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QUASI-CLASSICAL ANALYSIS OF NONLINEAR INTEGRABLE SYSTEMS

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Mathematical methods of quasi-classical analysis have been mostly developed for linear problems. A typical case is the WKBJ method to the classical Sturm-Liouville problem (or the time-independent Schrödinger equation)

$$\hbar^2 \frac{\partial^2 \psi}{\partial x^2} = f(x)\psi, \tag{1}$$

the Planck constant $\hbar$ playing the role of a small parameter. This method is based on the the WKBJ (or Liouville-Green) approximation

$$\psi \sim f^{-1/4} \exp[\pm \int^x f(y)^{1/2} dy], \tag{2}$$

which is known to give double asymptotics with respect to both $\hbar \to 0$ and $x \to \infty$ in a suitable domain.

We wish to extend the idea of quasi-classical analysis to nonlinear problems. This is inspired by recent progress of mathematical physics, in particular, two dimensional quantum gravity and its relation to Painlevé transcendents. In the simplest case, the

Painlevé equation of the first kind (PI) emerges therein with a Planck constant as:

$$\hbar^2 \frac{\partial^2 u}{\partial x^2} = u^2 - x. \tag{3}$$

Physicists seek for a solution of this kind of equations (sometimes called "string equations") satisfying suitable physical boundary conditions. The first stage of this analysis is to construct a formal solution of, say, the above PI, of the form

$$u = x^{1/2}(1 - \frac{1}{8}\hbar^2 x^{-5/2} + \cdots). \tag{4}$$

Even the coefficients of such a formal solution are known to carry important information (on the topology of a moduli space of algebraic curves), but this is so called "perturbative analysis." "Non-perturbative analysis" is directly related to global properties of solutions, in particular, asymptotics as $x \to \infty$. It is rather easy to see that a nonlinear version of "Stokes phenomena" can take place and plays a crucial role in such non-perturbative analysis.

Physicists' demonstration of this fact is to write a solution of PI as a perturbation of the lowest order approximation $u \sim x^{1/2}$ as

$$u = x^{1/2} + v \tag{5}$$

and to consider the equation

$$\frac{\partial^2 v}{\partial x} - 2x^{1/2} v = \frac{1}{4} x^{-3/2} + v^2 \tag{6}$$

for v. This is a nonlinear and inhomogeneous version of the Sturm-Liouville problem; the WKBJ solutions

$$v_0 \sim x^{-1/8} \exp[\pm \frac{\sqrt{32}}{5} x^{5/4}] \tag{6}$$

of the linearized and homogenized equation are expected to carry lowest order information on non-perturbative aspects of PI. Not only being rather crude, this method is also unsatisfactory in the sense that it does not reflect a number of special properties of Painlevé equations.

More systematic analysis indeed rely on such special properties of Painlevé equations. For instance, Painlevé equations, like other isomonodromy deformations, are characterized as deformations preserving monodromy data (Stokes coefficients, etc) of an associated linear problem. Therefore, if an explicit analytical expression of the monodromy data is available, one can derive some information on solutions of the nonlinear problem. This method of approach has indeed been applied successfuly to several important cases, applying the WKBJ method to the linear problem to give such an analytical expression of monodromy data.

Besides these nonlinear problems of isomonodromy deformations, issues of quasi-classical analysis can also be found in nonlinear problems of isospectral deformations, such as the KdV equation, the KP hierarchy, etc. In fact, these two problems are closely related. For instance, PI is known to emerge as a specializaton of a higher KdV equation, and its algebraic structure becomes more transparent if it is embedded into the so called KdV (or KP) hierarchy that consists of an infinite number of all possible higher KdV (or KP) flows. This standpoint, too, is a lesson learned from the recent progress of two dimensional quantum gravity.

Actually, it turns out that the KP hierarchy itself can become a stage of quasi-classical analysis. Below I give a list of my publication (with Takashi Takebe) written along this line of interests. Unfortunately, most results are limited to the formal level (formal expansion in $\hbar$), but I would like to consider them as a first step towards analytical treatment.

1. Area-Preserving Diffeomorphisms and Nonlinear Integrable Systems, by Kanehisa Takasaki, 16 pages (Paper: hep-th/9112041 Field: T Date:1991.12.17) Published in: *Topological and geometrical methods in field theory*, Turku, Finland, May 26 - June 1, 1991, J. Mickelsson and O. Pekonen eds. (World Scientific, Singapore, 1992).

