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On Strongly Closed Subgraphs of Highly Regular Graphs

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Abstract

A geodetically closed induced subgraph $\Delta$ of a graph $\Gamma$ is defined to be strongly closed if $\Gamma_i(\alpha) \cap \Gamma_1(\beta)$ stays in $\Delta$ for every $i$ and $\alpha, \beta \in \Delta$ with $\vartheta(\alpha, \beta) = i$. We study the existence conditions of strongly closed subgraphs in highly regular graphs such as distance-regular graphs or distance-biregular graphs.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. For a subset $\Delta \subset V(\Gamma)$, we identify $\Delta$ with the induced subgraph on $\Delta$. In particular, $\Gamma = V(\Gamma)$.

For two vertices $\alpha, \beta$ in $\Gamma$, let $\vartheta(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in $\Gamma$, i.e., the length of a shortest path connecting $\alpha$ and $\beta$ in $\Gamma$. We also write $\vartheta(\alpha, \beta)$, when no confusion occurs. Let

$$\Gamma_i(\alpha) = \{ \beta \in \Gamma | \vartheta(\alpha, \beta) = i \} \text{ and } \Gamma(\alpha) = \Gamma_1(\alpha).$$

For vertices $\alpha, \beta$ in $\Gamma$ with $\vartheta(\alpha, \beta) = i$, let

$$C(\alpha, \beta) = C_i(\alpha, \beta) = \Gamma_{i-1}(\alpha) \cap \Gamma(\beta),$$

$$A(\alpha, \beta) = A_i(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma(\beta),$$

$$B(\alpha, \beta) = B_i(\alpha, \beta) = \Gamma_{i+1}(\alpha) \cap \Gamma(\beta), \text{ and}$$

$$G(\alpha, \beta) = \bigcup_{j=0}^{i} \Gamma_j(\alpha) \cap \Gamma_{i-j}(\beta)$$

$$= \{ \gamma \in \Gamma | \vartheta(\alpha, \gamma) + \vartheta(\gamma, \beta) = \vartheta(\alpha, \beta) \}.$$

$G(\alpha, \beta)$ is the set of vertices which lie on a geodesic between $\alpha$ and $\beta$. For the cardinalities, we use lower case letters, i.e.,

$$c_i(\alpha, \beta) = |C_i(\alpha, \beta)|, \quad a_i(\alpha, \beta) = |A_i(\alpha, \beta)|, \quad \text{and} \quad b_i(\alpha, \beta) = |B_i(\alpha, \beta)|.$$

We also write $c_i(\alpha) \text{ [resp. } a_i(\alpha), b_i(\alpha)\text{]}$ if the number $c_i(\alpha, \beta) \text{ [resp. } a_i(\alpha, \beta), b_i(\alpha, \beta)\text{]}$ does not depend on the choice of $\beta$ under the condition $\vartheta(\alpha, \beta) = i$, and $c_i \text{ [resp. } a_i, b_i\text{]}$ if the number $c_i(\alpha, \beta) \text{ [resp. } a_i(\alpha, \beta), b_i(\alpha, \beta)\text{]}$ does not depend on the choices of $\alpha$ and $\beta$. 

under the condition $\partial(\alpha, \beta) = i$. In these cases we say for example that $c_i(\alpha)$ exists or $c_i$
exists.

A connected graph $\Gamma$ is said to be distance-regular if $c_i, a_i, b_i$ exist for all $i$.

A connected bipartite graph $\Gamma$ with a bipartition $P \cup L$ is said to be distance-biregular if $c_i(\alpha)$, $b_i(\alpha)$ exist for all $i$ and these numbers depend only on the part $\alpha$ belongs to.

For convenience, if $\Gamma = P \cup L$ is a bipartite graph, we also use notations like $c_i^P$, $b_i^P$, $c_i^L$, $b_i^L$, when the corresponding numbers depend only on the part the base point belongs to.

