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On Strongly Closed Subgraphs of Highly Regular Graphs

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Abstract

A geodetically closed induced subgraph $\Delta$ of a graph $\Gamma$ is defined to be strongly closed if $\Gamma_i(\alpha) \cap \Gamma(\beta)$ stays in $\Delta$ for every $i$ and $\alpha, \beta \in \Delta$ with $\partial(\alpha, \beta) = i$. We study the existence conditions of strongly closed subgraphs in highly regular graphs such as distance-regular graphs or distance-biregular graphs.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. For a subset $\Delta \subset V(\Gamma)$, we identify $\Delta$ with the induced subgraph on $\Delta$. In particular, $\Gamma = V(\Gamma)$.

For two vertices $\alpha, \beta$ in $\Gamma$, let $\partial_\Gamma(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in $\Gamma$, i.e., the length of a shortest path connecting $\alpha$ and $\beta$ in $\Gamma$. We also write $\partial(\alpha, \beta)$, when no confusion occurs. Let

$$\Gamma_i(\alpha) = \{ \beta \in \Gamma | \partial(\alpha, \beta) = i \} \quad \text{and} \quad \Gamma(\alpha) = \Gamma_1(\alpha).$$

For vertices $\alpha, \beta$ in $\Gamma$ with $\partial(\alpha, \beta) = i$, let

$$C(\alpha, \beta) = C_i(\alpha, \beta) = \Gamma_{i-1}(\alpha) \cap \Gamma(\beta),$$
$$A(\alpha, \beta) = A_i(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma(\beta),$$
$$B(\alpha, \beta) = B_i(\alpha, \beta) = \Gamma_{i+1}(\alpha) \cap \Gamma(\beta), \quad \text{and}$$
$$G(\alpha, \beta) = \bigcup_{j=0}^{i} \Gamma_j(\alpha) \cap \Gamma_{i-j}(\beta)$$
$$= \{ \gamma \in \Gamma | \partial(\alpha, \gamma) + \partial(\gamma, \beta) = \partial(\alpha, \beta) \}.$$

$G(\alpha, \beta)$ is the set of vertices which lie on a geodesic between $\alpha$ and $\beta$. For the cardinalities, we use lower case letters, i.e.,

$$c_i(\alpha, \beta) = |C_i(\alpha, \beta)|, \quad a_i(\alpha, \beta) = |A_i(\alpha, \beta)|, \quad \text{and} \quad b_i(\alpha, \beta) = |B_i(\alpha, \beta)|.$$

We also write $c_i(\alpha)$ [resp. $a_i(\alpha)$, $b_i(\alpha)$] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choice of $\beta$ under the condition $\partial(\alpha, \beta) = i$, and $c_i$ [resp. $a_i$, $b_i$] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choices of $\alpha$ and $\beta$. 
under the condition $\partial(\alpha, \beta) = i$. In these cases we say for example that $c_i(\alpha)$ exists or $c_i$
exists.

A connected graph $\Gamma$ is said to be distance-regular if $c_i, a_i, b_i$ exist for all $i$.

A connected bipartite graph $\Gamma$ with a bipartition $P \cup L$ is said to be distance-biregular
if $c_i(\alpha), b_i(\alpha)$ exist for all $i$ and these numbers depend only on the part $\alpha$ belongs to.

For convenience, if $\Gamma = P \cup L$ is a bipartite graph, we also use notations like $c^P_i, b^P_i,$
$c^L_i, b^L_i$, when the corresponding numbers depend only on the part the base point belongs to.

A subset $\Delta$ of a graph $\Gamma$ is said to be $C_i$-closed [resp. $A_i$-closed] if $C_i(\alpha, \beta) \subset \Delta$ [resp.
$A_i(\alpha, \beta) \subset \Delta$] for every pair of vertices $\alpha, \beta$ in $\Delta$ with $\partial_\Gamma(\alpha, \beta) = i$.

A subset $\Delta$ of $\Gamma$ is said to be geodetically closed if $C(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha, \beta$ in $\Delta$, i.e., $\Delta$ is $C_i$-closed for every $i$. In this case, we have $\partial_\Gamma(\alpha, \beta) = \partial_\Delta(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$. It is clear that $\Delta$ is geodetically closed if and only if $G(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha, \beta$ in $\Delta$.

A subset $\Delta$ of $\Gamma$ is said to be strongly closed if $C(\alpha, \beta) \subset \Delta$ and $A(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha, \beta$ in $\Delta$, i.e., $\Delta$ is both $C_i$-closed and $A_i$-closed for every $i$.

We call the induced subgraph on $\Delta$ a geodetically [resp. strongly] closed subgraph
when $\Delta$ is a geodetically [resp. strongly] closed subset.

By definition, every strongly closed subgraph is geodetically closed, in particular connected
if $\Gamma$ is connected. When $\Gamma$ is bipartite, every geodetically closed subgraph is
strongly closed and we do not need to distinguish these notions.

In most known distance-regular graphs, there are many nontrivial geodetically closed
subgraphs and in many cases they are even strongly closed. In some cases we can guarantee
the existence of strongly [or geodetically] closed subgraphs if we know a part of the
parameters $c_i, a_i$. See [6, 18, 19, 21, 24], and [5, Section 4.3]. We believe that the
investigation of strongly [or geodetically] closed subgraphs is a key in the study of distance-
regular graphs.

The first question is the following:

Is a strongly closed subgraph $\Delta$ of a distance-regular graph $\Gamma$ always distance-
regular?

By definition, the answer is 'yes' if $\Delta$ is regular. On the contrary, we can find counter
eamples easily. For example, if the girth of $\Gamma$ is large, we can construct a strongly closed
subgraph isomorphic to a tree.

Are there any other types of non-regular strongly closed subgraphs of distance-regular
graphs? Theorem 1.1 gives a solution to this problem.

We need a few more definitions to state the theorem.
Let $l(c, a, b) = |\{i|(c_i, a_i, b_i) = (c, a, b)|$ and $r(\Gamma) = l(c_1, a_1, b_1)$.
Let $d(\Gamma) = \max\{\partial(\alpha, \beta)|\alpha, \beta \in \Gamma\}$, and $k(\alpha) = |\Gamma(\alpha)| = b_0(\alpha, \alpha)$. 
Let $K_{k+1}$ denote the complete graph of valency $k$, and $M_k$ denote a Moore graph of valency $k$, which is known to be of diameter 2 and $k \in \{2, 3, 7, 57\}$.

For a graph $\Gamma$, $\Gamma'$ denotes a subdivision graph obtained by replacing each edge by a path of length $t$.

\[
\begin{align*}
\text{Figure 1.} & \\
K_4 & 2K_4 & 3K_4
\end{align*}
\]

**Theorem 1.1** Let $\Delta$ be a strongly closed subgraph of a distance-regular graph $\Gamma$. Then one of the following holds.

(i) $\Delta$ is a distance-regular graph,

(ii) $2 \leq d(\Delta) \leq r(\Gamma)$,

(iii) $\Delta$ is a distance-biregular graph with $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq d(\Delta)$. In particular, $r(\Gamma) \equiv d(\Delta) \equiv 0 \pmod{2}$; or

(iv) $\Delta$ is a subdivision graph of a complete graph or a Moore graph obtained by replacing each edge by a path of length three, i.e., $\Delta \simeq 3K_{l+1}$ or $3M_1$. In particular, $d(\Delta) = r(\Gamma) + 2 = 5$ or 8, and $a_1 = 0$, $a_{r+1} = a_{r+2} = a_{r+3} = 1$, where $r = r(\Gamma)$.

In particular, $(c_m, a_m, b_m) = (c_{m-1}, a_{m-1}, b_{m-1}) = (c_m, a_m, b_m)$ with $d(\Delta) = m$, except the case (i).

For the corresponding result when $\Gamma$ is a distance-biregular graph, see the following section.

It follows easily from Theorem 1.1 that if $c_2 \neq 1$, then every strongly closed subgraph in a distance-regular graph is distance-regular. Using this fact, one can prove the following theorem without difficulty, and it is useful when one wants to characterize a distance-regular graph $\Gamma$ by the structure of its antipode $\Gamma_d(\alpha)$.

**Theorem 1.2** ([28]) Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$. If $\Gamma_d(\alpha)$ is strongly closed for some $\alpha \in \Gamma$, then $\Gamma_d(\beta)$ is a clique for every vertex $\beta \in \Gamma$.

The second question is the following:
Can we find parametrical conditions for distance-regular graphs to have strongly closed subgraphs?

In this paper, we shall discuss this problem for the cases (iii) and (iv). Note that if $2 \leq m \leq \tau(\Gamma)$, then we can find a strongly closed subgraph $\Delta$ in $\Gamma$ of diameter $m$ which is, roughly speaking, isomorphic to a graph obtained by replacing each edge of a tree by a clique.

Case (iii) is treated in Sections 3 and 4. In this case we have $a_i = 0$ for $i \leq d(\Delta)$. Though we discuss in full generality, it seems more natural to state the results on bipartite graphs. The first result in this case is an improvement of a result of Ray-Chaudhuri and Sprague on pseudo-projective incidence systems.

