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<td>ANDRONIKOF, E.; TOSE, N.</td>
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Kyoto University
Introduction

Elliptic boundary value problems have their own long history. For the general system they were, however, first clearly formulated microlocally by M. Kashiwara and T. Kawai [K-K]. Their theorem has enjoyed many applications, for example, to solvability of operators of simple characteristics, hypoelliptic operators, and tangential Cauchy-Riemann systems. The theorem does not give, however, much information if we restrict ourselves in the space of distributions. This note aims at giving an analogous theorem of Kashiwara-Kawai type in case function spaces are tempered. See Theorem 3 in Section 1 for the main theorem. By this theorem, we can obtain many application to distribution boundary values of holomorphic functions (e.g. M. Uchida[U]). The result of this note was obtained while the second author was staying in Univ. de Paris VI and Univ. Paris XIII.
1. Main theorem

Let $M$ be a real analytic manifold of dimension $n$ with a complex neighborhood $X$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module on $X$ and assume that $\mathcal{M}$ is elliptic on $M$, i.e.

\[(1) \quad \text{char}(\mathcal{M}) \cap T^*_M X \subset T^*_X X.\]

Let $N$ be a real analytic submanifold of $M$ of codimension $d \geq 1$ in $M$, and $Y$ be a complexification of $N$ in $X$. We assume that $Y$ is non-characteristic for $\mathcal{M}$, i.e.

\[(2) \quad \text{char}(\mathcal{M}) \cap T^*_Y X \subset T^*_X X.\]

In this situation, we have the canonical morphisms

$$T^*_N M \xrightarrow{\rho} T^*_N X \xrightarrow{\omega} T^*_N X.$$ 

Under the above notation we have

**THEOREM 1.** The natural morphism

$$\mathbf{R}_\rho_* \mathbf{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, C^f_{N|X}) \xleftarrow{\sim} \mathbf{R} \text{Hom}_{\mathcal{D}_X}(\mathcal{M}, T-\mu_N(Db_M)) \otimes o_{N/M}$$

is an isomorphism.

In the above theorem $o_{N/M}$ denotes the relativs orientation sheaf of $N$ in $M$. The sheaf $C^f_{N|X}$ on $T^*_N X$ is the tempered version of $C_{N|X}$ and is given, with the tempered microlocalization due to E. Andronikof[An], by

$$C^f_{N|X} := T-\mu_N(\mathcal{O}_X) \otimes o_{M[n]}.$$ 

We remark that the above object in the derived category is concentrated in degree 0. For a point $\hat{z} \in T^*_N X$, the stalk of $C^f_{N|X}$ at $\hat{z}$ is given, with the aid of local cohomology with bounds, by

$$C^f_{N|X, \hat{z}} \simeq \lim_{\longrightarrow} H^n_{|Z|}(\mathcal{O}_X)_{\pi_X(\hat{z})}.$$ 

Here $\pi_X$ denotes the projection $\pi_X : T^*X \rightarrow X$ and the inductive limit is taken for all closed subanalytic sets $Z$ in $X$ satisfying the property

$$C_N(Z)_{\pi_X(\hat{z})} \subset \{ v \in T_N X; <\hat{z}, v > < 0 \} \cup \{0\}.$$
ELLIPTIC BOUNDARY VALUE PROBLEMS IN THE SPACE OF DISTRIBUTIONS

Refer here to Kashiwara-Schapira[K-S2] for the notion of normal cones $C_N(\cdot)$. The sheaf $T-\mu_N(Db_M)$ on $T^*_N M$ is also constructed by E. Andronikof[An]. We just explain that its stalk at $\overset{\circ}{x} \in T^*_N M$ is given by the isomorphism

$$T-\mu_N(Db_M)_{\overset{\circ}{x}} \simeq \lim_{\to Z} \Gamma_Z(Db_M)_{\pi_M(\overset{\circ}{x})}.$$  

Here the inductive limit is taken for any closed subanalytic set $Z$ in $M$ with the property

$$C_N(Z)_{\pi_M(\overset{\circ}{x})} \subset \{ v \in T_N M; <\overset{\circ}{x}, v > < 0 \} \cup \{0\}$$  

$$(\pi_M : T^* M \to M).$$

Next we give another theorem, which is analogous to Theorem 6.3.1 of Kashiwara-Shapira [K-S1] (refer also to Kashiwara-Kawai[K-K] where we find the theorem of [K-S1] in its original form).

