数理解析研究所講究録
第 845 巻 1993 年 19-25

INDEX THEOREM FOR ELLIPTIC PAIRS

PIERRE SCHAPIRA AND JEAN-PIERRE SCHNEIDERS
(Univ. Paris 6) (Univ. Paris XIII)

I. Elliptic pairs

In this paper we follow the notations of [K-S].

We define here an elliptic pair on a complex manifold X as the data of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F (better: objects of derived categories) satisfying:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X. \tag{1.1}$$

Theorem 1.1. *(Regularity) The natural morphism:*

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes D'F \longrightarrow \mathrm{R}\mathcal{H}om(\mathcal{M} \otimes F, \mathcal{O}_X)$$

is an isomorphism.

Recall that $D'F = \mathrm{R}\mathcal{H}om(F, \mathbb{C}_X)$. The proof follows immediately from the next results of [K-S]:

(i) setting $G = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, then

$$SS(G) = \mathrm{char}(\mathcal{M})$$

(ii) $D'F \otimes G \to \mathrm{R}\mathcal{H}om(F, G)$ is an isomorphism as soon as $SS(F) \cap SS(G) \subset T^*_X X$, F being constructible (This holds on a real manifold X). □

Theorem 1.2. *(Finiteness) Assume $(\mathcal{M}, F)$ is an elliptic pair with compact support. Then for all j, the vector spaces*

$$H^j(R\Gamma(X; \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)))$$

are finite dimensional.

Sketch of proof. Set

$$F_0 = R\Gamma(X; \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes D'F),$$
$$F_1 = R\Gamma(X; \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)).$$

We shall represent F_0 (resp. F_1) by a bounded complex of topological vector spaces of type (DFS) (resp. (FS)), the quasi-isomorphism $F_0 \xrightarrow{\sim} F_1$ (cf. Theorem 1.1) being continuous. The finiteness follows. Using cohomological descent, one reduces to the case where $\mathcal{M}$ is a bounded complex of free $\mathcal{D}_X$ modules

$$\mathcal{M} \simeq \cdots \to \mathcal{D}_X^{N_i} \underset{P_i}{\to} \cdots .$$

Moreover one represents F by a bounded complex

$$F \simeq \cdots \to \bigoplus_{\alpha_j} \mathbb{C}_{U_{\alpha_j, j}} \underset{\rho_j}{\to} \cdots$$

where the U's (appearing in $\mathbb{C}_{U_{\alpha_j, j}}$) are open subanalytic relatively compact subsets of X and satisfy

$$D'\mathbb{C}_U = \mathbb{C}_{\overline{U}}.$$

Finally one represents $\mathcal{O}_X$ by its Dolbeault resolution with coefficients in real analytic or C^∞-functions

$$\mathcal{O}_X \simeq \cdots \to \mathcal{A}_X^{(k)} \underset{\overline{\partial}}{\to} \cdots$$

or

$$\mathcal{O}_X \simeq \cdots \to C_X^{\infty,(k)} \underset{\overline{\partial}}{\to} \cdots .$$

Then F_0 (resp. F_1) is the simple complex associated to the triple complex with components isomorphic to finite sums of $\mathcal{A}(\overline{U})$ (resp. $C_X^\infty(U)$). □

Example 1.3. Assume X is the complexification of a real analytic manifold M. Then $(\mathcal{M}, D'\mathbb{C}_M)$ is elliptic iff $\mathcal{M}$ is elliptic on M and the regularity theorem recovers the well known isomorphism:

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$$

where $\mathcal{A}_M = \mathcal{O}_X \otimes \mathbb{C}_M$ is the sheaf of real analytic functions and $\mathcal{B}_M = \mathrm{R}\mathcal{H}om(D'\mathbb{C}_M, \mathcal{O}_X)$ is the sheaf of Sato's hyperfunctions. If M is compact, Theorem 1.2 recovers classical finiteness results.

II. Microlocal Euler class

We shall adapt Kashiwara's construction of the characteristic cycle of $\mathbb{R}$-constructible sheaves (see [K-S, Chapter IX]) to the case of elliptic pairs. Set

$$
\begin{aligned}
& n = \dim_{\mathbb{C}} X \\
& \Omega_X = \mathcal{O}_X^{(n)} \quad \text{(holomorphic } n\text{-forms)} \\
& \omega_X = or_X[2n] \simeq \mathbb{C}_X[2n] \quad (X \text{ is oriented}) \\
& \delta : X \cong \Delta \hookrightarrow X \times X, \quad \text{diagonal embedding} \\
& q_1, q_2 : \text{ projections } X \times X \to X.
\end{aligned}
$$

Let $(\mathcal{M}, F)$ be an elliptic pair with $\mathcal{M}$ a right $\mathcal{D}_X$-module. Set for short:

$$
\begin{aligned}
& \mathcal{P} = F \otimes \mathcal{M} \\
& \underline{D}\mathcal{M} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X[n]).
\end{aligned}
$$

Starting with Sato's isomorphism:

$$
\mathcal{D}_X^{\infty} \simeq \delta^! \mathcal{O}_{X\times X}^{(0,n)}[n] \tag{2.1}
$$

we get the morphism

$$
\delta_! \mathcal{D}_X \to \mathcal{O}_{X\times X}^{(0,n)}[n]
$$

hence

$$
\begin{aligned}
& \delta_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \\
& \quad \to \mathrm{R}\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes^{\mathbb{L}}_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}^{(0,n)}[n]) \\
& \quad \xleftarrow{\sim} D'F \otimes \underline{D}\mathcal{M} \boxtimes F \otimes \mathcal{M} \otimes^{\mathbb{L}}_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \\
& \quad := H.
\end{aligned}
$$

(Note that one uses the ellipticity hypothesis in order to construct the last isomorphism, as in Theorem 1.1).

