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Maslov classes of Lagrange varieties and Legendre varieties

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1. Introduction

There are many situations where so called Maslov indices and Maslov classes play important roles, for instance, in the asymptotic method of P.D.E., in representation theory, in geometric analysis and in symplectic topology.

In this note we give a global formula on Maslov classes appearing in the process of symplectic reduction, comparing the geometry of totally real submanifolds of a complex submanifold, and on the other hand we calculate Maslov indices appearing in projective geometry of curves in the context of contact geometry. Intimate considerations and proofs will be given in a forthcoming papers.

First let us review the classical Maslov-Arnol'd class in an easy manner.

Consider a (topological) symplectic vector bundle over a topological space. A subbundle of the symplectic bundle is called isotropic (resp. Lagrange, coisotropic) if its skew-orthogonal complement (with respect to the symplectic form) contains itself (resp. equals itself, is contained in itself). Remark that a symplectic bundle has a complex structure (unique up to homotopy) such that the symmetric form defined by the symplectic form and the complex structure is positive definite. Further the symplectic bundle has the Hermitian structure such that the imaginary part of the Hermitian form is equals to the symplectic form.

Now consider two oriented Lagrange subbunldes of the symplectic bundle. Take local oriented orthonormal frames of the first (resp. second) Lagrange bundle. Then there exists a local family of unitary matrices connecting the frame of the first to the frame of the second. Taking the determinants of the unitary matrices defines a mapping to the circle, which is glued to a global continuous mapping from the base space. Then, for each loop in the base space, we define the *Maslov index* by the mapping degree of the composition of the loop and the above determinant mapping. Thus we have a class in the first cohomology group of the base space with integer coefficient, which is called the *(oriented) Maslov class* (defined by the symplectic bundle and a pair of oriented Lagrange subbundles).

The following properties of Maslov classes are fundamental and easy to see:

(1) For three oriented Lagrange subbundles of a fixed symplectic bundle, the sum of the Maslov class of the first and the second, and the Maslov class of the second and the third, is equal to that of the first and the third.

(2) If an isomorphism of symplectic bundles maps a pair of Lagrange subbundles to another pair of Lagrange subbundles, then the corresponding Maslov classes coincide.

(3) For a continuous mapping, the Maslov class of the pull-back bundles of a symplectic bundle and a pair of Lagrange subbundles over the target space is equal to the pull-back of the Maslov cohomology class of that triplet.

We intend in this note to apply this tool to study Lagrange and Legendre variety.

A type of Lagrange variety in the cotangent bundle of a manifold is the graph of a closed multivalued one form on the manifold in some naive sense. (In general, a subset in a symplectic manifold should be called Lagrange if the regular points set is open dense and it is a Lagrange submanifold, that is, the maximal dimensional integral submanifold where the symplectic form vanishes.)

The graph of a closed one form on a manifold (in the usual sense) is an example of Lagrange submanifold of the cotangent bundle. Another important example is the conormal bundle of a submanifold: In general the conormal bundles of varieties with singularities form another class of Lagrange varieties in cotangent bundles.

To make the story more clear, we shall study Lagrange varieties via parametrizations of them.

A mapping from a manifold into a symplectic manifold is called *isotropic* if the pull-back of symplectic form is zero. Isotropic mappings arise particularly in the process of (local) symplectic reduction due to Marsden, Weinstein and Tulczyjew.

In a symplectic manifold consider a coisotropic submanifold, where the tangent bundle is a coisotropic subbundle of the restriction of the tangent bundle of the symplectic manifold. Then the skew-othogonal complement to the tangent bundle of the coisotropic submanifold in the restriction of the tangent bundle of the symplectic manifold with respect to the symplectic form is an integrable subundle: We call it the characteristic distribution of the coisotropic submanifold and the induced foliation the characteristic foliation. Locally at each point of the coisotropic submanifold, we have a submersion (quotient mapping) to the leaf space. Then the (local) leaf space has the unique symplectic structure up to symplectic diffeomorphisms such that the pull-back of the symplectic form is equal to the restriction of the original symplectic form. Furthermore consider an isotropic submanifold of the symplectic manifold contained in the above coisotropic submanifold. Then we see the local projection of the isotropic submanifold to the leaf space is an isotropic mapping, which is not neccesarily an immersion: It is an immersion at a point of the isotropic submanifold if and only if the tangent space to the isotropic submanifold at the point contains no characteristic directions (lines in the skew-orthogonal complement to the coisotropic submanifold at that point). Such a germ is called an isotropic map-germ arising from symplectic reduction. Conversely any global isotropic mappings are obtained by symplectic reduction processes where the projection to the leaf space is globally defined.

