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REGULARITY THEOREMS FOR HOLONOMIC MODULES

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0. INTRODUCTION

The regularity theorem for the ordinary differential equations has a long history. Malgrange [17] has shown the regular singularity of the system is equivalent to the convergence of its formal power series solutions. Ramis [18] extend the results to the irregular singular case, that is, the irregularity is characterized by the Gevrey growth order of its formal power series solutions. In the real domain, Komatsu [15] also has a similar result comparing ultra-distribution and hyperfunction solutions.

One of the important problems is to extend these results to the higher dimension. The deep study of holonomic systems due to Kashiwara-Kawai [10] and Kashiwara [9] established the regularity theorems for holonomic modules in the regular singular case. The purpose of this paper is to give several regularity theorem for the irregular singular holonomic modules.

1. PRELIMINARY

Let X be a complex manifold of dimension n and $\pi : T^*X \to X$ its cotangent bundle. Set $\overset{\circ}{T}{}^*X = T^*X \setminus T^*_XX$ and denote by $\overset{\circ}{\pi}$ the restriction of π to $\overset{\circ}{T}{}^*X$. We choose a local coordinate system of X as $(x_1, \cdots, x_n)$ and that of T^*X as $(x_1, \cdots, x_n; \xi_1, \cdots, \xi_n)$. T^*X is endowed with the sheaf $\mathcal{E}_X^\infty$ of micro-differential operators of infinite order constructed by Sato-Kashiwara-Kawai [19].

We denote by $\mathcal{E}_X$ (resp. $\mathcal{E}_X(m)$) the subsheaf of $\mathcal{E}_X^\infty$ consisting of micro-differential operators of finite order (resp. micro-differential operators of order at most m). For the theory of $\mathcal{E}_X$, refer to [19] and Schapira [20]. Now we define the subsheaf $\mathcal{E}_X^{(s)}$ of micro-differential operators of Gevrey growth order (s) for any $s \in (1, \infty)$.

Definition 1.1. *For an open subset U of T^*X, a sum $\sum_{i\in\mathbb{Z}} P_i(x,\xi) \in \mathcal{E}_X^\infty(U)$ belongs to $\mathcal{E}_X^{(s)}(U)$ if and only if $\{P_i\}_{i\in\mathrm{N}}$ satisfies the following estimate (1.1); for any compact set K of U, there exists a positive constant C_K such that*

$$\sup_K \mid P_i(x,\xi) \mid \leq \frac{C_K^i}{i!^s} \quad (i \geq 0). \tag{1.1}$$

For convenience, we set $\mathcal{E}_X^{(1)} := \mathcal{E}_X^\infty$ and $\mathcal{E}_X^{(\infty)} := \mathcal{E}_X$.

Next we review briefly the definition of the sheaf of holomorphic microfunctions in Gevrey class. Let Y be a complex submanifold of X and T^*_YX its conormal bundle. Then we define the subsheaf $\mathcal{C}_{Y|X}^{\mathbb{R},(s)}$ of the holomorphic microfunctions $\mathcal{C}_{Y|X}^{\mathbb{R}}$ as

$$\mathcal{C}_{Y|X}^{\mathbb{R},(s)} := \mathcal{E}_X^{(s)} \mathcal{C}_{Y|X}^{\mathbb{R},f}$$

where $\mathcal{C}_{Y|X}^{\mathbb{R},f}$ is the sheaf of tempered holomorphic microfunctions (for the definition, refer to Andronikof [1,2]). Remark that these sheaves are also defined by the functor $T-\mu^{(s)}_{\cdot}(\mathcal{O})$, which is a natural extension of tempered microlocalization functor $T-\mu_{\cdot}(\mathcal{O})$ constructed by Andronikof [1,2],

$$\mathcal{E}_X^{(s)} := \tau^{-1}\tau_* T - \mu_X^{(s)}(\mathcal{O}_{X\times X}) \otimes \Omega_X[\dim X]$$

where $\tau : \overset{\circ}{T}{}^*X \to P^*X$ is a canonical projection, and

$$\mathcal{C}_{Y|X}^{\mathbb{R},(s)} := T - \mu_Y^{(s)}(\mathcal{O}_X)[\operatorname{codim} Y].$$

For the definition and properties of the Gevrey microlocalization functor, refer to Honda [5].