Let $q = p^e$ be a prime power and $V$ be a $d$-dimensional vector space over $GF(q)$ when $q \neq 1$, and a $d$-element set when $q = 1$. Let $\binom{V}{i}_q$ denote the collection of $i$-dimensional subspaces of $V$ when $q \neq 1$, and the collection of $i$-subsets when $q = 1$.

Let $J_q(d, s, s - 1)$ denote a bipartite graph with a bipartition

$$\begin{bmatrix} V \\ s - 1 \end{bmatrix}_q \cup \begin{bmatrix} V \\ s \end{bmatrix}_q,$$

where $x \in \begin{bmatrix} V \\ s - 1 \end{bmatrix}_q$, $l \in \begin{bmatrix} V \\ s \end{bmatrix}_q$ is adjacent if and only if $x \subset l$. $J_q(d, s, s - 1)$ is a distance-biregular graph and is called an $(s, q, d)$-projective incidence structure in [24].

Throughout this paper, we make a convention that $(q^m - 1)/(q - 1) = m$, when $q = 1$.

**Theorem 1.3** Let $\Gamma$ be a connected bipartite graph of diameter at least five with a bipartition $P \cup L$. Suppose $c_2(x) = 1, c_3(x) = c_4(x) = q + 1$ for every $x \in P$, where $q$ is a fixed positive integer. Then $\Gamma$ is a biregular graph of valencies $k^P = k(x)$, and $k^L = k(l)$, where $x \in P, l \in L$. If $c_6(x)$ exists for every $x \in P$ and does not depend on the choice of $x \in P$, then one of the following holds.

(i) $\Gamma \simeq J_q(d, s, s - 1)$, where $k^L = (q^s - 1)/(q - 1), k^P = (q^{d-s+1} - 1)/(q - 1)$, or

(ii) $d(\Gamma) \leq 7, q \neq 1, k^P, k^L \leq 3q - 1$.

In particular, $q$ is a power of a prime if $k^P \geq 3q$ or $k^L \geq 3q$.

In [20] Koolen conjectured that under the hypothesis slightly stronger than that of Theorem 1.3, (i) or $d(\Gamma) \leq 4$ holds. Hence Theorem 1.3 gives an affirmative (but not complete) solution to the conjecture. For the detailed information on the case (ii), see Section 3.

Ray-Chaudhuri and Sprague obtained only the case (i) under an additional hypothesis $q^2 + q + 1 \leq k^L$. So in this paper we shall treat the case when the valency is not so large
compared with $c^P_r$, using a result of Terwilliger in [30]. In any case, as we can guess from the conclusion, one of the keys is to show that every pair of vertices of distance four determines a geodetically closed (hence, strongly closed but not regular) subgraph of diameter four assuming that the valency $k^L$ is not so small. See Section 3.

Let $\Gamma$ be a distance-biregular graph with a bipartition $P \cup L$. Assume $r$ is even and

$$c^P_r = 1 < c^P_{r+1} = c^P_{r+2}.$$  

This is one of the typical cases corresponding to Theorem 1.1.(iii). By Theorem 1.3, if $r = 2$ and $d(\Gamma) \geq 8$, then $\Gamma$ contains a strongly closed subgraph, which is distance-biregular of diameter four. It seems unlikely to have $r > 4$ and $r = 4$ is rare. We do not have a proof, but we can prove that $r \leq 4$ if $\Gamma$ contains a strongly closed subgraph of diameter $r + 2$. See Section 3. In Section 4, we treat the case $c^P_{r+1} = 2$ with $r = 4$ and prove the following.

**Theorem 1.4** Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_2(x) = c_3(x) = c_4(x) = 1$, $c_5(x) = c_6(x) = 2$ for every $x \in P$. Then $\Gamma$ is a biregular graph of valencies $k^P$ and $k^L$. If $\alpha$, $\beta$ be vertices in $\Gamma$ with $\partial(\alpha, \beta) = 5$, then there is a strongly closed subgraph $\Delta$ containing $\alpha$ and $\beta$ isomorphic to $2M_{k^P}$. In particular, $k^P \in \{2, 3, 7, 57\}$, if $d(\Gamma) \geq 5$.

We can show under the hypothesis in Theorem 1.4 that $c^L_i$ exists for $i = 1, 2, 3, 4, 5, 6$, $c^L_1 = \cdots c^L_4 = 1$ and $c^L_5 = c^L_6 = 2$. Hence Theorem 1.4 implies that $k^L \in \{2, 3, 7, 57\}$ as well. When $k^P = 2$ or $k^L = 2$, $\Gamma$ itself is a subdivision graph of a Moore graph isomorphic to $2M_k$ for some $k$. When $k^P = k^L = 3$, Foster graph is an example. We do not know any other examples. It may be possible to classify $\Gamma$ satisfying the condition of Theorem 1.4.

Case(iv) in Theorem 1.1 is treated in Section 5, under an additional condition $c_{r+3} = 1$.

**Theorem 1.5** Let $\Gamma$ be a distance-regular graph of valency $k > 2$ satisfying the following.

$$ (c_r, a_r, b_r) = (1, 0, k - 1), $$

$$ (c_{r+1}, a_{r+1}, b_{r+1}) = (c_{r+2}, a_{r+2}, b_{r+2}) = (1, 1, k - 2), $$

$r \geq 1$ and $c_{r+3} = 1$. Then $r \equiv 0 \pmod{3}$, and the following holds.

(1) If $r = 3$, then for every $\alpha$, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 3$, there is a strongly closed subgraph $\Delta$ containing $\alpha$, $\beta$ isomorphic to $3K_{k+1}$.

(2) If $r = 6$, then for every $\alpha$, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 6$, there is a strongly closed subgraph $\Delta$ containing $\alpha$, $\beta$ isomorphic to $3M_k$. In particular $k \in \{3, 7, 57\}$.

The first part $r \equiv 0 \pmod{3}$ is due to Boshier-Nomura [4]. It is known that if $l(1, 0, k - 1) = r \geq 1$, then $l(1, 1, k - 2) \leq 3$ and if $l(1, 1, k - 2) = 3$, then $c_{r+4} > 1$ [4, 13].
It is worth mentioning that both results Theorem 1.4 and 1.5 are related to circuit chasing technique. See [26] for a result related to Theorem 1.4.

We use intersection diagrams as our tools. We refer those who are not familiar with them to [4, 13, 14, 16, 23, 25, 26] and [5, Section 5.10] for example.

For subsets $A$, $B$ of $\Gamma$ let $e(A, B)$ denote the number of edges between $A$ and $B$, and
\[ e(x, A) = e(\{x\}, A). \]

$\Gamma^{(i)}$ will denote the distance-$i$-graph on $\Gamma$, i.e., the graph defined on the vertex set $V(\Gamma)$ of $\Gamma$ such that $\alpha$ and $\beta$ are adjacent if and only if $\partial_i(\alpha, \beta) = i$.

We write $\alpha \sim \beta$ when $\alpha \in \Gamma(\beta)$.

## 2 Strongly Closed Subgraphs

We shall prove Theorem 1.1 and related results in this section. The key of the proof is the determination of graphs such that $c_i$'s and $a_i$'s exist. Problems in similar settings are discussed in [12, 30, 20].

**Proposition 2.1** Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P$ exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. If $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$, with $r + 1 \leq m$, then the following hold.

1. If $c_i^P = c_{i-1}^L$ for some $i \leq m$, then $c_i^P$ exists and $c_{i-1}^P = c_i^L$. In particular, $c_1^P, \ldots, c_r^P$ exist and $c_1^P = \cdots = c_r^P = 1$.

2. If $c_1^P, \ldots, c_{2i}^P$ exist and $2i + 1 \leq m$, then $c_{2i+1}^P$ exists and $c_{2j}^P c_{2j+1}^P = c_{2j}^L c_{2j+1}^L$ for all $j \leq i$.

3. If $r$ is even, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover $c_{r+1}^P$ exists and $c_{r+1}^P = c_{r+1}^L$.

4. If $r$ is odd, and $c_{r+1}^L$ exists, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover,
\[ (c_{r+1}^P - 1)(b_0^L - 1) = (c_{r+1}^L - 1)(b_0^P - 1). \]

5. Suppose $\Gamma$ is biregular of valencies $k^P = b_0^P$ and $k^L = b_0^L$. Then $|P| k^P = |L| k^L$. Moreover, if $c_1^P, \ldots, c_{2i}^P$ exist with $2i \leq m$, then $b_s^P$, $b_t^L$ exist for $s \leq m$, $t \leq 2i$ and $b_{2j}^P b_{2j}^P = b_{2j-1}^P b_{2j}^P$, for all $j \leq i$.

We can obtain the following theorem as a direct corollary by applying Proposition 2.1 to $\Delta$.