In any case we have Maslov classes by taking quotient bundles in stead of quotient spaces. In §2 we give a global formula for Maslov classes in this situation.

Now we turn our attention to Legendre varieties.

From several evidences we should formulate the notion of a "front hypersurface" by the property that the Nash modification projects to the hypersurface itself finitely to one. The Nash modification, in this case, is the closure of the lift of the regular points set in the projective cotangent bundle of the manifold where the hypersurface lies in: The projective cotangent bundle is identified with the totality of contact elements (tangent hyperplanes) of the base space and it has the natural contact structure: A velocity vector from a contact element is contained in the contact distribution if and only if the projection of the vector is contained in the contact element.

The tangent hyperplanes to the regular points form a Legendre submanifold, that is, the maximal dimensional integral submanifold of the contact distribution defined over the projective cotangent bundle. Though the notion of Legendre variety have not established yet, the closure of this natural lift might be regarded as a Legendre variety. (In fact a definition of a Legendre variety is that it contains an open dense Legendre submanifold.) The Legendre variety thus obtained by the Nash modification has singularities in general. In the case when the Nash modification is non-singular, then the hypersurface turns out the projection of a Legendre submanifold. Thus the front hypersurface is, in this case, a wave front set in the sense of Arnol'd. Remark that, for a generic Legendre submanifold, the projection is finite to one. In the above definition of front hypersurfaces we allow singularities for Nash modifications. To make the definition non-trivial, it is necessary the finite condition as imposed.

Exactly we utilize parametrizations of Legendre varieties to formulate notions above mentioned as follows: A mapping from an n-dimensional manifold to an $n+1$-dimensional manifold (say, of class C^∞) is called a *front mapping* if the regular points set is dense and, for each point of the source, the images of the tangent spaces of regular points converge to a *tangent hyperplane* as regular points tend to that point, and the tangent hyperplanes depend smoothly on the points in source space. If we associate the tangent hyperplane thus defined to the point of the source, then we have a C^∞ lift of the mapping to the projective cotangent space of the target space. This lift is an integral mapping (or an isotropic mapping) in the sense that the image of tangent space to each point is contained in the contact distribution of the projective cotangent bundle. Under this formulation, if a front mapping is finite to one, then the image of the lift projects finitely to one, so this formulation, in this case, fits to the naive consideration mentioned before.

Now consider the Nash modification of the Legendre variety itself, which would be called the Legendre-Nash modification. In many situations we deal with, we observe the Legendre-Nash modification has at least continuous section outside a submanifold of codimension two of the source space: We call it "the essential singular locus" of the (parametrized) Legendre variety. Then the Maslov index of a loop around the essential singular locus has the meaning as the obstruction to extend the section beyond the essential singular locus. An important example of front hypersurface is the developable of a curve in an affine space or in an projective space. We treat this object in §3.

We add two related but independent topics in following Appendices.

2. Maslov classes for symplectic reductions

Let (M, ω) be a symplectic manifold of dimension $2(n+k)$, $C \subset M$ be an involutive submanifold of codimension k. The skew-orthogonal complement $(TC)^{\perp}$ of TC in $TM|C$ with respect to ω is an integral subbundle of dimension k. This defines the characteristic foliation on C. Set $\tilde{E} = TC/(TC)^{\perp}$, the normal bundle of the foliation, which is a symplectic bundle over C.

Let $N^n \subset C$ be an isotropic submanifold ($TN \subset (TN)^{\perp}$) of M. The singular locus of N is defined by

$$\Sigma = \{x \in N \mid T_xN \cap (T_xC)^{\perp} \neq \{0\}\},$$

that is, the set of points where T_xN contains a characteristic direction in $(T_xC)^{\perp}$.

Our purpose is to investigate the singular behavior of the "Lagrnage distribution" defined by N and C: $(TN + (TC)^{\perp})/(TC)^{\perp}|N$ in $E = \tilde{E}|N$.

REMARK: Let $f : N \to (M', \omega')$ be an isotropic mapping. Set $M = T^*N \times M' \supset C = N \times M' \supset graph f (\cong N)$.. Then Σ is equal to the singular locus of f as mapping.

EXAMPLE: A typical isotropic mapping with singularities is the open Whitney umbrella: Set $f = (\xi_1, x_1, \xi_2, x_2) = (v^3/3, u, uv, v^2/2) : \mathbb{R}^2, 0 \to T^*\mathbb{R}^2$. Then $f^*\theta = d(uv^3/3)$, where $\theta = \xi_1 dx_1 + \xi_2 dx_2$ is the Liouville form.