Let V be a regular or maximally degenerate involutive submanifold of codimension $d \geq 1$ in $\overset{\circ}{T}{}^*X := T^*X \backslash T_X^*X$. We define the subsheaf $I_V \subset \mathcal{E}_X(1)$ by

$$I_V := \{P \in \mathcal{E}_X(1); \delta_1(P)\,|_V \equiv 0\}.$$

Here we denote the symbol map of degree 1 by $\delta_1(\cdot)$. Now we define the sheaf of rings $\mathcal{E}_V^{(\sigma)}$ in $\overset{\circ}{T}{}^*X$ for a rational number $\sigma \in [1,\infty)$.

$$\mathcal{E}_V^{(\sigma)} := \sum_{n\geq 0} \mathcal{E}_X\left(\frac{(1-\sigma)n}{\sigma}\right) I_V^n.$$

In case $\sigma = 1$, this sheaf coincides with the sheaf $\mathcal{E}_V$ defined in Kashiwara-Oshima [12] and [10].

We list up the main properties of the sheaf $\mathcal{E}_V^{(\sigma)}$.

(1) $\mathcal{E}_V^{(\sigma)}$ is a subring of $\mathcal{E}_X$.
(2) $\mathcal{E}_X(0) \subset \mathcal{E}_V^{(\sigma)}$, and $\mathcal{E}_V^{(\sigma)}$ is a left and right $\mathcal{E}_X(0)$ module.
(3) $\mathcal{E}_V^{(\sigma)}$ is a sheaf of Noetherian ring, and any coherent $\mathcal{E}_X$ module is pseudo-coherent over $\mathcal{E}_V^{(\sigma)}$.
(4) If $P \in \mathcal{E}_V^{(\sigma)}$, then its formal ajoint operator P^* belongs to $\mathcal{E}_V^{(\sigma)}$.

Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$ modules in a neighborhood of $p \in \overset{\circ}{T}{}^*X$. We first define the weak irregularity of $\mathcal{M}$ at a smooth point of its support. Given $p \notin \operatorname{supp}(\mathcal{M})_{sing} \cup T_X^*X$.

Definition 1.2. *$\mathcal{M}$ has weak irregularity at most σ at p if and only if $\mathcal{M}$ satisfies the following conditions.*

There exist an open neighborhood U of p, maximally degenerate involutive submanifold V with its singular locus $\operatorname{supp}(\mathcal{M})$, *and an $\mathcal{E}_V^{(\sigma)}$ modules $\mathcal{M}_0$ in U which generates $\mathcal{M}$ over $\mathcal{E}_X$ and is finitely generated over $\mathcal{E}_X(0)$ at any point of a dense subset in* $\operatorname{supp}(\mathcal{M}) \cap U$.

Next we define weak irregularity in the general case.

Definition 1.3.

(1) A holonomic $\mathcal{E}_X$ module $\mathcal{M}$ has weak irregularity at most σ at p if and only if there exist an open neighborhood U of p and a closed analytic subset $Z \supset \mathrm{supp}(\mathcal{M})_{sing}$ with $\mathrm{codim}\, Z \geq \dim X + 1$, $\mathcal{M}$ has weak irregularity at most σ at any point in $U \backslash Z \cap \overset{\circ}{T^*}X$.

(2) A holonomic $\mathcal{D}_X$ module $\mathcal{N}$ has weak irregularity at most σ if and only if $\mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{N}$ has irregularity at most σ at any point in $\overset{\circ}{T^*}X$.

2. Statement of main theorem

Main Theorem. *Let $U \subset T^*X$ be a $\mathbb{C}^\times$ conic open set, $\mathcal{M}$ a holonomic $\mathcal{E}_X$ modules in U and $\sigma \geq 1$ a rational number. The following conditions (1),(2) and (3) are equivalent.*

(1) There exists a holonomic $\mathcal{E}_X$ module $\mathcal{M}_{reg}$ with regular singularities satisfying

$$\mathcal{E}_X^{(s)} \otimes_{\mathcal{E}_X} \mathcal{M} \simeq \mathcal{E}_X^{(s)} \otimes_{\mathcal{E}_X} \mathcal{M}_{reg}$$

in U for all $s \in [1, \frac{\sigma}{\sigma-1}]$.