**Theorem 2.2** Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P$, $c_i^L$ exist for $i = 1, \ldots, m$. Let $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$ with $r + 1 \leq m$. If $\Delta$ is a geodetically closed subgraph of $\Gamma$ of diameter $m$, then $\Delta$ is a distance-biregular graph.
Remark. For a distance-biregular graph $\Gamma = P \cup L$, let $d^P = \max\{\partial(x, \alpha) | \alpha \in \Gamma\}$, where $x \in P$, and $d^L = \max\{\partial(l, \alpha) | \alpha \in \Gamma\}$, where $l \in L$. In Theorem 2.2, if $d^{P\cap\Delta} \geq d^{L\cap\Delta}$, then $k^{P\cap\Delta} = c^P_m$. But we cannot determine the other valency when $d^{P\cap\Delta} > d^{L\cap\Delta}$.

Proposition 2.3 Let $\Gamma$ be a connected graph. Suppose $c_i$ exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. Suppose $c_1 = \cdots = c_r = 1$, $a_1, \ldots, a_r$ exist and $a_1 = \cdots = a_r$ and either $c_{r+1} > 1$ or $c_{r+1} = 1$ and $a_{r+1}$ exists with $a_{r+1} \neq a_1$, where $2 \leq r + 1 \leq m$. Then one of the following holds.

(i) $\Gamma$ is regular.

(ii) $\Gamma$ is a bipartite biregular graph such that $r \equiv 0 (\text{mod} 2)$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$.

(iii) $\Gamma \simeq 3K_{k+1}$ or $3M_k$, where $k$ is the largest valency of a vertex in $\Gamma$. In particular, $r = 3$ or $6$.

Lemma 2.4 Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_d$, $c_{d-1}$, $a_d$, $a_{d-1}$ exist. Then $\Gamma$ is regular of valency $c_d + a_d$ if and only if $(c_{d-1}, a_{d-1}) \neq (c_d, a_d)$.

Lemma 2.5 Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$ and $m < d$. Suppose $\Gamma$ has a strongly closed subgraph of diameter $m$ containing $\alpha$ and $\beta$ for every pair of vertices $\alpha$, $\beta$ with $\partial(\alpha, \beta) = m$. Then for all $\gamma, \delta \in \Gamma$ with $\partial(\gamma, \delta) \leq m + 1$, $C(\gamma, \delta)$ is a coclique.

Now we prove Theorem 1.1 under weaker conditions.

Theorem 2.6 Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_i$'s and $a_i$'s exist for all $i = 1, \ldots, m$, where $m \leq d$. Let

$$r = r(\Gamma) = \max\{i | (c_1, a_1) = (c_2, a_2) = \cdots = (c_i, a_i)\}.$$

If $\Gamma$ contains a strongly closed subgraph $\Delta$ of diameter $m$, then one of the following holds.

(i) $\Delta$ is a distance-regular graph,

(ii) $2 \leq m \leq r$,

(iii) $\Delta$ is a distance-biregular graph and that $r \equiv m \equiv 0 (\text{mod} 2)$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$, or

(iv) $\Delta \simeq 3K_{l+1}$ or $3M_l$ and $m = r + 2 = 5$ or $8$, $a_1 = \cdots = a_r = 0$, $c_1 = \cdots = c_{r+2} = a_{r+1} = a_{r+2} = 1$. 

Proof. Since \( \Delta \) is a strongly closed subgraph of \( \Gamma \), we can apply Proposition 2.3 to the subgraph \( \Delta \). If \( r \geq m \), then (i) or (ii) holds.

Assume \( r + 1 \leq m \). Then \( \Delta \) is one of the types in Proposition 2.3. If \( \Delta \) is regular, then \( \Delta \) is distance-regular as \( c_{i}' \)s and \( a_{i}' \)s exist for \( i \leq d(\Delta) = m \). Suppose \( \Delta \) is not regular. Since \( \Delta \) is strongly closed, \( k(\alpha) = k(\beta) \) if \( \alpha, \beta \in \Delta \) and \( \partial(\alpha, \beta) = m \). So if \( \Delta \) is a bipartite biregular graph, \( \Delta \) is distance-regular as \( c_{i}' \)s and \( a_{i}' \)s exist for \( i \leq d(\Delta) = m \).

Lemma 2.7 Let \( \Gamma \) be a distance-biregular graph with a bipartition \( P \cup L \). Suppose \( k^{P}, k^{L} \geq 2 \). Let \( d = d(\Gamma) \),

\[
d^{P} = \max\{\partial(x, \alpha)|x \in P, \alpha \in \Gamma\}, \quad d^{L} = \max\{\partial(l, \alpha)|l \in L, \alpha \in \Gamma\},
\]

and \( r(\Gamma) = \max\{i|c_{i}^{P} = 1\} \). Then \( r(\Gamma) = \max\{i|c_{i}^{L} = 1\} \) and the following are equivalent.

(i) \( r(\Gamma) + 2 = d^{P} + 1 = d^{L} = d \).

(ii) \( d = d^{L} = r(\Gamma) + 2, c_{d-1}^{L} = c_{d}^{L} \) with \( d \) even.

In this case \( \Gamma \) is a Moore geometry and \( d = 4 \) or 6. If \( d = 4 \), \( \Gamma \) is nothing but a nonsymmetrical 2-\((|P|, k^{L}, 1)\) design. If \( d = 6 \), then the incidence graph on \( P \) is a strongly regular graph with parameters \( (v, k, \lambda, \mu) = (|P|, k^{P}(k^{L} - 1), k^{L} - 2, 1) \).

For the diameter bound of Moore geometries, see [8, 7, 10, 11] and [5, Section 6.8]

Remark. In the case Theorem 2.6.(iii), the smallest possible value for \( m \) is \( r + 2 \) if the minimum valency is at least 2. By the previous lemma, we have \( r = 2 \) or 4. We treat these cases in the following sections. But it may be possible to give a bound of \( r = r(\Gamma) \) of distance-regular graphs satisfying \( a_{1} = 0, c_{r+1} = c_{r+2} \) with \( r \) even, by showing the existence of geodetically closed subgraphs of diameter \( r + 2 \), i.e., graphs discussed in the previous lemma.

3 A Refinement of a Theorem of Ray-Chaudhuri and Sprague

In [24], Ray-Chaudhuri and Sprague proved the following theorem in the context of incidence systems.

Theorem 3.1 Let \( \Gamma \) be a connected bipartite graph with a bipartition \( P \cup L \). For some positive integer \( q \), suppose \( c_{2}(x) = 1, c_{3}(x) = c_{4}(x) = q + 1 \) for every \( x \in P \). Then \( \Gamma \) is
biregular of valencies $k^P$ and $k^L$. If $k^P > q+1$ and $k^L \geq q^2 + q + 1$, then $\Gamma \simeq J_q(d, s, s-1)$, where $s$ and $d$ are real numbers defined by

$$k^L = (q^s - 1)/(q - 1), \quad k^P = (q^{d-s+1} - 1)/(q - 1).$$

In particular, $q$ is a power of a prime number and both $s$ and $d$ are integers.

The first part of this section is the following: By reviewing the proof of Ray-Chaudhuri and Sprague, we show that we can conclude either $d(\Gamma) \leq 4$ or $\Gamma \simeq J_q(d, s, s-1)$ if we can construct a geodetically closed subgraph of diameter 4 having vertices of valency $q+1$ and that such a subgraph exists if one of the valencies $k^P$ or $k^L$ is at least $3q$. Roughly speaking, we want to decrease the lower bound of the condition on the valencies in the hypothesis from $q^2 + q + 1$ to $3q$.

Before we start, we prepare a proposition.

**Proposition 3.2** Let $\Gamma$ be a connected regular graph of valency $k$ and diameter $d$. Suppose the distance-2-graph $\Delta = \Gamma^{(2)}$ is distance-regular of diameter $\bar{d}$. If each pair of vertices $\alpha, \beta$ at distance three in $\Gamma$ is contained in a shortest circuit of odd length $2m+1$, then $\bar{d} = m$ and a connected component of $\Delta_{\bar{d}}(\alpha)$ is a clique of size $k$. Moreover, $\Delta_{\bar{d}}(\alpha)$ is connected if and only if $d = \bar{d}$ and $\Gamma$ is a generalized Odd graph, i.e., a distance-regular graph such that $a_i = 0$, $i = 1, \ldots, d - 1$ and $a_d \neq 0$.

**Proof.** Firstly, we have $a_1 = \cdots = a_{m-1} = 0$, $m \geq 3$. And we have the following.

$$\Delta_1(\alpha) = \Gamma_2(\alpha), \quad \Delta_{m-1}(\alpha) \supset \Gamma_3(\alpha), \quad \Delta_m(\alpha) \supset \Gamma_1(\alpha).$$

Let $\beta \in \Gamma_1(\alpha)$. Then $\Delta_{m+1}(\alpha) \cap \Delta_1(\beta) = \emptyset$, $\bar{d} = m$. Moreover,

$$\Gamma_1(\alpha) \setminus \{\beta\} \subset \Delta_1(\beta) \cap \Delta_{\bar{d}}(\alpha) \subset \Gamma_1(\alpha) \setminus \{\beta\}.$$ 

Hence $a_{\bar{d}} = k - 1$ and a connected component of $\Delta_{\bar{d}}(\alpha)$ containing $\beta$ is a clique of size $k$.