EXAMPLE: Let X be an $(n+k)$-manifold. Set $M = T^*X$. Then a coisotropic submanifold $C \subset T^*X$ of codimension k is regarded as a *Hamilton-Jacobi equation*. Let $S \subset X$ be an n-submanifold and $t : S \to T^*X$ be an isotropic section: $N = t(S) \subset C$ is an initial condition of the *generalized Cauchy problem* for the Hamilton-Jacobi equation. Let $x \in N - \Sigma$, then locally the union of leaves through N form a Lagrange submanifold which are regarded as a solution of the equation. The following result can be applied to this situation.

Now we assume N is oriented and compact without boundary and assume that each component S of Σ has an open neighborhood U_S in N such that $\overline{U}_S$ is a manifold with boundary $\partial\overline{U}_S$, $\overline{U}_S$ is a deformation retract of S and $E|\overline{U}_S$ has an oriented Lagrange subbundle L'_S.

Then, by Lefshetze duality and Poincaré duality we have a class m_S in $H^2(N; \mathbb{Z})$ from the Maslov class of the triplet $(E|U_S - S, L|U_S - S, L'_S|U_S - S)$ in $H^1(U_S - S; \mathbb{Z})$. Then we have a type of residue formula:

THEOREM. *The sum of m_S over all components of Σ is equal to the first Chern class $c_1(E)$.*

This result generalizes an equatity in [IO] in the symplectic case and is proved similarly when we use a perturbation of N and observe complex points of thus obtained submanifold.

Recently Professor Tatsuo Suwa has suggested that the formula should be further generalized in connection with the results of Vaiseman, Lehmann and himself: The author would like to achieve such generalizations in a forthcoming paper.

3. Developable of a curve and Maslov index

The ruled surface by the tangent lines to a space curve is called the developable surface of the curve. In general the developable of a curve in $(n+1)$-dimensional projective space is defined as the hypersurface "ruled" by osculating $(n-1)$-subspaces to the curve. Let $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$ be a C^∞ parametrized curve, where M is a one-dimensional manifold and $n \geqq 1$. We call the germ γ_p at a point $p \in M$ of finite osculation-type (or simply, of finite type) if γ is represented by $x_i = t^{a_i} + o(t^{a_i}), 1 \leqq i \leqq n+1$, for a C^∞ coordinate t of (M, p) and an affine coordinate $(x_1, \dots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ centered at $\gamma(p)$, where each a_i is a natural number and $1 \leqq a_1 < \cdots < a_{n+1}$. Then $\mathsf{A} = (a_1, a_2, \dots, a_{n+1})$ is a local projective invariant of the germ γ_p and we call A the type of γ_p; type$(\gamma_p) = \mathsf{A}$. A point $p \in M$ is called an ordinary point if type$(\gamma_p) = (1, 2, \dots, n, n+1)$, and, otherwise, it is called a special point. Special points of finite type are isolated in M.

For each $p \in M$ where γ_p is of finite type and for each i, $(0 \leqq i \leqq n+1)$, we set $O_i(\gamma, p) = \{x_{i+1} = \cdots = x_{n+1} = 0\} \subset T_{\gamma(p)}\mathbf{R}P^{n+1}$ under the above affine representation of γ_p. The corresponding projective subspace of $\mathbf{R}P^{n+1}$ through $\gamma(p)$ of dimension i is also denoted by $O_i(\gamma, p)$. Further we define the osculating i-bundle $O_i(\gamma) = \bigcup_{p \in M} O_i(\gamma, p)$ in the pullback bundle $\gamma^{-1}T\mathbf{R}P^{n+1}$. The natural parametrization dev$(\gamma) : O_{n-1}(\gamma) \longrightarrow \mathbf{R}P^{n+1}$ defined by $(p, q) \mapsto q$, where $q \in O_{n-1}(\gamma, p) (\subset \mathbf{R}P^{n+1})$, is called a developable of γ. The germ dev$(\gamma)_p$ of dev(γ) at $(p, 0)$ is determined up to the projective equivalence by the projective class of γ_p.