Hence, setting

$$\Lambda = \mathrm{char}(\mathcal{M}) + SS(F)$$

we get

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{P}) &\to \delta^! H \\ &\xrightarrow{\sim} \mathbb{R}\pi_* R\Gamma_\Lambda \mu_\Delta(H) \\ &\to \mathbb{R}\pi_* R\Gamma_\Lambda \mu_\Delta(\delta_*\delta^{-1}H). \end{aligned}$$

On the other hand one constructs the trace morphism

$$\delta^{-1}H \to \omega_X,$$

and we finally obtain

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{P}) \to \mathbb{R}\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X.$$

Applying $H^0 R\Gamma(X;\cdot)$ we get

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{P},\mathcal{P}) \to H^{2n}_\Lambda(T^*X;\mathbb{C}_{T^*X}). \tag{2.2}$$

Definition 2.1. The microlocal Euler class $\mu eu(\mathcal{M},F)$ of the elliptic pair $(\mathcal{M},F)$ is the image of id by the morphism (2.2).

$$\mu eu(\mathcal{M},F) \in H^{2n}_{\mathrm{char}(\mathcal{M})+SS(F)}(T^*X;\mathbb{C}_{T^*X}).$$

We also set

$$\begin{aligned} eu(\mathcal{M},F) &= \mu eu(\mathcal{M},F)|_{T^*_XX} \\ \mu eu(\mathcal{M}) &= \mu eu(\mathcal{M},\mathbb{C}_X) \\ \mu eu(F) &= \mu eu(\Omega_X,F). \end{aligned}$$

We checks easily that $\mu eu(F)$ coincides with the Kashiwara's characteristic cycle of F, (up to sign). Let Y be another complex manifold, $(\mathcal{N}, G)$ an elliptic pair on Y. Set

$$\begin{aligned} n_X &= \dim_{\mathbb{C}} X, \ n_Y = \dim_{\mathbb{C}} Y, \\ \Lambda_X &= \mathrm{char}(\mathcal{M}) + SS(F), \\ \Lambda_Y &= \mathrm{char}(\mathcal{N}) + SS(G). \end{aligned}$$

We checks easily that

$$\mu eu(\mathcal{M} \boxtimes \mathcal{N}, F \boxtimes G) = \mu eu(\mathcal{M}, F) \boxtimes \mu eu(\mathcal{N}, G).$$

Now let $f : Y \to X$ be a morphism of complex manifolds, and consider the associated maps

(2.3) $$T^*Y \underset{{}^t f'}{\leftarrow} Y \times_X T^*X \underset{f\pi}{\rightarrow} T^*X$$

Assume f is proper on $\Lambda_Y \cap T^*_Y Y$ (i.e. f_π is proper on ${}^t f'^{-1}(\Lambda_Y)$) or ${}^t f'$ is proper on $f_\pi^{-1} \Lambda_X$ (i.e. f is noncharacteristic for Λ_X). Then we have natural maps (see [K-S, Chapter IX, §3])

$$\begin{aligned} f_\mu &: H^{2n_Y}_{\Lambda_Y}(T^*Y; \mathbb{C}_{T^*Y}) \to H^{2n_X}_{f_\pi^t f'^{-1}\Lambda_Y}(T^*X; \mathbb{C}_{T^*X}) \\ f^\mu &: H^{2n_X}_{\Lambda_X}(T^*X; \mathbb{C}_{T^*X}) \to H^{2n_Y}_{{}^t f' f_\pi^{-1}\Lambda_X}(T^*Y; \mathbb{C}_{T^*Y}). \end{aligned}$$

In particular, if Λ_1 and Λ_2 are closed and conic in T^*X and satisfy $\Lambda_1 \cap \Lambda_2^a \subset T^*_X X$, we define the convolution

$$* : H^{2n}_{\Lambda_0}(T^*X; \mathbb{C}_{T^*X}) \times H^{2n}_{\Lambda_1}(T^*X; \mathbb{C}_{T^*X}) \to H^{2n}_{\Lambda_0 + \Lambda_1}(T^*X; \mathbb{C}_{T^*X})$$

as $* = \delta^\mu \circ \boxtimes$.