If $\Delta_{\bar{d}}(\alpha)$ is connected, as $\Delta$ is distance-regular, $\Delta_{\bar{d}}(\gamma) = \Gamma_1(\gamma)$ is a clique of size $k$ in $\Delta$ for every $\gamma \in \Gamma$. Hence $\Gamma$ is a generalized Odd graph. See [1], [2, Section III.4], and [5, Section 4.2].

In the following we also treat the case when $\Gamma$ is a $k$-regular with the same conditions on $c_i$'s as those in Theorem 3.1.

Let $q$ be a positive integer and $r$ a positive even integer. A connected graph $\Gamma$ is said to be a $P(r, q)$-graph if $c_i, a_j$ exist for $1 \leq i \leq r + 2$, $1 \leq j \leq r + 1$ and they satisfy

$$c_1 = \cdots = c_r = 1, \quad a_1 = \cdots = a_{r+1} = 0, \quad c_{r+1} = c_{r+2} = q + 1.$$

**Lemma 3.3** Let $q$ be a positive integer and $r$ an even positive integer. The following hold.
(1) Let $\Gamma$ be a connected bipartite graph of diameter at least $r+1$ with a bipartition $P \cup L$. If $c^P_i$ exists for $1 \leq i \leq r+2$, and $c^P_1 = \cdots = c^P_r = 1$, $c^P_{r+1} = c^P_{r+2} = q+1$, then $\Gamma$ is a $P(r, q)$-graph.

(2) Let $\Gamma$ be a $P(r, q)$-graph. Then one of the following holds.

(i) $\Gamma$ is a bipartite biregular (possibly regular) graph; or

(ii) $\Gamma$ is a nonbipartite regular graph, i.e., a regular graph containing a circuit of odd length.

**Proof.** (1) This follows from Proposition 2.1.(1), (2).

(2) This follows from Proposition 2.3.

Let $\Gamma$ be a $P(r, q)$-graph of diameter at least $r+1$. According to the previous lemma, there are two possibilities.

(i) $\Gamma$ is a bipartite graph with a bipartition $P \cup L$ and biregular of valencies $k^P$ and $k^L$.

(ii) $\Gamma$ is a nonbipartite graph and regular of valency $k$. In this case, let $\Gamma = P = L$.

We give a list of known $P(r, q)$-graphs, which is not a polygon. $r = 2$ for the first three examples and $r = 4$ for the rest.

1. $J_q(d, s, s-1)$.
2. $O_k$, the Odd graph of valency $k$, (nonbipartite).
3. $2M_7$, the doubled Hoffman-Singleton graph, $(d = 5, q = 5)$.
4. $2M_k$, $k = 3, 7$, $(d = 6, q = 1)$.
5. Foster graph, that is the three fold cover of the incidence graph of $GQ(2, 2)$, the generalized quadrangle of order $(2, 2)$, $(d = 8, q = 1)$.

In this section we study $P(2, q)$-graphs. Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five.

For $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = 2$ and $\gamma \in C(\alpha, \beta)$, let

$$T(\alpha, \beta) = \Gamma_2(\alpha) \cap \Gamma_2(\beta) \cap \Gamma_3(\gamma).$$

We say $\Gamma$ satisfies the condition $\#^L$ [resp. $\#^P$], if $\delta, \eta \in T(\alpha, \beta)$ implies $\partial(\delta, \eta) \leq 2$ for all $\alpha, \beta \in L$ [resp. $P$] with $\partial(\alpha, \beta) = 2$.

The condition above is called 'Pasch's axiom' in [24].

**Lemma 3.4** (1) If $k^L \geq 3q$ or $q = 1$, then $\Gamma$ satisfies the condition $\#^L$. 

(2) If \( k^P \geq 3q \) or \( q = 1 \), then \( \Gamma \) satisfies the condition \#^P.

Proof. By symmetry it suffices to prove (1).

Let \( m_1, m_2 \in L \) with \( \partial(m_1, m_2) = 2 \) and \( \{x\} = C(m_1, m_2) \). Let \( T = T(m_1, m_2) \).

If \( l \in T \), then \( C(m_2, l) \subset \Gamma_3(m_1) \). Hence

\[
|T| = |T(m_1, m_2)| = b_2^L(c_3^L - 1) = (k^L - 1)q.
\]

Suppose the condition \#^L fails. Then there exist \( l, l' \in T \) with \( \partial(l, l') = 4 \). Let \( \{x_i\} = C(l, m_i), \{x_i'\} = C(l', m_i), i = 1, 2 \). Since \( c_3 = c_4 = q + 1 \), for \( i, j = 1, 2 \),

\[
x_i' \in C(l, l') = C(x_j, l') \quad \text{or} \quad \partial(x_i', x_j) = 2.
\]

So we have that

\[
x_1' \in C(x_2, m_1) \setminus \{x, x_1\}, \quad x_2' \in C(x_1, m_2) \setminus \{x, x_2\}.
\]

Hence \( |T \cap \Gamma_4(l)| \leq (q - 1)^2 \). Similarly, \( |T \cap \Gamma_4(l')| \leq (q - 1)^2 \). In particular, \( q \neq 1 \). Thus

\[
(q + 1)^2 = |\Gamma_2(l) \cap \Gamma_2(l')| \geq |T \cap \Gamma_2(l) \cap \Gamma_2(l')| + |\{m_1, m_2\}| \geq |T| + |\{m_1, m_2\}| - |T \cap \Gamma_4(l)| - |T \cap \Gamma_4(l')| \geq (k^L - 1)q + 2 - 2(q - 1)^2.
\]

So \( 3q^2 - 2q + 1 \geq (k^L - 1)q \) or \( k^L \leq 3q - 1 + \frac{1}{q} \). Since \( q \neq 1 \), \( k^L \leq 3q - 1 \), as desired.

For \( m_1, m_2 \in \Gamma_2(l) \) with \( m_1 \neq m_2 \), we write \( m_1 \approx m_2 \) if \( \partial(m_1, m_2) = 2 \) and \( C(m_1, m_2) \subset \Gamma_3(l) \), or equivalently if \( m_2 \in T(l, m_1) \). Since the relation \( \approx \) is symmetric, it defines a graph on \( \Gamma_2(l) \).

Let \( L_1(l, m) \) be a connected component in \( \Gamma_2(l) \) containing \( m \) with respect to \( \approx \). Let

\[
L(l, m) = \{l\} \cup L_1(l, m), \quad P(l, m) = \bigcup_{n \in L(l, m)} \Gamma(n), \quad \Delta(l, m) = P(l, m) \cup L(l, m).
\]

Lemma 3.5 Suppose \( \Gamma \) satisfies the condition \#^L. Then for \( l, m \in L \) with \( \partial(l, m) = 2 \), \( \Delta = \Delta(l, m) \) is a geodetically closed subgraph of \( \Gamma \) of diameter 4.

Proof. Since \( \Gamma \) satisfies the condition \#^L, we have \( \partial(m_1, m_2) \leq 2 \), if \( m_1, m_2 \in T(l, m) \). Hence we can prove the assertion without difficulty.

Let \( D = \{\Delta(l, m) | \partial(l, m) = 2, \ l, \ m \in L\} \).

Corollary 3.6 If \( \Gamma \) satisfies the condition \#^L, then the following hold.

(1) \( L(l, m) \) is a maximal clique in \( \Gamma^{(2)} \).
(2) If \(l, m \in \Delta_1 \cap \Delta_2 \cap L\), then \(\Delta_1 = \Delta_2\) or \(l = m\).

(3) \(\Delta\) is a bipartite biregular graph of valencies \(q + 1\) on \(P(l, m)\) and \(k^L\) on \(L(l, m)\).

(4) \(|L(l, m)| = qk^L + 1\).

(5) \(|\{\Delta \in D|l \in \Delta\}| = (k^P - 1)/q\) for every \(l \in L\).

Let \(\Pi\) be a bipartite graph on \(L \cup D\) with adjacency defined as follows: For \(l \in L, \Delta \in D, l \in \Delta\) and the valency of \(l\) in \(\Delta\) is \(k^L\). Note that \(k^L > q + 1\) as \(d(\Gamma) \geq 5\).

**Lemma 3.7** If \(\Gamma\) satisfies the condition \(\#^L\), then \(\Pi\) is a \(P(2, q)\)-graph of valencies \((k^P - 1)/q\) on \(L\) and \(qk^L + 1\) on \(D\).

**Proposition 3.8** Let \(\Gamma\) be a \(P(2, q)\)-graph of diameter at least five satisfying the condition \(\#^L\). Then one of the following holds.

(i) \(\Gamma \simeq J_q(d, s, s - 1)\), where \(k^L = (q^s - 1)/(q - 1)\), \(k^P = (q^{d-s+1} - 1)/(q - 1)\), or

(ii) \(\Gamma\) is a regular nonbipartite graph of valency \(k\) and \(\Gamma^{(2)}\) is isomorphic to a connected component of the distance-2-graph of \(J_q(2s - 3, s - 2, s - 3)\), where \(k = (q^{s-1} - 1)/(q - 1)\). Moreover, if each pair of vertices of \(\Gamma\) at distance three is contained in a shortest circuit of odd length, then \(q = 1\) and \(\Gamma\) is isomorphic to an Odd graph.