THEOREM 2. Let $\tilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi^{-1}_X D_X} \pi^{-1}_X \mathcal{M}$. Then the natural morphism

$$\mathbf{RHom}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, C^f_{N|X}) \leftarrow \mathbf{RHom}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{E}_{X-Y}) \bigotimes^L_{\text{End}(\mathcal{E}_{X-Y})} \mathbf{RHom}_{\mathcal{E}_X}(\mathcal{E}_{X-Y}, C^f_{N|X})$$

is an isomorphism outside of $T^*_N X \cap T^*_Y X$. This entails an isomorphism

$$\mathbf{RHom}_{\mathcal{E}_X}(\mathcal{M}, C^f_{N|X}) \simeq \mathbf{RHom}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X-Y}) \bigotimes^L_{p^{-1}\mathcal{E}_Y} p^{-1} C^f_N$$

on $T^*_N X \setminus T^*_Y X$ where $p$ is the canonical morphism

$$p : T^*_N X \setminus T^*_Y X \to T^*_N Y.$$  

In the above theorem, the object $C^f_N$ on $T^*_N Y$ is the sheaf of temperate microfunctions. This is a subsheaf of $C_N$ and describes microlocal analytic singularities of distributions on $N$. By the notation of E. Andronikof[An], this sheaf is defined as

$$C^f_N := T-\mu_N(O_Y)[n-d] \otimes \sigma_{N/Y}.$$  

The proof of this theorem is essentially the same as in Theorem 6.3.1 of [K-S1] and relies on the division theorem of temperate microfunctions with holomorphic parameters with respect to microdifferential operators. We also remark that only the non-characterlicity of $Y$ is utilized in its proof.

By combining the above theorems into one, we get the main theorem of this note. Let $q$ denote the restriction of $\rho$ to $\overset{\circ}{T}^*_N X \setminus T^*_M X$; $q : \overset{\circ}{T}^*_N X \setminus T^*_M X \to T^*_N M$ and $p$ the projection $\overset{\circ}{T}^*_N X \setminus T^*_Y X \to \overset{\circ}{T}^*_N Y$. Then we have
THEOREM 3. We have a canonical isomorphism on $T_N^* Y$

$$Rq_* \left( R\text{Hom}_{\mathcal{E}_X} (\tilde{\mathcal{M}}, \mathcal{E}_{X-Y})_{\tilde{T}_N X} \otimes_{p^{-1} \mathbb{C}_N^f} \right) \simeq R\text{Hom}_{\mathcal{D}_X} (\mathcal{M}, T-\mu_N(Db_M)) \otimes or_{N/M}. $$

2. Idea of Proof

What is left to us is now to construct the morphism in Theorem 1 and to show it an isomorphism.

First we construct a commutative diagram

$$\begin{array}{ccc}
R\rho! C_{N|X}^f \otimes or_{N/X} & \longrightarrow & T-\mu_N(A_M) \\
\downarrow & & \downarrow \\
R\rho_* C_{N|X}^f \otimes or_{N/X} & \longrightarrow & T-\mu_N(Db_M)
\end{array}$$

