reader.

6. A tempered version of Theorem 1

Let M, X, $\mathcal{M}$ be again as in section 1. In particular, $\mathcal{M}$ is an elliptic system of differential equations on X. Let $\mathcal{D}b_X$ be the sheaf of Schwartz's distributions on X.

Recently Andronikof and Tose [AT1] have proved an analogue of the celebrated formula of [KK] in elliptic boundary value problems for tempered distributions. By their result, we have in particular :

Theorem [AT1]. *For $j < d$,*

$$H^j \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}(\mathcal{M}|_M),\, T\text{-}\mu_M(\mathcal{D}b_X)) = 0.$$

Here $T\text{-}\mu_M(\mathcal{D}b_X)$ is the tempered microlocalization of $\mathcal{D}b_X$ along M due to Andronikof; this is, by the definition, the Fourier-Sato transform of the conic $\tau^{-1}(\mathcal{D}_X|_M)$-submodule $T\text{-}\nu_M(\mathcal{D}b_X)$ of $H^0\nu_M(\mathcal{D}b_X)$. For an open conic subset U of $T_M X$, we have

$$\Gamma(U,\, T\text{-}\nu_M(\mathcal{D}b_X)) \cong \varinjlim_{V} \Gamma_{t-M}(V,\, \mathcal{D}b_X),$$

where V ranges through the family $\mathcal{V}_U$ of the open subsets of X satisfying $C_M(X \setminus V) \cap U = \varnothing$, and

$$\begin{aligned}\Gamma_{t-M}(V,\, \mathcal{D}b_X) = \{\, f \in \mathcal{D}b_X(V) \mid & \text{ For any } u \in U, \\ & \text{there is an open subset } V' \text{ of } V \text{ such that } C_M(X \setminus V') \not\ni u \\ & \text{and } f|_{V'} \text{ is tempered at every point of } \overline{V'} \}.\end{aligned}$$

Since $\mathcal{M}$ is coherent over $\mathcal{D}_X$, we have :

$$\begin{aligned}&\Phi(\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}(\mathcal{M}|_M),\, T\text{-}\nu_M(\mathcal{D}b_X))) \\ &\qquad \cong \mathbf{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}(\mathcal{M}|_M),\, T\text{-}\mu_M(\mathcal{D}b_X))\end{aligned}$$

and

$$\begin{aligned}&H^0(U,\, \mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}(\mathcal{M}|_M),\, T\text{-}\nu_M(\mathcal{D}b_X))) \\ &\qquad \cong \varinjlim_{V} \Gamma_{t-M}(V,\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\, \mathcal{D}b_X))\end{aligned}$$

Hence, in virtue of the theorem [AT1] above, by the same argument as in section 5 with $G = \mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}(\mathcal{M}|_M),\, T\text{-}\nu_M(\mathcal{D}b_X))$, the following tempered version of Theorem 1 is obtained :

Theorem 6.1. *Let U and $\widetilde{U}$ be as in Theorem 1. Then*

$$(6.1)\qquad \varinjlim_{\widetilde{V}\in\mathcal{V}_{\widetilde{U}}} \Gamma_{t-M}(\widetilde{V}, H^0(\mathcal{A}_X{}^\bullet)) \longrightarrow \varinjlim_{V\in\mathcal{V}_U} \Gamma_{t-M}(V, H^0(\mathcal{A}_X{}^\bullet))$$

is an isomorphism, where $H^0(\mathcal{A}_X{}^\bullet) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_X)$.

Remark. — (6.1) is nothing but morphism (1.1) with a growth condition.

7. Concluding remarks

Let X, M be as in section 2. We follow the notations of section 2. Let $\pi : T^*_M X \to M$ be the conormal bundle of M in X,

$$\mu_M : \mathrm{D^b}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbf{R}^+}(T^*_M X)$$

the microlocalization functor [KS].