**Proof.** Firstly, note that \(J_q(d, s, s - 1) \simeq J_q(d, d - s + 1, d - s)\), if we take the dual interchanging \(P\) and \(L\).

Suppose \(\Gamma\) is bipartite. Since \(d(\Gamma) \geq 5\), \(k^P, k^L > q + 1\). By Theorem 3.1, (i) holds if \(k^P \geq q^2 + q + 1\), using the first remark above.

Assume \(k^P < q^2 + q + 1\). Since \(\Gamma\) satisfies the condition \(\#^L\), \(\Pi\) is a \(P(2, q)\)-graph of valencies \((k^P - 1)/q\) on \(L\). Since \((k^P - 1)/q < q + 1\), \(\partial_{\Pi}(l, m) \leq 2\) for all \(l, m \in L\). Hence \(\partial_{\Gamma}(l, m) \leq 2\) for all \(l, m \in L\), which is not the case.

Suppose \(\Gamma\) is not bipartite. By the previous lemma, \(\Pi\) is a bipartite \(P(2, q)\)-graph of valencies \((k - 1)/q\) on \(L\) and \(qk + 1\) on \(D\).

Suppose \((k - 1)/q \leq q + 1\). Since \(d(\Gamma) \geq 5\), there are vertices \(l_0, l_1, l_2, l_3\) such that

\[\partial(l_0, l_1) = \partial(l_1, l_2) = \partial(l_2, l_3) = 2, \quad \partial(l_0, l_2) = 4.\]

Since \(|\Pi_3(l_0) \cap \Pi(l_2)| = q + 1, (k - 1)/q = q + 1\) and \(\Delta(l_2, l_3) \in \Pi_3(l_0) \cap \Pi(l_2)\). So there is a vertex \(l \in \Delta(l_2, l_3)\) such that \(\partial(l, l_3) = \partial(l_0, l) = 2\). Hence \(\partial(l_3, l_0) \leq 4\). In particular \(d(\Gamma) = 5\), \(a_5\) exists and \(a_5 = 0\). Since \(\Gamma\) is not bipartite, we may assume that \(\partial(l_0, l_3) = 3\). Then \(|\Pi_2(l_3) \cap \Pi_2(l_0)| = 0\). This is a contradiction.

Thus \((k - 1)/q > q + 1, qk + 1 > q^2 + q + 1\). Hence by Theorem 3.1, \(\Gamma \simeq J_q(d, s, s - 1)\), where \(qk + 1 = (q^s - 1)/(q - 1), (k - 1)/q = (q^{d-s+1} - 1)/(q - 1)\).
Therefore $k = (q^{s-1} - 1)/(q - 1)$ and $d = 2s - 3$. Since $\partial_{\Pi}(l, m) = 2$ if and only if $\partial_{\Pi}(l, m) = 2$, $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $\Pi$ on $L$.

If $\Gamma$ satisfies the additional condition in (ii), we can apply Proposition 2.2. If $q \neq 1$, then $\Gamma^{(2)}$ is a Grassman graph, which is also called a $q$-analogue of Johnson graph. But in this case it is easy to check that the antipode is connected, while it is not a clique. Hence $q = 1$ and $\Gamma^{(2)} \simeq J(2s - 3, s - 2)$. Thus $\Gamma$ is an Odd graph.

In the following, we investigate the case when $\Gamma$ does not satisfy $\#^L$. By symmetry proved in Lemma 3.3, we may assume that $\Gamma$ does not satisfy $\#^P$ either. Hence by Lemma 3.4, we need only to consider the case $k^P$, $k^L \leq 3q - 1$.

The key to analyze this case is the following proposition proved by Terwilliger. We kept the notations in [30], where $M_i$ is no longer a Moore graph.

**Proposition 3.9 ([30])** Let integers $c$, $p$ and $s$ all be at least 2. Suppose the vertices of some graph $\Gamma$ can be partitioned into $s + 1$ disjoint sets $V_{\Gamma} = \bigcup_{i=0}^{s} M_i$, where for any $u, v \in V_{\Gamma}$, $u \in M_i$, $v \in M_j$ and $(u, v) \in E_{\Gamma}$ implies $|i - j| \leq 1$. For $i = 1$ or $s$, let $l_i$ and $L_i$ denote the minimum and maximum number of vertices in $M_{i-1}$ any vertex in $M_i$ is adjacent to, and for $i = 0$ or $s - 1$, let $r_i$ and $R_i$ denote the minimum and maximum number of vertices in $M_{i+1}$ any vertex in $M_i$ is adjacent to. Also assume

(i) $\partial(u, v) = s$ for some $u \in M_0$ and $v \in M_s$,

(ii) for integers $0 \leq i, j \leq s$ and for any $u \in M_i$ and $v \in M_j$, there are either $c$ or $0$ paths of length $s$ connecting them if $|j - i| = s$, and either $0$ or $1$ paths of length $|j - i|$ connecting them if $1 \leq |j - i| \leq s - 1$, and

(iii) for any $u, v \in V_{\Gamma}$ with $u \in M_1$, $v \in M_{s-1}$, and $\partial(u, v) > s - 2$, there are at most $p$ paths $\{u = v_0, v_1, \ldots, v_{s-1}, v_s = v\}$, where either $v_1 \in M_0$ or $v_{s-1} \in M_s$.

Then

$$\frac{p}{c - 1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1}.$$

**Proposition 3.10** Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five. If $c_5^P$ exists, then $c_5$ exists, i.e., $c_5^P = c_5^L$, $c_5 > q + 1$ and the following hold.

1. If $d(\Gamma) \geq 7$, then $c_5 \geq 2q + 1$.
2. If $\alpha, \beta, \gamma \in \Gamma$ with $\partial(\alpha, \beta) = 8$, $\partial(\alpha, \gamma) = 3$, $\partial(\gamma, \beta) = 5$, then $k(\gamma) \geq 3q + 2$.
3. For $\alpha \in \Gamma$ let $j = k(\alpha) - c_5$. If $a_4 = 0$, then

$$k(\alpha) \geq \frac{2q + j + 3 + \sqrt{4jq^2 + (j - 1)^2}}{2}.$$

In particular, if $j \geq 4$, then $k(\alpha) \geq 3q + 4$. 
Proof. It follows from Proposition 2.1.(2) that $c_5$ exists.

(1) Let $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = 7$. Let

$$M_i = \Gamma_{2+i}(\alpha) \cap \Gamma_{5-i}(\beta), \ i = 0, 1, 2, 3.$$ 

Apply Proposition 3.9.

(2) Since $d \geq 8$, we can apply (1). We have

$$k(\gamma) \geq c_3(\alpha, \gamma) + c_5(\beta, \gamma) \geq 3q + 2.$$ 

(3) Let $\alpha \in \Gamma$ and $M_i = \Gamma_{i+2}(\alpha), \ i = 0, 1, 2, 3$. Apply Proposition 3.8.

We now summarize our results in this section, from which we have Theorem 1.3 as a corollary.

Theorem 3.11 Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five. Suppose $c_5$ exists. Then $\Gamma$ is a bipartite biregular graph of valencies $k^P$ and $k^L$, or a regular graph of valency $k = k^P = k^L$ and one of the following holds.

(i) $\Gamma \simeq J_q(d, s, s-1)$, where $k^L = (q^s - 1)/(q-1)$, $k^P = (q^{d-s+1} - 1)/(q-1)$,

(ii) $\Gamma$ is a regular nonbipartite graph of valency $k$ and the distance-2-graph $\Gamma(2)$ is isomorphic to a connected component of the distance-2-graph of $J_q(2s-3, s-2, s-3)$, where $k = (q^{s-1} - 1)/(q-1)$. Moreover, if each pair of vertices of $\Gamma$ at distance three is contained in a shortest circuit of odd length, then $q = 1$ and $\Gamma$ is isomorphic to an Odd graph; or

(iii) $d(\Gamma) \leq 7$ and $k^P, k^L \leq 3q - 1$, $q \neq 1$. Moreover if $a_4 = 0$, then $\Gamma$ is bipartite and $k^P - c_5, k^L - c_5 \leq 3$. In particular, if $\Gamma$ is not bipartite and $a_4$ exists, then $d(\Gamma) \leq 6$.

Corollary 3.12 Let $\Gamma$ be a distance-regular graph of valency $k$. Suppose $c_2 = 1$, $c_3 = c_4 = q + 1$ and $a_1 = a_2 = a_3 = 0$ for some positive integer $q$. Then one of the following holds.

(i) $\Gamma \simeq J_q(2s - 1, s - 2, s - 3)$, where $k = (q^s - 1)/(q-1)$.

(ii) $\Gamma \simeq O_k$, an Odd graph of valency $k$; or

(iii) $d(\Gamma) \leq 7$, and the equality holds only if $\Gamma$ is bipartite.