Denote the dual projective space of $\mathbf{R}P^{n+1}$ by $\mathbf{R}P^{n+1*}$, which is the space of all hyperplanes in $\mathbf{R}P^{n+1}$. Then we naturally identify $\mathbf{R}P^{n+1**}$ with $\mathbf{R}P^{n+1}$. For a curve $\gamma : M \longrightarrow \mathbf{R}P^{n+1}$, γ_p being of finite type at each point $p \in M$, we define the dual $\gamma^* : M \longrightarrow \mathbf{R}P^{n+1*}$ of γ by $p \mapsto O_n(\gamma, p)$. We describe the developable of γ by the dual curve γ^* in $\mathbf{R}P^{n+1*}$:

LEMMA ([S1]).
(0) *γ^* is a C^∞ map.*
(1) *If γ_p is of type $\mathsf{A} = (a_1, \dots, a_{n+1})$, then γ^*_p is also of finite type $\mathsf{A}^* = (a_{n+1} - a_n, a_{n+1} - a_{n-1}, \dots, a_{n+1} - a_1, a_{n+1})$.*
(2) *$O_i(\gamma^*, p) = O_{n-i}(\gamma, p)^*$, the dual of $O_{n-i}(\gamma, p)$, $0 \leqq i \leqq n$.*
(3) *$\gamma^{**} = \gamma$.*
(4) *dev(γ) is identified with front(γ^*) : $O_{n-1}(\gamma) = O_1(\gamma^*)^* \longrightarrow \mathbf{R}P^{n+1}$ defined by $(p, q) \mapsto q$, where $q \in O_1(\gamma^*, p)^* \subset \mathbf{R}P^{n+1**} = \mathbf{R}P^{n+1}$.*

Set $\mathsf{A}^* = \mathsf{B} = (b_1, \dots, b_{n+1})$ and $a_0 = 0$. Then $b_i = a_{n+1} - a_{n+1-j}, 1 \leqq i \leqq n+1$.

Set $Q = \{(p, q) | p \in q^*\} \subset \mathbf{R}P^{n+1} \times \mathbf{R}P^{n+1*}$. The both natural identifications $Q \cong PT^*\mathbf{R}P^{n+1}$ and $Q \cong PT^*\mathbf{R}P^{n+1*}$ induce the same contact structure on Q, [S1]. Then front(γ^*) lifts to an isotropic mapping $O_1(\gamma^*)^* \longrightarrow Q$ naturally. Therefore the developable is regarded to be a wave front set of a Legendre variety, which in general has singular points.

We take an affine representaive of $\gamma^*_p : y_i = y_i(t)$ with the order of $y_i(t) = b_i$. We define the affine coordinate $x = (x_1, \dots, x_{n+1})$ of $\mathbf{R}P^{n+1}$ correspondingly such that the projective duality is described by $\sum_{j=0}^{n+1} x_j y_{n+1-j} = 0$, with $x_0 = y_0 = 1$. Set

$$F(x,t) = y_{n+1}(t) + x_1 y_n(t) + \cdots + x_n y_1(t) + x_{n+1} = \sum_{j=0}^{n+1} x_j y_{n+1-j}(t).$$

The one-parameter family of osculating hyperplanes of γ near p is defined by $F = 0$. Then $O_1(\gamma^*)^*$ is obtained when we solve the system of equations $F = \partial F/\partial t = 0$ first for $t \neq 0$, and then extend to $t = 0$. Thus we have

$$x_n(x',t) = -(1/\dot{y}_1)(\dot{y}_{n+1} + x_1 \dot{y}_n + \cdots + x_{n-1}\dot{y}_2),$$

where $x' = (x_1, \ldots, x_{n-1})$, and $x_{n+1}(x',t)$ is determined by

$$\partial x_{n+1}/\partial t = -y_1 \partial x_n/\partial t, \quad x_{n+1} \in t^r E_{x',t},$$

where $r = b_1 = a_{n+1} - a_n$. The developable is then parametrized by the germ $f : \mathbf{R}^{n-1} \times \mathbf{R}, 0 \longrightarrow \mathbf{R}P^{n+1}, \gamma(p)$ defined by $(x',t) \mapsto (x', x_n(x',t), x_{n+1}(x',t))$. Remark that the singular locus $\Sigma(f) \subset \mathbf{R}^{n-1} \times \mathbf{R}, (0,0)$ of f is equal to $\{\partial x_n/\partial t = 0\}$. Therefore $\Sigma(f)$ contains the component $\{t = 0\}$ if and only if $s = a_n - a_{n-1} > 1$.