Let U be an open conic subset of $T_M X$, with *convex* (non-empty) fibres on M. Then we have a canonical isomorphism [KS, prop.3.7.12]

$$(7.1)\qquad \mathbf{R}\Gamma(U, \nu_M(F)) \cong \mathbf{R}\Gamma_\gamma(T^*_M X, \mu_M(F) \otimes \pi^! k_M)$$

for $F \in \mathrm{Ob}(\mathrm{D^b}(X))$, where $\gamma = U^{\circ a}$. From this isomorphism, we get a canonical morphism

$$(7.2)\qquad \begin{aligned} \mathbf{R}\Gamma(U, \nu_M(F)) &\longrightarrow \mathbf{R}\Gamma(T^*_M X, \mu_M(F) \otimes \pi^! k_M) \\ &\cong \mathbf{R}\Gamma(M, j^! F[d] \otimes or_{M|X}). \end{aligned}$$

Such a description of the boundary value morphism is given in [ST, sect.4]. This is compatible with morphism (2.1); in fact, we have :

Lemma 7.1. *There is a canonical commutative diagram :*

$$(7.3)\qquad \begin{array}{ccc} \mathbf{R}\Gamma(U, \nu_M(F)) & \xrightarrow{\sim} & \mathbf{R}\Gamma_\gamma(T^*_M X, \mu_M(F) \otimes \pi^! k_M) \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma(U, \tau^! \mathbf{R}\tau_! \nu_M(F)) & \xrightarrow{\sim} & \mathbf{R}\Gamma(T^*_M X, \mu_M(F) \otimes \pi^! k_M). \end{array}$$

Assume now that $H^j\mu_M(F) = 0$ for $j < d$. Then, noting also that $H^j\nu_M(F) = 0$ for $j < 0$, we have from (7.3) :

$$\begin{array}{ccc} \Gamma(U,\, H^0\nu_M(F)) & \xrightarrow{\sim} & \Gamma_\gamma(T_M^*X,\, H^d\mu_M(F) \otimes \pi^{-1}or_{M|X}) \\ {\scriptstyle b_U}\downarrow & & \downarrow \\ \Gamma(M,\, H_M^d(F) \otimes or_{M|X}) & \xrightarrow{\sim} & \Gamma(T_M^*X,\, H^d\mu_M(F) \otimes \pi^{-1}or_{M|X}). \end{array}$$

By this diagram, it is quite easy to prove a microlocal version of Epstein's edge-of-the-wedge theorem in elliptic boundary value problems :

Proposition 7.2. *Let M, X, $\mathcal{M}$ be as in section 1. Let U_1, U_2 be open conic subsets of T_MX, with convex (non-empty) fibres on M. Then the sequence*

$$\Gamma(U_1+U_2,\, H^0\nu_M(\mathcal{A}_X{}^\bullet)) \longrightarrow \Gamma(U_1,\, H^0\nu_M(\mathcal{A}_X{}^\bullet)) \oplus \Gamma(U_2,\, H^0\nu_M(\mathcal{A}_X{}^\bullet)) \xrightarrow{b_{U_1}-b_{U_2}} \Gamma(M,\, H_M^d(\mathcal{A}_X{}^\bullet) \otimes or_{M|X})$$

is exact, where $\mathcal{A}_X{}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\, \mathcal{A}_X)$.

For a general edge-of-the-wedge theorem of Martineau type (i.e., for N convex, open infinitesimal wedge domains $U_1, \cdots, U_N$ with the edge on M), the suppleness of the sheaf $H^d\mu_M(\mathcal{A}_X{}^\bullet)$ seems to be necessary (*cf.* [ST, sect.4]). We finally remark that, in virtue of the result of [AT1] (*cf.* theorem of section 6), one can replace

$$H^0\nu_M(\mathcal{A}_X{}^\bullet) = H^0\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}(\mathcal{M}|_M),\, \nu_M(\mathcal{A}_X))$$

in the proposition above by

$$H^0\mathbf{R}\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}(\mathcal{M}|_M),\, T\text{-}\nu_M(\mathcal{D}b_X))\ ;$$

this gives a tempered version of generalized Epstein's theorem in elliptic boundary value problems.

Acknowledgements. I thank E. Andronikof for discussions on the result of [AT1].

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