Koolen [20] conjectured the following:

If $\Gamma$ is a distance-biregular graph of diameter at least 5 such that $c_i$ exists for all $i$, and $c_2 = 1$, $c_3 = c_4 > 2$, then $\Gamma \simeq J_q(d, s, s-1)$. 

Our results asserts that $d(\Gamma) \leq 7$ and the parameters are restricted very much. It is known that if $d(\Gamma) = 5$ or 7, then $\Gamma$ is distance-regular, under the assumption of the conjecture above. See [9, 20].

We also note that for $d(\Gamma) = 5$, the doubled Moore graph satisfy the hypothesis with $c_5 = q+2$. Moreover if it’s valency is not 3, say 7, then it does not come from $J_q(d, s, s-1)$. So this gives a counter example to the conjecture above.

4 \textit{P}(r, 1)\textit{-graphs}

According to the remark following Lemma 3.3, a \textit{P}(r, 1)\textit{-graph is a connected graph $\Gamma$, which is either a bipartite biregular graph with a bipartition $P \cup L$ or a nonbipartite regular graph such that}

$$c_1 = \cdots = c_r = 1, a_1 = \cdots = a_{r+1} = 0, c_{r+1} = c_{r+2} = 2,$$

where $r$ is an even positive integer. In this section we study \textit{P}(r, 1)\textit{-graphs and we show the following when $r = 4$. We do not know any \textit{P}(r, 1)\textit{-graphs with $r > 4$.}

\textbf{Theorem 4.1} Let $\Gamma$ be a \textit{P}(4, 1)\textit{-graph of diameter at least four and $\alpha, \gamma \in \Gamma$ with $\partial(\alpha, \gamma) = 4$. Then there is a geodetically closed subgraph $\Delta$ containing $\alpha, \gamma$ isomorphic to $2M_{k(\alpha)}$. Here $k(\alpha)$ denotes the valency of $\alpha$ in $\Gamma$. In particular, $k(\alpha) \in \{2, 3, 7, 57\}$.

Let $\Gamma$ be a \textit{P}(r, 1)\textit{-graph with $r \geq 4$.}

Fix a vertex $\alpha \in \Gamma$. For $\gamma, \delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma, \delta) = 2$ and $C(\gamma, \delta) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing $\gamma$ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on $C_\gamma$ defined by the relation $\approx$. For $\gamma, \delta \in \Gamma$ with $\partial(\gamma, \delta) = r$, and $0 \leq i \leq r$, let

$$\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta, \alpha), \beta(\delta) = g_2(\delta, \alpha), \text{ and } \gamma(\delta) = g_4(\delta, \alpha).$$

Firstly we note that the intersection diagram with respect to $x$, $l$ with $\partial(x, l) = 1$ has the following shape, where $D^i_j = \Gamma_i(x) \cap \Gamma_j(l)$. See the properties (a) $\sim$ (e) below.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) {$\{x\} = D^0_1 \rightarrow \cdots \rightarrow D^{r-1}_r \rightarrow D^r_{r+1} \rightarrow D^{r+1}_{r+2}$};
\node at (0,-1) {$\{l\} = D^0_1 \rightarrow \cdots \rightarrow D^{r-1}_r \rightarrow D^r_{r+1} \rightarrow D^{r+2}_{r+2}$};
\end{tikzpicture}
\caption{Figure 2.}
\end{figure}
(a) \( D_i^i = \emptyset \), for \( 1 \leq i \leq r+1 \).

(b) For \( y \in D_i^{i+1}, z \in D_i^{i+1}, e(y, D_i^{i-1}) = e(z, D_i^{i-1}) = 1, 1 \leq i \leq r \).

(c) For \( y \in D_r^{r+1}, z \in D_r^{r+1}, e(y, D_r^{r}) = e(z, D_r^{r}) = 2 \).

(d) For \( y \in D_r^{r+1}, z \in D_r^{r+1}, e(y, D_r^{r+1}) = e(z, D_r^{r+1}) = 1 \).

(e) \( e(D_i^{i+1}, D_i^{i+1}) = 0, 1 \leq i \leq r-1 \) and \( i = r+1 \).

The following two lemmas are related to circuit chasing technique. See [4, 13, 14] and [5, Section 5.10].

**Lemma 4.2** Let \( x_0 \sim x_1 \sim \cdots \sim x_{2r+2t} = x_0 \) be a circuit of length \( 2r + 2t \). i.e., a closed path and \( x_{i-1} \neq x_{i+1} \), \( i = 1, \ldots, 2r + 2t - 1 \) and \( x_{2r+2t-1} \neq x_1 \). Suppose

\[
x_r, x_{r+2}, \ldots, x_{r+2t} \in \Gamma_r(x_0), x_{r+1}, x_{r+3}, \ldots, x_{r+2t-1} \in \Gamma_{r+1}(x_0).
\]

Set \( D_j^i = \Gamma_i(x_0) \cap \Gamma_j(x_1) \). Then the following hold.

1. \( t \geq 1 \) and \( x_r \in D_{r-1}^{r}, x_{r+1} \in D_{r+1}^{r+1}, x_{r+2} \in D_{r+1}^{r+1} \).

2. If \( t \geq 2 \), then \( x_{r+3} \in D_{r+1}^{r+1} \) and \( x_{r+4} \in D_{r+1}^{r+1} \).

3. If \( t = 2 \), then the mutual distance of the vertices in the circuit is uniquely determined. In particular,

\[
\partial(x_2, x_{r+2}) = \partial(x_2, x_{r+4}) = r, \partial(x_2, x_{r+5}) = r + 1.
\]

4. If \( t = 3 \), then \( x_{r+5} \in D_{r+2}^{r+1}, x_{r+6} \in D_{r+1}^{r+1} \) and

\[
\partial(x_2, x_{r+4}) = \partial(x_2, x_{r+6}) = \partial(x_4, x_{r+6}) = r, \partial(x_4, x_{r+5}) = \partial(x_4, x_{r+7}) = r + 1.
\]

**Proof.** In the following, we use (a) \( \sim (e) \) to determine the locations of \( x_j \)'s in the diagram with respect to an edge \( x_{i-1} \sim x_i \), using the information on the distances from \( x_{i-1} \).

1. Since \( x_{i-1} \neq x_{i+1} \), for all \( i \), and \( c_1 = \cdots = c_r = 1, t \geq 1 \). It is clear that \( x_r \in D_{r-1}^{r} \). Since \( x_{r+1} \in \Gamma_{r+1}(x_0) \cap \Gamma(x_r), x_{r+1} \in D_{r+1}^{r+1} \). \( x_r \neq x_{r+2} \in \Gamma_r(x_0) \cap \Gamma(x_{r+1}) \) implies that \( x_{r+2} \in D_{r+1}^{r+1} \).

2. Since \( x_{r+2} \in D_{r+1}^{r+1} \) and \( e(x_{r+2}, D_{r+1}^{r+1}) = 1 \) with \( x_{r+1} \in D_{r+1}^{r+1} \cap \Gamma(x_{r+2}), x_{r+3} \in D_{r+2}^{r+1} \), \( x_{r+4} \in D_{r+1}^{r+1} \).

3. It is easy to determine the mutual distances as follows.

\[
\begin{array}{cccccccc}
\text{x}_r & \text{x}_{r+1} & \text{x}_{r+2} & \text{x}_{r+3} & \text{x}_{r+4} & \text{x}_{r+5} \\
x_0 & r & r+1 & r & r+1 & r & r-1 \\
x_1 & r-1 & r & r+1 & r+2 & r+1 & r \\
x_2 & r-2 & r-1 & r & r+1 & r & r+1 \\
\end{array}
\]
Now the distance pattern with respect to $x_2$ is the same as that with respect to $x_0$, the mutual distance of the vertices in the circuit is uniquely determined and the assertion follows.

(4) We do the same as in (3).

\[
\begin{array}{cccccccccccc}
  & x_r & x_{r+1} & x_{r+2} & x_{r+3} & x_{r+4} & x_{r+5} & x_{r+6} & x_{r+7} & x_{r+8} & x_{r+9} \\
x_0 & r & r+1 & r & r+1 & r & r+1 & r & r-1 & r-2 & r-3 \\
x_1 & r-1 & r & r+1 & r+2 & r+1 & r & r-1 & r & r-2 \\
x_2 & r-2 & r-1 & r & r+1 & r & r+1 & r & r-1 & r \\
x_3 & r-3 & r-2 & r-1 & r & r+1 & r+2 & r+1 & r & r \\
x_4 & r-4 & r-3 & r-2 & r-1 & r & r+1 & r & r+1 & r \\
\end{array}
\]

Note that since $x_{r+7} \in D_{r}^{-1}$, $x_{r+5}$ cannot be in $D_{r}^{r+1}$.

**Lemma 4.3** Let $y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4$ be a path of length four such that $y_{i-1} \neq y_{i+1}$, $i = 1, \ldots, 3$. Suppose $y_0, y_4 \in \Gamma_r(\alpha)$. Then one of the following holds.