Let $\gamma : M \to \mathbf{R}P^{n+1}$ be a curve of finite type with special points $t^{(1)}, \ldots, t^{(\ell)}$ of type $a^{(1)}, \ldots, a^{(\ell)}$, and $f : M \times \mathbf{R}P^{n-1} \to \mathbf{R}P^{n+1}$ be the natural parametrization of the developable of γ. Then f lifts to an isotropic map $\tilde{f} : M \times \mathbf{R}P^{n-1} \to Q = PT^*\mathbf{R}P^{n+1}$. Let $\Lambda(Q)$ denotes the space of Legendre planes of Q. Then we have

PROPOSITION. *The isotropic map $\tilde{f}$ is smoothly lifts to $M \times \mathbf{R}P^{n-1} - \Sigma \to \Lambda(Q)$ where $\Sigma = \cup_j \Sigma^{(j)} = \cup_j \{t^{(j)}\} \times O_{n-2}(t^{(j)})$, where j runs over with $r^{(j)} > s^{(j)}$. Further the Maslov index around $\Sigma^{(j)}$ is equal to ± 1 if $r^{(j)} - s^{(j)}$ is odd, and 0 otherwise.*

EXAMPLE: For a curve of type $(1,2,4)$, we have $r = 2, s = 1$. Thus the Maslov index of the developable of type $(1,2,4)$ is equal to ± 1. (In this case the developable is called the composed umbrella).

Appendix. Universal spaces for non-oriented Maslov classes

Let M be a colsed C^∞ manifold of dimension m, and $\pi : E \to M$ a Hermitian bundle of real rank $2n$ with the complex structure J and the Hermitian form H. then the real part g of H is a metric on E and the imaginary part Ω of H is a symplectic structure on E.

Now let L be a Lagrangian subbundle of the symplectic bundle (E, Ω). Denote by $\Lambda(E)$ the totality of Lagrange planes of E, and by $\pi' : \Lambda(E) \to N$ the canonical projection from $\Lambda(E)$ to N.

For the standard symplectic vector space $\mathbb{C}^n$, we denote by $\Lambda(n) \cong U(n)/O(n)$ the set of Lagrangian planes of $\mathbb{C}^n$.

Consider the classifying space $BO(n)$; the set of n-planes in $\mathbb{R}^{N+1}$ for a sufficiently large N. Further consider the space $EO(n)$; the set of frames of n-planes in $\mathbb{R}^{N+1}$. Then we have the principal $O(n)$-bundle $\Pi : EO(n) \to BO(n)$: For $\mathbf{f} = (f_1, \ldots, f_n) \in EO(n)$ and $A \in O(n)$, denote by $\mathbf{f}A = (f_1, \ldots, f_n)A$ the frame transformed by A.

Set $X_n = EO(n) \times_{O(n)} \Lambda(n)$, which is obtained by identifying $(\mathbf{f}, \lambda) \sim (\mathbf{f}A, A^{-1}\lambda), \mathbf{f} \in EO(n), \lambda \in \Lambda(n), A \in O(n)$. Then we set $\Pi' : X_n \to BO(n), \Pi'[\mathbf{f}, \lambda] = \Pi[\mathbf{f}]$.

Let $\psi_L : M \to BO(n)$ the classifying map of L. Then there exist an isomorphism ρ between the frame bundle of L and $\psi_L^* EO(n)$. Define $\phi_L : \Lambda(E) \to X_n$ as follows: For $\lambda \in \Lambda(E), \pi(\lambda) = x \in M$ and for an orthonormal frame $\mathbf{f}$ of L_x, there exists $A \in U(n)$ such that $\mathbf{f}A$ is a frame of λ. By ρ, $\mathbf{f}$ can be regarded as $\mathbf{f} \in \Pi^{-1}(\psi_L(x))$. Then we define $\phi_L(\lambda)$ by the class of $(\mathbf{f}, A)$ in X_n. We have then $\Pi' \circ \phi_L = \psi_L \circ \pi'$.

Let L' be another Lagrangian subbundle of (E, Ω). Then there associated a section $s_{L'} : M \to \Lambda(E)$ and thus $\phi_L \circ s_{L'} : M \to X_n$.

Let $h \in H^*(X_n, \mathbb{Z})$. Then we set $m_h(E; L, L') = (\phi_L \circ s_{L'})^* h$ and call it the *Maslov class* of triple $(E; L, L')$ relatively to h.

For the canonical map $Det^2 : X_n \to O(n)\backslash U(n)/O(n) \to S^1$ defined by taking the square of determinant, define $e \in H^1(X_n; \mathbb{Z})$ by $e = (Det^2)^* 1$, where $1 \in H^1(S^1; \mathbb{Z})$ is the canonical generator. Then the usual (non-oriented) Maslov-Arnol'd class is equal to $m_e(E; L, L')$.