Abstract: Present state of the study of nonlinear "integrable' systems related to the group of area-preserving diffeomorphisms on various surfaces is overviewed. Roles of area-preserving diffeomorphisms in 4-d self-dual gravity are reviewed. Recent progress in new members of this family, the SDiff(2) KP and Toda hierarchies, is reported. The group of area-preserving diffeomorphisms on a cylinder plays a key role just as the infinite matrix group GL(∞) does in the ordinary KP and Toda lattice hierarchies. The notion of tau functions is also shown to persist in these hierarchies, and gives rise to a central extension of the corresponding Lie algebra.

2. SDIFF(2) TODA EQUATION – HIERARCHY, τ FUNCTION, AND SYMMETRIES, by Kanehisa Takasaki and Takashi Takebe, 16 pages (Paper: hep-th/9112042 Field: T Date:1991.12.17)

   Published in: Lett. Math. Phys. 23 (1991), 205-214.

Abstract: A continuum limit of the Toda lattice field theory, called the SDiff(2) Toda equation, is shown to have a Lax formalism and an infinite hierarchy of higher flows. The Lax formalism is very similar to the case of the self-dual vacuum Einstein equation and its hyper-Káhler version, however now based upon a symplectic structure and the group SDiff(2) of area preserving diffeomorphisms on a cylinder $S^1 \times \mathbf{R}$. An analogue of the Toda lattice tau function is introduced. The existence of hidden SDiff(2) symmetries are derived from a Riemann-Hilbert problem in the SDiff(2) group. Symmetries of the tau function turn out to have commutator anomalies, hence give a representation of a central extension of the SDiff(2) algebra.

3. SDIFF(2) KP HIERARCHY, by Kanehisa Takasaki and Takashi Takebe, 34 pages (Paper: hep-th/9112046 Field: T Date:1991.12.18)

   Published in: Proceesings of RIMS Research Project 1991 "Infinite Analysis," Int. J. Mod. Phys. A7, Suppl. 1B (1992), 889-922.

Abstract: An analogue of the KP hierarchy, the SDiff(2) KP hierarchy, related to the group of area-preserving diffeomorphisms on a cylinder is proposed. An improved Lax formalism of the KP hierarchy is shown to give a prototype of this new hierarchy. Two important potentials, S and τ, are introduced. The latter is a counterpart of the tau function of the ordinary KP hierarchy. A Riemann-Hilbert problem relative to the group of area-diffeomorphisms gives a twistor theoretical description (nonlinear graviton construction) of general solutions. A special family of solutions related to topological minimal models are identified in the framework of the Riemann-Hilbert problem. Further, infinitesimal symmetries of the hierarchy are constructed. At the level of the tau function, these symmetries obey anomalous commutation relations, hence leads to a central extension of the algebra of infinitesimal area-preserving diffeomorphisms (or of the associated Poisson algebra).

4. W ALGEBRA, TWISTOR, AND NONLINEAR INTEGRABLE SYSTEMS, by Kanehisa Takasaki, 15 pages (Paper: hep-th/9206030 Field: T Date:1992.06.06 Rev.:1992.07.24) Published in: "Algebraic Analysis and Number Theory," RIMS Kokyuroku 810 (September, 1992).

Abstract: W algebras arise in the study of various nonlinear integrable systems such as: self-dual gravity, the KP and Toda hierarchies, their quasi-classical (or dispersionless) limit, etc. Twistor theory provides a geometric background for these algebras. Present state of these topics is overviewed. A few ideas are proposed to unify these nonlinear systems as deformations of self-dual gravity including quantum deformations.

5. QUASI-CLASSICAL LIMIT OF KP HIERARCHY, W-SYMMETRIES AND FREE FERMIONS, by Kanehisa Takasaki and Takashi Takebe, 11 pages (Paper: hep-th/9207081 Field: T Date:1992.07.24)

Abstract: This paper deals with the dispersionless KP hierarchy from the point of view of quasi-classical limit. Its Lax formalism, W-infinity symmetries and general

solutions are shown to be reproduced from their counterparts in the KP hierarchy in the limit of $\hbar \to 0$. Free fermions and bosonized vertex operators play a key role in the description of W-infinity symmetries and general solutions, which is technically very similar to a recent free fermion formalism of $c = 1$ matrix models.

6. QUASI-CLASSICAL LIMIT OF TODA HIERARCHY, by Kanehisa Takasaki and Takashi Takebe, in preparation

Abstract: Previous results on quasi-classical limit of the KP hierarchy and its W-infinity symmetries are extended to the Toda hierarchy. The Planck constant $\hbar$ now emerges as the spacing unit of difference operators in the Lax formalism. Basic notions, such as dressing operators, Baker-Akhiezer functions and tau function, are redefined. $W_{1+\infty}$ symmetries of the Toda hierarchy are realized by suitable rescaling of the Date-Jimbo-Kashiara-Miwa vertex operators. These symmetries are contracted to $w_{1+\infty}$ symmetries of the dispersionless hierarchy through their action on the tau function.

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