(i) $y_2 \in \Gamma_{r-2}(\alpha)$,

(ii) $y_1 \in \Gamma_{r-1}(\alpha)$ or $y_3 \in \Gamma_{r-1}(\alpha)$ and $\alpha(y_0) \neq \alpha(y_4)$,

(iii) $y_1, y_3 \in \Gamma_{r+1}(\alpha), y_2 \in \Gamma_r(\alpha)$ and $\alpha(y_0) \neq \alpha(y_4)$,

(iv) $y_2 \in \Gamma_{r+2}(\alpha)$ and $\alpha(y_0) = \alpha(y_4)$, while $\beta(y_0) \neq \beta(y_4)$, or

(v) $y_2 \in \Gamma_{r+2}(\alpha)$ and $\alpha(y_0) \neq \alpha(y_4)$, $\partial(\beta(y_0), y_4) = r+2$.

By Lemma 4.2 and 4.3, we can prove the following concerning the connected component in $\Gamma_r(\alpha)$ with respect to $\approx$.

**Lemma 4.4** Let $\{\alpha_1, \ldots, \alpha_k(\alpha)\} = \Gamma(\alpha), \gamma \in \Gamma_r(\alpha), C = C_\gamma$. Let $S_i = \{\delta \in C|\alpha(\delta) = \alpha_i\}$. Then the following hold.

(1) For $\delta \in S_i, |\Pi(\delta) \cap S_j| = 1 - \delta_{i,j}$ and $S_i \subseteq \Gamma_{r-2}(\beta(\delta))$. In particular, $\Pi$ is a $k(\alpha)$-partite $(k(\alpha) - 1)$-regular graph.

(2) Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in $\Pi$. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3), \delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.

If $r = 4$, $\gamma(\delta) = \delta$ for every $\delta \in \Pi$. So by Lemma 4.4, we have the following.

**Lemma 4.5** If $r = 4$, then the following holds.

(1) If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4$ such that $\delta_0 \approx \delta_4 \approx \delta_3$. 

\[
\begin{array}{cccccccccccc}
  & x_r & x_{r+1} & x_{r+2} & x_{r+3} & x_{r+4} & x_{r+5} & x_{r+6} & x_{r+7} & x_{r+8} & x_{r+9} \\
x_0 & r & r+1 & r & r+1 & r & r+1 & r & r-1 & r-2 & r-3 \\
x_1 & r-1 & r & r+1 & r+2 & r+1 & r & r-1 & r & r-2 \\
x_2 & r-2 & r-1 & r & r+1 & r & r+1 & r & r-1 & r \\
x_3 & r-3 & r-2 & r-1 & r & r+1 & r+2 & r+1 & r & r \\
x_4 & r-4 & r-3 & r-2 & r-1 & r & r+1 & r & r+1 & r \\
\end{array}
\]
(2) If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) = \alpha(\delta_3)$, then $\beta(\delta_0) = \beta(\delta_3)$.

(3) If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ with $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, then there exists $\delta_5$ such that $\delta_0 \approx \delta_5 \approx \delta_4$.

(4) $d(\Pi) \leq 3$ and if $\partial(\delta, \delta') = 3$, then $\beta(\delta) = \beta(\delta')$.

Proof. (1) Since $\gamma(\delta) = \delta$ for every $\delta \in \Pi$, (1) is a direct consequence of Lemma 4.4.(2).

(2) This follows from Lemma 4.4.(1).

(3) By (2), $\beta(\delta_0) = \beta(\delta_3) \neq \beta(\delta_1) = \beta(\delta_4)$. Now $\delta_3, \beta(\delta_1) \in \Gamma_4(\delta_0)$, and there is a path of length 4,

$$y_0 = \delta_3 \sim y_1 \sim y_2 = \delta_4 \sim y_3 \sim y_4 = \beta(\delta_1),$$

where $y_1 \in C(\delta_3, \delta_4)$, $y_3 = g_1(\alpha, \delta_4)$.

It is easy to check that $y_1, y_3 \in \Gamma_5(\delta_0)$ and that $g_1(\delta_3, \delta_0) \neq g_1(\beta(\delta_1), \delta_0)$. Hence by Lemma 4.3.(iii) or (v) occurs.

If (v) occurs, $\partial(\beta(\delta_0), \delta_4) = 6$, which is not the case. Hence $\partial(\delta_0, \delta_4) = 4$.

Let $\delta_0 = z_0 \sim z_1 \sim z_2 \sim z_3 \sim z_4 = \delta_4$ be a path connecting $\delta_0$ and $\delta_4$. Then by Lemma 4.3, we have (iii) as $\partial(\beta(\delta_0), \delta_4) = 4$. Hence we can set $z_2 = \delta_5$.

(4) This follows from (1), (2) and (3).

Proof of Theorem 4.1. Let $r = 4$ and

$$L(\alpha, \gamma) = \{\alpha\} \cup \bigcup_{\delta \in C_{\gamma}} (\Gamma_2(\alpha) \cap \Gamma_2(\delta)) \cup C_{\gamma},$$

$$P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_1(\delta),$$

$$\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)$$

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$.

We shall show in the sequel that $\Delta$ is a geodetically closed subgraph isomorphic to $2M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_{k(\alpha)}\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_{k(\alpha)})\} \cup C_{\gamma}.$$

By Lemma 4.5, the distance-2-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

If $k(\alpha) = 2$, there is nothing to prove. Assume $k(\alpha) > 2$.

$\partial(\beta(\gamma), \gamma_2) = 4$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k(\alpha)-1}\} \subset \Gamma_4(\beta(\gamma)), \]
there is a vertex $\delta_i' \in \Pi(\delta_i) \cap \Gamma_2(\beta(\gamma))$ for each $i$. Since the girth of $\Gamma$ is 10, we can conclude that the valency of $\beta(\gamma)$ in the distance-2-graph induced on $L(\Delta)$ equals $k(\alpha)$. By Lemma 4.5, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that $\Delta$ is geodetically closed subgraph of $\Gamma$ isomorphic to $2M_{k(\alpha)}$ easily.

This completes the proof of Theorem 4.1.

We remark that in the final step, we can also apply [5, Theorem 1.17.1] to determine the regularity of the distance-2-graph induced on $L(\Delta)$. See the proof of [5, Proposition 4.3.11].

5 Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. We can follow the proof in the previous section step by step, replacing each path of length 2 by a path of length 3.

Let $\Gamma$ be a graph satisfying the hypothesis in Theorem 1.5.

Fix a vertex $\alpha \in \Gamma$. For $\gamma$, $\delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma, \delta) = 3$. Then $C(\gamma, \delta) \cup C(\delta, \gamma) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing $\gamma$ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on $C_\gamma$ defined by the relation $\approx$. Hence $C$ is a connected component of the distance-3-graph of $\Gamma$ induced on the set $\Gamma_r(\alpha)$.

For $\gamma$, $\delta \in \Gamma$ with $\partial(\gamma, \delta) = r$, and $0 \leq i \leq r$, let

$$\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta, \alpha), \quad \alpha'(\delta) = g_2(\delta, \alpha), \quad \beta(\delta) = g_3(\delta, \alpha), \quad \text{and} \quad \gamma(\delta) = g_6(\delta, \alpha).$$

Firstly we note that the intersection diagram with respect to $x$, $y$ with $\partial(x, y) = 1$ has the following shape, where $D^i_j = \Gamma_i(x) \cap \Gamma_j(y)$. See the properties (a) $\sim$ (g) below.

$$\{x\} = D^0_1 \quad \cdots \quad D^r_{r+1} \quad \cdots \quad D^{r+1}_{r+2} \quad \cdots \quad D^{r+2}_{r+3} \quad \cdots$$

$$\{y\} = D^0_1 \quad \cdots \quad D^r_{r-1} \quad \cdots \quad D^{r+1}_{r+2} \quad \cdots \quad D^{r+2}_{r+3} \quad \cdots$$

Figure 3.

(a) $D^i_i = \emptyset$, for $1 \leq i \leq r$.

(b) For $y \in D^{i+1}_i$, $z \in D^i_{i+1}$, $e(y, D^i_{i-1}) = e(z, D^i_{i-1}) = 1$, $1 \leq i \leq r + 2$. 

(c) For \( y \in D_{i}^{i+1}, \ z \in \acute{D}_{i+1}^{i}, \ e(y, D_{i}^{i+1}) = e(z, D_{i+1}^{i}) = 0, \ 1 \leq i \leq r \) and \( e(y, D_{i}^{i+1}) = e(z, D_{i+1}^{i}) = 1, \ i = r+1, \ r+2. \)

(d) For \( y \in D_{r+1}^{r+1}, \ e(y, D_{r+1}^{r+1}) = e(y, D_{r+1}^{r}) = 1 \) and \( e(y, D_{r+1}^{r+1}) = e(y, \ddagger^{1}1) = 0. \)

(e) For \( y \in D_{r+1}^{r+1}, \ z \in D_{r+1}^{r+1}, \ e(y, D_{r+1}^{r+1}) = e(z, D_{r+1}^{r+1}) = 1. \)

(f) For \( y \in D_{r+2}^{r+2}, \ e(y, D_{r+1}^{r+1}) = e(y, D_{r+2}^{r+2}) = 1. \)

(g) \( e(D_{i}^{i+1}, D_{i+1}^{i}) = 0, \ 1 \leq i \leq r+2. \)

We again apply circuit chasing technique.