A subset $\Delta$ of a graph $\Gamma$ is said to be $C_i$-closed [resp. $A_i$-closed] if $C_i(\alpha, \beta) \subset \Delta$ [resp. $A_i(\alpha, \beta) \subset \Delta$] for every pair of vertices $\alpha$, $\beta$ in $\Delta$ with $\partial_{\Gamma}(\alpha, \beta) = i$.

A subset $\Delta$ of $\Gamma$ is said to be geodetically closed if $C(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$, i.e., $\Delta$ is $C_i$-closed for every $i$. In this case, we have $\partial_{\Gamma}(\alpha, \beta) = \partial_{\Delta}(\alpha, \beta)$ for all $\alpha$, $\beta \in \Delta$. It is clear that $\Delta$ is geodetically closed if and only if $G(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$.

A subset $\Delta$ of $\Gamma$ is said to be strongly closed if $C(\alpha, \beta) \subset \Delta$ and $A(\alpha, \beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$, i.e., $\Delta$ is both $C_i$-closed and $A_i$-closed for every $i$.

We call the induced subgraph on $\Delta$ a geodetically [resp. strongly] closed subgraph when $\Delta$ is a geodetically [resp. strongly] closed subset.

By definition, every strongly closed subgraph is geodetically closed, in particular connected if $\Gamma$ is connected. When $\Gamma$ is bipartite, every geodetically closed subgraph is strongly closed and we do not need to distinguish these notions.

In most known distance-regular graphs, there are many nontrivial geodetically closed subgraphs and in many cases they are even strongly closed. In some cases we can guarantee the existence of strongly [or geodetically] closed subgraphs if we know a part of the parameters $c_i$, $a_i$. See [6, 18, 19, 21, 24], and [5, Section 4.3]. We believe that the investigation of strongly [or geodetically] closed subgraphs is a key in the study of distance-regular graphs.

The first question is the following:

**Is a strongly closed subgraph $\Delta$ of a distance-regular graph $\Gamma$ always distance-regular?**

By definition, the answer is 'yes' if $\Delta$ is regular. On the contrary, we can find counter examples easily. For example, if the girth of $\Gamma$ is large, we can construct a strongly closed subgraph isomorphic to a tree.

Are there any other types of non-regular strongly closed subgraphs of distance-regular graphs? Theorem 1.1 gives a solution to this problem.

We need a few more definitions to state the theorem.

Let $l(c, a, b) = |\{i|(c_i, a_i, b_i) = (c, a, b)\}|$ and $r(\Gamma) = l(c_1, a_1, b_1)$.

Let $d(\Gamma) = \max\{\partial(\alpha, \beta) | \alpha, \beta \in \Gamma\}$, and $k(\alpha) = |\Gamma(\alpha)| = b_0(\alpha, \alpha)$.
Let $K_{k+1}$ denote the complete graph of valency $k$, and $M_k$ denote a Moore graph of valency $k$, which is known to be of diameter 2 and $k \in \{2, 3, 7, 57\}$.

For a graph $\Gamma$, $'\Gamma$ denotes a subdivision graph obtained by replacing each edge by a path of length $t$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{example.png}
\caption{Figure 1.}
\end{figure}

**Theorem 1.1** Let $\Delta$ be a strongly closed subgraph of a distance-regular graph $\Gamma$. Then one of the following holds.

(i) $\Delta$ is a distance-regular graph,

(ii) $2 \leq d(\Delta) \leq r(\Gamma),

(iii) $\Delta$ is a distance-biregular graph with $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq d(\Delta)$. In particular, $r(\Gamma) \equiv d(\Delta) \equiv 0 \pmod{2}$; or

(iv) $\Delta$ is a subdivision graph of a complete graph or a Moore graph obtained by replacing each edge by a path of length three, i.e., $\Delta \simeq 3K_{l+1}$ or $3M_l$. In particular, $d(\Delta) = r(\Gamma) + 2 = 5$ or 8, and $a_1 = 0$, $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$, where $r = r(\Gamma)$.

In particular, $(c_{m-1}, a_{m-1}, b_{m-1}) = (c_m, a_m, b_m)$ with $d(\Delta) = m$, except the case (i).