On the classifying space $BO(n)$, we consider the tautological n-bundle $\mathcal{L} \subset BO(n) \times \mathbb{R}^{N+1}$. Set $\mathcal{E}' = \mathcal{L} \otimes \mathbb{C} \subset BO(n) \times \mathbb{C}^{N+1}$. Set $\mathcal{E} = \Pi'^* \mathcal{E}', \mathcal{L}_1 = \Pi'^* \mathcal{L}$. Define $\mathcal{L}_2$, for $[\mathbf{f}, A] \in X_n$, $\mathcal{L}_{2,[\mathbf{f},A]} = \langle \mathbf{f}A \rangle \subset \mathcal{E}_{[\mathbf{f},A]}$.

Then $\mathcal{L}_1, \mathcal{L}_2$ are Lagrange subbundle of $\mathcal{E}$ and we easily see that $m_h(\mathcal{E}; \mathcal{L}_1, \mathcal{L}_2) = h$ for any $h \in H^*(X_n; \mathbb{Z})$.

Appendix. Lagrange-Nash modification and Maslov-Arnol'd-Fuks homology

Let $N \subset \mathbb{C}^n$ be a singular Lagrange variety and $\Sigma \subset N$ the singular locus of N. Consider the Gauss map $\phi : N - \Sigma \to \Lambda(n)$ defined by $\phi(x) = T_x(N - \Sigma)$, for $x \in N - \Sigma$. In the Lagrange Grassmannian $\Lambda(n)$, we set, for $k = 1, 2, \ldots,$

$$\sigma_k = \{\lambda \in \Lambda(n) \mid \dim \lambda \cap \sqrt{-1}\mathbb{R}^n \geqq k\}.$$

Then σ_{2k-1} defines $m_k \in H^{k(2k-1)}(\Lambda(n); \mathbb{Z})$, the Maslov-Arnol'd-Fuks cohomology class, via the Poincaré duality. Then we set

$$m_k(N) = \delta\phi^* m_k \in H^{k(2k+1)+1}(N, N - \Sigma; \mathbb{Z}).$$

We denote by $\tilde{N}$ the closure of the graph of $\phi : N - \Sigma \to \Lambda(n)$ in $N \times \Lambda(n)$. Denote by $\pi : \tilde{N} \to N$ and $\tilde{\phi} : \tilde{N} \to \Lambda(n)$ the first and second projection respectively. Then $\pi|\tilde{N} - \pi^{-1}(\Sigma) : \tilde{N} - \pi^{-1}(\Sigma) \to N - \Sigma$ is a diffeomorphism. Consider the decomposition

$$H^*(N - \Sigma) \longleftarrow H^*(\tilde{N}) \longleftarrow H^*(\Lambda(n)).$$

Let $M_n(\mathbb{C})$ denote the set of $\mathbb{R}$-linear maps $h : \mathbb{R}^n \to \mathbb{C}^n$, and $I_n \subset M_n(\mathbb{C})$ denote the set of isotropic linear maps. Then I_n is a real algebraic set in $M_n(\mathbb{C})$ of codimension $(1/2)n(n-1)$. Set $\Sigma_k = \{h \in I_n \mid \dim(\ker h) \geqq k\}$.

Define $\Phi : I_n - \Sigma_1 \to \Lambda(n)$ by $\Phi(h) = h(\mathbb{R}^n)$. Denote by $\mathcal{N}(I_n)$ the closure of the graph of Φ in $I_n \times \Lambda(n)$. Then $\mathcal{N}(I_n)$ is non-singular and the second projectin $\tilde{\Phi} : \mathcal{N}(I_n) \to \Lambda(n)$ is a homotopy equivalence. Then we have a further decomposition:

$$H^{*+1}(N, N - \Sigma) \leftarrow H^*(N - \Sigma) \leftarrow H^*(\tilde{N}) \leftarrow H^*(\mathcal{N}(I_n)) \cong H^*(\Lambda(n)).$$

Remark that, for the first projection $\pi : \mathcal{N}(I_n) \to I_n$, $\pi^{-1}(\Sigma_k - \Sigma_{k+1}) \to \Sigma_k - \Sigma_{k+1}$ is a fibration with fiber $F_k \cong \Lambda(k)$, $(k = 0, 1, 2, \ldots)$. In fact,

$$F_k = \{\lambda \in \Lambda(n) \mid \lambda \supset \rho\} \cong \Lambda(\rho^\perp/\rho) \cong \Lambda(k),$$

where $\rho \subset \mathbb{R}^n \subset \mathbb{C}^n$ is an $(n-k)$-dimensional subspace.