Lemma 5.1 Let \( x_0 \sim x_1 \sim \cdots \sim x_{2r+3t} = x_0 \) be a circuit of length \( 2r+3t \). i.e., a closed path and \( x_{i-1} \neq x_{i+1}, \ i = 1, \ldots, 2r+3t-1 \) and \( x_{2r+3t-1} \neq x_1. \) Suppose \( x_r, x_{r+3}, \ldots, x_{r+3t} \in \Gamma_r(x_0), \ x_{r+1}, x_{r+2}, x_{r+4}, x_{r+5}, \ldots, x_{x+3t-2}, x_{x+3t-1} \in \Gamma_{r+1}(x_0). \) Set \( D_j^i = \Gamma_i(x_0) \cap \Gamma_j(x_1). \) Then the following hold.

(1) \( t \geq 1 \) and \( x_r \in D_{r-1}^r, \ x_{r+1} \in D_{r}^{r+1}, \ x_{r+2} \in D_{r+1}^{r+1} \) and \( x_{r+3} \in D_{r+1}^r. \)

(2) If \( t \geq 2, \) then \( x_{r+2} \in D_{r+1}^{r+1} \) and \( x_{r+5} \in D_{r+2}^{r+1} \) and \( x_{r+6} \in D_{r+1}^r. \)

(3) If \( t = 2, \) then the mutual distance of the vertices in the circuit is uniquely determined. In particular, \( r \equiv 0 (\text{mod } 3), \) and

\[ \partial(x_3, x_{r+3}) = \partial(x_3, x_{r+6}) = r, \ \partial(x_3, x_{r+7}) = r + 1. \]

(4) Suppose \( r \geq 6. \) If \( t = 3, \) then \( x_{r+7}, x_{r+8} \in D_{r+2}^{r+1} \), \( x_{r+9} \in D_{r+1}^r \) and

\[ \partial(x_3, x_{r+6}) = \partial(x_3, x_{r+9}) = \partial(x_6, x_{r+9}) = r, \ \partial(x_6, x_{r+8}) = \partial(x_6, x_{r+10}) = r + 1. \]

Lemma 5.2 Let \( y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4 \sim y_5 \sim y_6 \) be a path of length 6 such that \( y_{i-1} \neq y_i+1, \ i = 1, \ldots, 5. \) Suppose \( y_0, y_6 \in \Gamma_r(\alpha). \) Then one of the following holds.

(i) \( y_3 \in \Gamma_{r-3}(\alpha), \)

(ii) \( y_1, y_2, y_4, y_5 \in \Gamma_{r+1}(\alpha), \ y_3 \in \Gamma_r(\alpha) \) and \( \alpha(y_0) \neq \alpha(y_6), \)

(iii) \( y_3 \in \Gamma_{r+2}(\alpha) \) and \( y_5 \in \Gamma_{r+1}(\alpha) \cap \Gamma_{r+1}(\alpha(y_9)), \) while \( \partial(\beta(y_0), y_9) \geq r + 1. \)

Lemma 5.3 Let \( \{\alpha_1, \ldots, \alpha_k\} = \Gamma(\alpha), \ \gamma \in \Gamma_r(\alpha), \ C = C_{\gamma}. \) Let \( S_i = \{\delta \in C \mid \alpha(\delta) = \alpha_i\}. \) Then the following hold.

(1) \( \text{For } \delta \in S_i, \ |\Pi(\delta) \cap S_j| = 1 - \delta_{i,j} \text{ and } S_i \subset \Gamma_{r-3}(\beta(\delta)). \text{ In particular, } \Pi \text{ is a } k\text{-partite } (k-1)\text{-regular graph.} \)
(2) Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in $\Pi$. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.

**Lemma 5.4** If $r = 6$, then the following holds.

1. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4$ such that $\delta_0 \approx \delta_4 \approx \delta_3$.

2. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) = \alpha(\delta_3)$, then $\beta(\delta_0) = \beta(\delta_3)$.

3. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ with $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, then there exists $\delta_5$ such that $\delta_0 \approx \delta_5 \approx \delta_4$.

4. $d(\Pi) \leq 3$ and if $\partial_{\Pi}(\delta, \delta') = 3$, then $\beta(\delta) = \beta(\delta')$.

**Proof of Theorem 1.5.** Suppose $r = 3$. Let

$$L(\alpha, \gamma) = \{\alpha\} \cup C_{\gamma},$$

$$P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_1(\delta),$$

$$\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)$$

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$. Clearly $L(\Delta)$ is a maximal clique in the distance-3-graph of $\Gamma$, and the assertion follows easily from Lemma 5.3.

Let $r = 6$ and

$$L(\alpha, \gamma) = \{\alpha\} \cup \bigcup_{\delta \in C_{\gamma}} (\Gamma_3(\alpha) \cap \Gamma_3(\delta)) \cup C_{\gamma},$$

$$P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_1(\delta),$$

$$\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)$$

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$.

We shall show in the sequel that $\Delta$ is a geodetically closed subgraph isomorphic to \(3M_{k(\alpha)}\).

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_k\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_k)\} \cup C_{\gamma}.$$ 

By Lemma 5.4, the distance-3-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

If $\partial(\beta(\gamma), \gamma_2) = 6$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k-1}\} \subset \Gamma_6(\beta(\gamma)),$$
there is a vertex $\delta_i \in \Pi(\delta_i) \cap \Gamma_3(\beta(\gamma))$ for each $i$. Since the girth of $\Gamma$ is 15, we can conclude that the valency of $\beta(\gamma)$ in the distance-3-graph induced on $L(\Delta)$ equals $k$. By Lemma 5.4, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that $\Delta$ is geodetically closed easily.

This completes the proof of Theorem 1.5.

6 Concluding Remarks

It may be too optimistic to expect a classification of $P(r, q)$-graphs or the graphs similar to those discussed in the previous section in the near future. But we believe that the investigation of such graphs plays a key role to give an absolute bound of the girth of distance-biregular graphs or distance-regular graphs.

We list several problems, which we want to see solved.

1. Study geodetically closed subgraphs of distance-regular graphs and prove results corresponding to Proposition 2.3 and Theorem 2.6, especially when $a_1 \neq 0$. See [20].

2. Classify $P(r, q)$-graphs.

   a) For $r = 2$, it may be possible to improve Lemma 3.4 to have $2q$ as the lower bound. Then we have $d \leq 5$, by Proposition 3.10.

   b) For $q = 1$, the classification implies a classification of distance-biregular graphs with vertices of valency three, [26]. Hence we can obtain an absolute diameter bound of distance-regular graphs of order $(s, 2)$, i.e., those with $\Gamma(x) \simeq 3 \cdot K_s$. See [17, 3, 15, 31].

3. Let $\Gamma$ be a bipartite biregular graph with a bipartition $P \cup L$, or a regular graph with $\Gamma = P = L$. For a positive integer $q$ and a positive odd integer $r$, we call $\Gamma$ a $P(r, q)$-graph, if it is a connected graph such that

\[ c^P_1 = \cdots = c^P_r = 1, \quad a_1 = \cdots = a_{r+1} = 0, \quad c^P_{r+1} = q + 1 \quad \text{and} \quad c^L_{r+1} = c^P_{r+2}. \]

Classify them. If $q = 1$, then $\Gamma$ is a thin generalized polygon by a result in [26].

4. Study a distance-regular graphs $\Gamma$ with $r = r(\Gamma)$, $c_{r+1} = c_{r+2} = 1$, and clarify the correspondence with $P(r, q)$-graphs. In particular, show $r \leq 6$ in Theorem 1.5.

5. Let $\Gamma$ be a connected graph of diameter $d$. For a subset $I \subset \{1, \ldots, d\}$, let $\Gamma^{(I)}$ denote the distance-$I$-graph, i.e., $V(\Gamma^{(I)}) = V(\Gamma)$, and $\alpha$, $\beta$ are adjacent in $\Gamma^{(I)}$ if and only if $\partial(\alpha, \beta) \in I$. Study $\Gamma$ such that at least one of the connected components of $\Gamma^{(I)}$ is distance-regular of diameter at least three. To start with, assume $\Gamma^{(I)}$
is connected. It is not hard to determine parametrical conditions if $\Gamma$ itself is a distance-regular graph. In particular, classify distance-regular graphs $\Gamma$ such that $\Gamma^{(2)}$ is distance-regular of diameter $d(\Gamma) \neq d(\Gamma^{(2)}) \geq 3$. See Proposition 3.2 and [27, 29].

6. Give a geometrical classification of Moore graphs. One of the reasons, we could not obtain the results for $P(r, 1)$-graphs with $r \geq 6$, is a lack of such classification. We believe that this is one of the keys when we develop structure theories of distance-regular graphs just as the group theoretical proof of Burnside's $p^aq^b$ theorem gave a breakthrough to the classification of finite simple groups.

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