For the corresponding result when $\Gamma$ is a distance-biregular graph, see the following section.

It follows easily from Theorem 1.1 that if $c_2 \neq 1$, then every strongly closed subgraph in a distance-regular graph is distance-regular. Using this fact, one can prove the following theorem without difficulty, and it is useful when one wants to characterize a distance-regular graph $\Gamma$ by the structure of its antipode $\Gamma_{d}(\alpha)$.

**Theorem 1.2** ([28]) Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$. If $\Gamma_d(\alpha)$ is strongly closed for some $\alpha \in \Gamma$, then $\Gamma_d(\beta)$ is a clique for every vertex $\beta \in \Gamma$.

The second question is the following:
Can we find parametrical conditions for distance-regular graphs to have strongly closed subgraphs?

In this paper, we shall discuss this problem for the cases (iii) and (iv). Note that if $2 \leq m \leq r(\Gamma)$, then we can find a strongly closed subgraph $\Delta$ in $\Gamma$ of diameter $m$ which is, roughly speaking, isomorphic to a graph obtained by replacing each edge of a tree by a clique.

Case (iii) is treated in Sections 3 and 4. In this case we have $a_i = 0$ for $i \leq d(\Delta)$. Though we discuss in full generality, it seems more natural to state the results on bipartite graphs. The first result in this case is an improvement of a result of Ray-Chaudhuri and Sprague on pseudo-projective incidence systems.

Let $q = p^e$ be a prime power and $V$ be a $d$-dimensional vector space over $GF(q)$ when $q \neq 1$, and a $d$-element set when $q = 1$. Let $\begin{bmatrix} V \\ i \end{bmatrix}_q$ denote the collection of $i$-dimensional subspaces of $V$ when $q \neq 1$, and the collection of $i$-subsets when $q = 1$.

Let $J_q(d, s, s - 1)$ denote a bipartite graph with a bipartition

$$\begin{bmatrix} V \\ s - 1 \end{bmatrix}_q \cup \begin{bmatrix} V \\ s \end{bmatrix}_q,$$

where $x \in \begin{bmatrix} V \\ s - 1 \end{bmatrix}_q$, $l \in \begin{bmatrix} V \\ s \end{bmatrix}_q$ is adjacent if and only if $x \subset l$. $J_q(d, s, s - 1)$ is a distance-biregular graph and is called an $(s, q, d)$-projective incidence structure in [24].

Throughout this paper, we make a convention that $(q^m - 1)/(q - 1) = m$, when $q = 1$.

**Theorem 1.3** Let $\Gamma$ be a connected bipartite graph of diameter at least five with a bipartition $P \cup L$. Suppose $c_2(x) = 1, c_3(x) = c_4(x) = q + 1$ for every $x \in P$, where $q$ is a fixed positive integer. Then $\Gamma$ is a biregular graph of valencies $k^P = k(x)$, and $k^L = k(l)$, where $x \in P, l \in L$. If $c_5(x)$ exists for every $x \in P$ and does not depend on the choice of $x \in P$, then one of the following holds.

(i) $\Gamma \simeq J_q(d, s, s - 1)$, where $k^L = (q^s - 1)/(q - 1), k^P = (q^{d-s+1} - 1)/(q - 1)$, or

(ii) $d(\Gamma) \leq 7, q \neq 1, k^P, k^L \leq 3q - 1$.

In particular, $q$ is a power of a prime if $k^P \geq 3q$ or $k^L \geq 3q$.

In [20] Koolen conjectured that under the hypothesis slightly stronger than that of Theorem 1.3, (i) or $d(\Gamma) \leq 4$ holds. Hence Theorem 1.3 gives an affirmative (but not complete) solution to the conjecture. For the detailed information on the case (ii), see Section 3.

Ray-Chaudhuri and Sprague obtained only the case (i) under an additional hypothesis $q^2 + q + 1 \leq k^L$. So in this paper we shall treat the case when the valency is not so large
compared with \( c_i^p \), using a result of Terwilliger in [30]. In any case, as we can guess from the conclusion, one of the keys is to show that every pair of vertices of distance four determines a geodetically closed (hence, strongly closed but not regular) subgraph of diameter four assuming that the valency \( k^L \) is not so small. See Section 3.