Set $m_k^* = [F_{2k-1}] \in H_{k(2k-1)}(\mathcal{N}(I_n))$, then

$$\langle m_k^*, \tilde{\Phi}^* m_k \rangle = \langle \tilde{\Phi}_* m_k^*, m_k \rangle = \pm 1.$$

Consider the parametrized version of the construction above: Let $N \to T^*\mathbb{R}^n$ be an isotropic mapping. Then $Tf : N \to I_n$ is defined by $x \mapsto T_x f : T_x N \to T_x T^*\mathbb{R}^n \cong \mathbb{C}^n$. (Here I_n is regarded as the natural bundle over N.) Tf naturally lifts to $\tilde{Tf} : N_f \to \mathcal{N}(I_n)$, where N_f is the Lagrange-Nash modification of N with respect to f. Then we have (cf. [I3])

PROPOSITION. *If f is generic with at most kernel rank one, then*

(1) Tf is transverse to $\Sigma_1 - \Sigma_2$.

(2) $\tilde{Tf}_* : H_*(N_f) \to H_*(\mathcal{N}(I_n))$ maps $[\pi^{-1}x]$ to m_1^*, where $x \in \Sigma$.

(3) The natural mapping $H_*(N - \Sigma) \to H_*(N_f)$ maps $[\partial D^2]$ to $\pm 2[\pi^{-1}x]$, where D^2 is a small transversal disk with center x to $\Sigma_1 - \Sigma_2$.

REFERENCES

[AM] R. Abraham, J.E. Marsden, "Foundation of Mechanics, 2nd ed.," Benjamin, 1978.

[A1] V.I. Arnol'd, *On a characteristic class enterning in the quantization conditions*, Funkt. Anal. Prilozhen. **1** (1967), 1–14.

[A2] V.I. Arnol'd, *Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, Funkt. Anal. Prilozhen. **6-4** (1972), 3-25.

[A3] V.I. Arnol'd, *Lagrangian manifolds with singularities, asymptotic rays, and the open swallowtail*, Funct. Anal. Appl. **15-4** (1981), 235–246.

[A4] V.I. Arnol'd, *Singularities in variational calculus*, J. Soviet Math. **27-3** (1984), 2679 – 2713.

[A5] V.I. Arnol'd, "Singularities of Caustics and Wave Fronts," Kluwer Academic Publishers, 1990.

[AGV] V.I. Arnol'd, S.M. Gusein-Zade, A.N. Varchenko, "Singularities of Differentiable Maps I," Birkhäuser, 1985.

[Au] M. Audin, *Classes caractéristiques lagrangiennes*, Preprint.

[C] J. Cleave, *The form of the tangent developable at points of zero torsion on space curves*, Math. Proc. Camb. Phil. Soc. **88** (1980), 403–407.

[Da] A.A. Davydov, *The normal form of slow motions of an equation of relaxation type and fibrations of binormal surfaces*, Math. USSR Sbornik **60-1** (1988), 133–141.

[De] Ph. Delanoë, *L'opérateur de Monge-Ampère réel et la géométrie des sous-variétés*, in "in Geometry and Topology of Submanifolds," ed. by J.-M. Morvan and L. Verstraelen, World Scientific, 1989, pp. 49–72.

[Du] J.P. Dufour, *Familles de courbes planes différentiables*, Topology **22-4** (1983), 449 – 474.

[F] D. Fujiwara, *A construction of fundamental solution of Schrödinger's equation on the sphere*, J. Math. Soc. Japan **28** (1976), 483–505.

[Fuk] D.B. Fuks, *Maslov-Arnol'd characteristic classes*, Dokl. Akad. Nauk SSSR **178–2** (1968), 303–306.

[Gir] E. Giroux, *Formes génératrices d'immersions lagrangiennes*, C. R. Acad. Sci. Paris **306** (1988), 761–764.

[Giv1] A.B. Givental', *Lagrangian imbeddings of surfaces and unfolded Whitney umbrella*, Funkt. Anal. Prilozhen **20-3** (1986), 35–41.

[Giv2] A.B. Givental', *Singular Lagrangian varieties and thier Lagrangian mappings*, in "Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. (Contemporary Problems of Mathematics) 33," VITINI, 1988, pp. 55-112.

[Gr] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82–5** (1985), 307–347.

[GS] V. Guillemin, Sh. Sternberg, "Geometric Asymptotics," Math. Surveys 14, Amer. Math. Soc., 1977.

[H] L. Hörmander, *Fourier integral operators I*, Acta. Math. **127** (1971), 79–183.

[I1] G. Ishikawa, *Families of functions dominated by distributions of C-classes of mappings*, Ann. Inst. Fourier **33-2** (1983), 199–217.

[I2] G. Ishikawa, *Parametrization of a singular Lagrangian variety*, Trans. Amer. Math. Soc. **331–2** (1992), 787–798.