(2) For any submanifold $Y \subset X$ and any $s \in [1, \frac{\sigma}{\sigma-1}]$, we have

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R},(s)}_{Y|X})|_U \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\mathbb{R}}_{Y|X})|_U.$$

(3) $\mathcal{M}$ has weak irregularity at most σ in U.

Using the same technique of the proof, we deduce the following corollary. Let M be a real analytic manifold with its complexification X.

Corollary 1. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$ modules at $p \in T^*X$ with weak irregularity at most σ. Then we have the isomorphisms for all $s \in [1, \frac{\sigma}{\sigma-1}]$,*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M^{(s)}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$$

where $\mathcal{C}_M$ and (resp. $\mathcal{C}_M^{(s)}$) is the sheaf of microfunctions (resp. microfunctions of Gevrey class (s)).

In the case that $\mathcal{M}$ is regular singular and the solution sheaf is tempered microfunctions, this result is already obtained by Andronikof [3]. Finally we remark that applying the functor τ_* to the result (2) of the main theorem, we can recover the results of Laurent [16].

Corollary 2 [16]. *Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$ modules at $p \in T^*X$ with weak irregularity at most σ. Then we have the isomorphisms for all $s \in [1, \frac{\sigma}{\sigma-1}]$ and for any submanifold $Y \subset X$,*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{(s)}_{Y|X}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^{\infty}_{Y|X}).$$

References

[1] E. Andronikof, *Microlocalisation tempérée des distributions et des fonctions holomorphes I*, C.R.Acad.Sci 303, 347-350 (1986).

[2] E. Andronikof, *Microlocalisation tempérée des distributions et des fonctions holomorphes II*, C.R.Acad.Sci 304, 511-514 (1987).

[3] E. Andronikof, *On the C^∞-singularities of regular holonomic distributions*, Ann. Inst. Fourier, 42, 695-704 (1992).

[4] N.Honda, *On the reconstruction theorem of holonomic modules in Gevrey classes*, Publ. RIMS, 27, Kyoto Univ., 923-943 (1991).

[5] N.Honda, *Microlocalization in Gevrey classes*, in preparation.

[6] N.Honda, *Regularity theorems for holonomic modules*, in preparation.

[7] M.Kashiwara, *On the maximally overdetermined systems of linear differential equations, I*, Publ. RIMS, 10, Kyoto Univ., 563-579 (1975).

[8] M.Kashiwara, *On the holonomic systems of linear differential equations II*, Inventiones Math., 49, 121-135 (1978).

[9] M.Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. RIMS, 20, Kyoto Univ., 319-365 (1984).

[10] M.Kashiwara, T.Kawai, *On the holonomic systems of microdifferential equations III*, Publ. RIMS, 17, Kyoto Univ., 813-979 (1981).

[11] M.Kashiwara, T.Kawai, *Second microlocalization and asymptotic expansions*, Lect. Notes Phys. 126, 21-76 (1980).

[12] M.Kashiwara, T.Oshima, *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math., 106, 145-200 (1977).

[13] M.Kashiwara, P.Schapira, *Microlocal study of sheaves*, Astérisque, 128 (1985).

[14] M.Kashiwara, P.Schapira, *Sheaves on manifolds*, Grundlehren der Math., 292, Springer-Verlag (1990).

[15] H.Komatsu, *On the regularity of hyperfunction solutions of linear ordinary differential equations with real analytic coefficients*, J. Fac. Sci. Univ. Tokyo, Sec. IA, 20, 107-119 (1973).

[16] Y. Laurent, *Théorie de la deuxième microlocalisation dans le domine complexe*, Progress in Mathematics, 53, Birkhäuser (1985).

[17] B.Malgrange, *Sur les points singuliers des équations différentielles*, Enseignement Math., 20, 147-176 (1974).

[18] J.-P. Ramis, *Devissage Gevrey*, Asterisque, 173-204 (1978).

[19] M.Sato, T.Kawai, M.Kashiwara, *Hyperfunctions and pseudodifferential equations*, Lecture Notes in Math., 287, Springer-Verlag, 265-529 (1973).

[20] P.Schapira, *Microdifferential systems in the complex domain*, Grundlehren der Math., 269, Springer-verlag (1985).