(A)

where $T-\mu_N(A_M)$ is the tempered microlocalization of the sheaf $A_M$ along $N$ and is constructed by E. Andronikof[A]. This object is the Fourier transform of the tempered specialization $T-\nu_N(A_M)$ whose stalk at $\overset{\circ}{v} \in T_N M$ is given by

$$T-\nu_N(A_M)_{\overset{\circ}{v}} \simeq \lim_{\longrightarrow U} \{ u \in \mathcal{A}(U); \text{ u is tempered on M as a distribution} \}.$$ 

Here $U$ in the inductive limit ranges through any open subanalytic set in $M$ with the property

$$\overset{\circ}{v} \notin C_N(M \setminus U).$$

To construct (A), it is sufficient to construct its image by the inverse Fourier transformation

$$\begin{array}{ccc}
\iota^{-1} T-\nu_N(\mathcal{O}_X) \otimes or_{N/X} & \longrightarrow & T-\nu_N(A_M) \\
\downarrow & & \downarrow \\
\iota^! T-\nu_N(\mathcal{O}_X) \otimes or_{N/X} & \longrightarrow & T-\nu_N(Db_M)
\end{array}$$

(A')

Here $\iota$ is the canonical embedding

$$\iota : T_N M \longrightarrow T_N X,$$

and $T-\nu_N(\mathcal{O}_X)$ is the tempered specialization of the sheaf $\mathcal{O}_X$ along $N$, which is concentrated in degree 0. The stalk of $T-\nu_N(\mathcal{O}_X)$ at $\overset{\circ}{v} \in T_N X$ is given by

$$T-\nu_N(\mathcal{O}_X)_{\overset{\circ}{v}} \simeq \lim_{\longrightarrow U} \{ u \in \mathcal{O}(U); \text{ u can be extended to X as a distribution} \}.$$
where $U$ runs through all open subanalytic sets in $X$ with $\bar{v} \not\in C_N(M \setminus U)$. The diagram (A') can be constructed easily if we scrutinize the construction by E. Andronikof[An].

Next we apply $R\underline{Hom}_{D_X}(\mathcal{M}, \cdot)$ to the diagram (A') and obtain the commutative diagram

$$
\begin{array}{c}
R\underline{Hom}_{D_X}(\mathcal{M}, \iota^{-1}T-v_N(\mathcal{O}_X)) \otimes or_{N/X} \xrightarrow{\Phi_1} R\underline{Hom}_{D_X}(\mathcal{M}, T-v_N(\mathcal{A}_M)) \\
\Phi_4 \\
\Phi_3 \\
R\underline{Hom}_{D_X}(\mathcal{M}, \iota^!T-v_N(\mathcal{O}_X)) \otimes or_{N/X} \xleftarrow{\Phi_3} R\underline{Hom}_{D_X}(\mathcal{M}, T-v_N(\mathcal{D}b_M)).
\end{array}
$$

It is easy to see from the ellipticity of $\mathcal{M}$ that $\Phi_4$ and $\Phi_2$ are isomorphisms. (To show $\Phi_4$ is an isomorphism, it is easier to consider its image by Fourier transformation). Thus to prove that $\Phi_3$ and thus its image by Fourier transformation are isomorphisms, it suffices to show that $\Phi_1$ is an isomorphism. The problem for $\Phi_1$ can be reduced to the case where $\mathcal{M}$ is a single equation; i.e. $\mathcal{M} = D_X/D_XP$. Moreover it is sufficient to show that

$$
\underline{Hom}_{D_X}(D_X/D_XP, \iota^{-1}T-v_M(\mathcal{O}_X)) \otimes or_{N/X} \longrightarrow \underline{Hom}_{D_X}(D_X/D_XP, T-v_N(\mathcal{A}_M))
$$

is surjective. This problem can be solved by using the construction of the elementary solution of $P$ by means of Radon transformation and microdifferential operators.
REFERENCES


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E. Andronikof
Département de Mathématiques, Univ. Paris XIII
93430 Villetaneuse, France

N. Tose
Mathematics, General Education, Keio Univ.
4-1-1 Hiyoshi, Yokohama 223, Japan