[I3] G. Ishikawa, *The local model of an isotropic map-germ arising from one dimensional symplectic reduction*, Math. Proc. Camb. Phil. Soc. **111–1** (1992), 103–112.

[I4] G. Ishikawa, *Maslov class of an isotropic map-germ arising from one dimensional symplectic reduction.* To appear in Advanced Studies in Pure Math.

[I5] G. Ishikawa, *Determinacy of envelope of the osculating hyperplanes to a curve.* To appear in Bull.

London. Math. Soc.

[I6] G. Ishikawa, *Developable of a curve and its determinacy relative to osculation-type.* Preprint, Hokkaido University

[IO] G. Ishikawa, T. Ohmoto, *Local invariants of singular surfaces in an almost complex four-manifold.* To appear in Ann. Global Anal. Geom.

[J] S. Janeczko, *Generating families for images of Lagrangian submanifolds and open swallowtails*, Math. Proc. Camb. Phil. Soc. **100** (1986), 91–107.

[KS] M. Kashiwara, P. Schapira, "Microlocal Study of Sheaves," Astérisque, 128, Soc. Math. de France, 1985.

[K] M.É. Kazaryan, *Singularities of the boundary of fundamental systems, flat points of projective curves, and Schubert cells*, in "Itogi Nauki Tekh., Ser. Sorrem. Probl. Mat. (Comtemporary Problems of Mathematics) 33," VITINI, 1988, pp. 215–232.

[L] J.A. Lees, *Defining Lagrangian immersions by phase functions*, Trans. Amer. Math. Soc. **250** (1979), 213–222.

[Leh1] D. Lehmann, *Classes caractéristiques et J-connexité des espaces de connexions*, Ann. Inst. Fourier **24–3** (1974), 267–306.

[Leh2] D. Lehmann, *Résidus des connexions à singularités et classes caractéristiques*, Ann. Inst. Fourier **31–1** (1981), 83–98.

[Leh3] D. Lehmann, *Résidus des sous-variétés invariantes d'un feuilletage singulier*, Ann. Inst. Fourier **41–1** (1991), 211–258.

[LV] G. Lion, M. Vergne, "The Weil representation, Maslov index and Theta series," Progress in Math. 6, Birkhäuser, 1980.

[M1] B. Malgrange, "Ideals of Differentiable Functions," Oxford Univ. Press, 1966.

[M2] B. Malgrange, *Frobenius avec singularités, 2. Le cas general*, Invent. Math. **39** (1977), 67–89.

[Ma] V.P. Maslov, "Theory of Perturbations and Asymptotic Methods," Izdat. Moskow Gos. Univ., Moskow, 1965.

[Mo1] D. Mond, *On the tangent developable of a space curve*, Math. Proc. Camb. Phil. Soc. **91** (1982), 351–355.

[Mo2] D. Mond, *Singularities of the tangent developable surface of a space curve*, Quart. J. Math. Oxford **40** (1989), 79–91.

[Morim] T. Morimoto, *La géométrie des équations de Monge-Ampère*, C. R. Acad. Sc. Paris **289** (1979), 25–28.

[Morin] B. Morin, *Formes canonique des singularités d'une application différentiable*, C. R. Acad. Sc. Paris **260** (1965), 5662–5665.

[MN] J.M. Morvan, L. Niglio, *Classes caractéristique des couples de sous-fibrés lagrangiens*, Ann. Inst. Fourier **36** (1986), 193–209.

[S1] O.P. Scherbak, *Projectively dual curves and Legendre singularities*, Sel. Math. Sov. **5-4** (1986), 391–421.

[S2] O.P. Shcherbak, *Wavefront and reflection groups*, Russian Math. Surveys **43-3** (1988), 149–194.

[V1] I. Vaisman, "Symplectic Geometry and Secondary Characteristic Classes," Progress in Math. 72, Birkhäuser, 1987.

[V2] I. Vaisman, *Residues of Chern and Maslov classes*, in "Geometry and Topology II," ed. by M. Boyom, J.-M. Morvan, L. Verstraelen, World Scientific, 1990, pp. 370–385.

[Vas] V.A. Vassilyev, "Lagrange and Legendre Characteristic Classes," Gordon and Breach Sci. Publ..

[W] A. Weinstein, "Lectures on symplectic manifolds," Regional Conference Series in Math. 29, Amer. Math. Soc., 1977.

[Z] V.M. Zakalyukin, *Generating ideals of Lagrangian varieties*, in "Theory of singularities and its applications," ed. by V.I. Arnol'd, Adv. in Soviet Math, vol.1, Amer. Math. Soc., 1990, pp. 201–210.