Theorem 2.2. *Assume f is proper on* $\mathrm{supp}(\mathcal{N}, G)$, $\underline{f}_*(G \otimes \mathcal{N}) = F \otimes \mathcal{M}$, $\underline{f}_*(\mathcal{D}' G \otimes \underline{\mathcal{D}}' \mathcal{N}) = D' F \otimes \underline{D} \mathcal{M}$ *(+ some extra condition that we do not discuss). Then*

$$\mu eu(\mathcal{M}, F) = f_\mu(\mu eu(\mathcal{N}, G)).$$

Theorem 2.3. *Assume f is non characteristic for $\mathcal{M}$ and $\delta \circ f$ is non characteristic for* $\mathrm{char}(\mathcal{M}) \times SS(F)$. *Then $(\underline{f}^{-1}\mathcal{M}, f^{-1}F)$ is elliptic on Y and*

$$\mu eu(\underline{f}^{-1}\mathcal{M}, f^{-1}F) = f^{\mu}(\mu eu(\mathcal{M}, F)).$$

From Theorems 2.2 and 2.3 one deduces

Corollary 2.4. Let $(\mathcal{M}, F)$ be an elliptic pair with compact support. Then

$$\begin{aligned}\chi(R\Gamma(X; F \otimes \mathcal{M} \otimes^{L}_{\mathcal{D}_X} \mathcal{O}_X)) &= \int_X eu(\mathcal{M}, F) \\ &= \int_{T^*X} \mu eu(\mathcal{M}) \cup \mu eu(F)\end{aligned}$$

where $\chi(\cdot)$ denotes the Euler-Poincaré index and $\cup$ the cup-product.

III Microlocal Chern class.

If Z is a closed complex analytic subset of a complex manifold X, denote by $K_Z^{an}(X)$ the Grothendieck group of the derived category of coherent $\mathcal{O}_X$-modules on X with cohomology supported by Z. Denote by $ch(\cdot)$ and $eu(\cdot)$ the Chern character and the total Euler class

$$K_Z^{an}(X) \to H_Z^{ev}(X; \mathbb{C}_X)$$

related by the formula

$$eu(F) = ch(F) \cup td_X(TX) \tag{3.1}$$

where $td_X(\cdot)$ is the Todd class. It is well-known that $ch(\cdot)$ is compatible to external product and inverse image and $eu(\cdot)$ to external product and proper direct image; this is the Riemann-Roch-Grothendieck theorem). Now let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module endowed with a good filtration and let $\Lambda \subset T^*X$ and assume that

$$\mathrm{char}(\mathcal{M}) \subset \Lambda.$$

We denote by $\sigma_\Lambda(\mathcal{M})$ the element of $K_\Lambda^{an}(T^*X)$ associated with $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{gr}\,\mathcal{D}_X} \pi^{-1}\,\mathrm{gr}\,\mathcal{M}$.

Definition 3.1. We set for a left $\mathcal{D}_X$-module $\mathcal{M}$

$$\mu ch_\Lambda(\mathcal{M}) = ch(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(T^*X)$$

and for a right $\mathcal{D}_X$-module $\mathcal{N}$:

$$\mu ch_\Lambda(\mathcal{N}) = ch(\sigma_\Lambda(\mathcal{N})) \cup \pi^* td_X(TX).$$

One denotes by $\mu ch^j_\Lambda(\mathcal{M}) \in H^j_\Lambda(T^*X; \mathbb{C}_{T^*X})$ the j-th component. Note that for a left $\mathcal{D}_X$-module $\mathcal{M}$, $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a right $\mathcal{D}_X$-module and

$$\mu ch_\Lambda(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \mu ch_\Lambda(\mathcal{M}). \tag{3.2}$$

Using the Riemann-Roch-Grothendieck theorem one checks easily that $\mu ch(\cdot)$ is compatible to

· external product

· non characteristic inverse image

· proper direct image

(this last point is due to Laumon [La]). Since $\mu eu(\mathcal{M})$ has similar properties, it is natural to state:

Conjecture 3.2. (i) $\mu ch^j_\Lambda(\mathcal{M}) = 0$ for $j > 2\dim_{\mathbb{C}} X$,
(ii) $\mu ch^{2n}_\Lambda(\mathcal{M}) = \mu eu(\mathcal{M})$.

Note that $\mu ch^j_\Lambda(\mathcal{M}) = 0$ for $j < 2d$, d being the codimension of Λ and $\mu ch^{2d}_\Lambda(\mathcal{M})$ is the d-cycle of $\mathcal{M}$ on Λ. In particular if $\mathcal{M}$ is holonomic, Conjecture 3.2, (ii) is true. Conjecture 3.2 would give a new proof and a generalization of the Atiyah-Singer theorem, as well as its relative version of [B-M].

Bibliography

[B-M] Boutet de Monvel, L. and Malgrange, B., *Le théorème de l'indice relatif*, Ann. Sci. Ec. Norm. Sup. **23**, 151-192.

[K-S] Kashiwara, M. and Schapira, P., *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer-Verlag (1990).

[La] Laumon G., *Sur la catégorie dérivée des $\mathcal{D}$-modules filtrés*, LNM **1016**, 151-237, Springer-Verlag (1983).

[S-Sc] Schapira P. and Schneiders J.-P., *Paires elliptiques I, II*, C.R.Acad. Sci. **311** (1990), 83-86,; **312** (1991), 81-84, and articles in preparation.