Let \( \Gamma \) be a distance-biregular graph with a bipartition \( P \cup L \). Assume \( r \) is even and

\[
c_r^P = 1 < c_{r+1}^P = c_{r+2}^P.
\]

This is one of the typical cases corresponding to Theorem 1.1.(iii). By Theorem 1.3, if \( r = 2 \) and \( d(\Gamma) \geq 8 \), then \( \Gamma \) contains a strongly closed subgraph, which is distance-biregular of diameter four. It seems unlikely to have \( r > 4 \) and \( r = 4 \) is rare. We do not have a proof, but we can prove that \( r \leq 4 \) if \( \Gamma \) contains a strongly closed subgraph of diameter \( r + 2 \). See Section 3. In Section 4, we treat the case \( c_{r+1}^P = 2 \) with \( r = 4 \) and prove the following.

**Theorem 1.4** Let \( \Gamma \) be a connected bipartite graph with a bipartition \( P \cup L \). Suppose

\[
c_2(x) = c_3(x) = c_4(x) = 1, c_5(x) = c_6(x) = 2 \quad \text{for every} \quad x \in P.
\]

Then \( \Gamma \) is a biregular graph of valencies \( k^P \) and \( k^L \). If \( \alpha, \beta \) be vertices in \( \Gamma \) with \( \partial(\alpha, \beta) = 5 \), then there is a strongly closed subgraph \( \Delta \) containing \( \alpha \) and \( \beta \) isomorphic to \( 2M_{k^P} \). In particular, \( k^P \in \{2, 3, 7, 57\} \), if \( d(\Gamma) \geq 5 \).

We can show under the hypothesis in Theorem 1.4 that \( c_i^L \) exists for \( i = 1, 2, 3, 4, 5, 6 \), \( c_1^L = \cdots c_4^L = 1 \) and \( c_5^L = c_6^L = 2 \). Hence Theorem 1.4 implies that \( k^L \in \{2, 3, 7, 57\} \) as well. When \( k^P = 2 \) or \( k^L = 2 \), \( \Gamma \) itself is a subdivision graph of a Moore graph isomorphic to \( 2M_k \) for some \( k \). When \( k^P = k^L = 3 \), Foster graph is an example. We do not know any other examples. It may be possible to classify \( \Gamma \) satisfying the condition of Theorem 1.4.

Case(iv) in Theorem 1.1 is treated in Section 5, under an additional condition \( c_{r+3} = 1 \).

**Theorem 1.5** Let \( \Gamma \) be a distance-regular graph of valency \( k > 2 \) satisfying the following.

\[
(c_r, a_r, b_r) = (1, 0, k - 1),
\]

\[
(c_{r+1}, a_{r+1}, b_{r+1}) = (c_{r+2}, a_{r+2}, b_{r+2}) = (1, 1, k - 2),
\]

\( r \geq 1 \) and \( c_{r+3} = 1 \). Then \( r \equiv 0 \quad (\text{mod} \ 3) \), and the following holds.

1. If \( r = 3 \), then for every \( \alpha, \beta \in \Gamma \) with \( \partial(\alpha, \beta) = 3 \), there is a strongly closed subgraph \( \Delta \) containing \( \alpha, \beta \) isomorphic to \( 3K_{k+1} \).

2. If \( r = 6 \), then for every \( \alpha, \beta \in \Gamma \) with \( \partial(\alpha, \beta) = 6 \), there is a strongly closed subgraph \( \Delta \) containing \( \alpha, \beta \) isomorphic to \( 3M_k \). In particular \( k \in \{3, 7, 57\} \).

The first part \( r \equiv 0 \ (\text{mod} \ 3) \) is due to Boshier-Nomura [4]. It is known that if \( l(1, 0, k - 1) = r \geq 1 \), then \( l(1, 1, k - 2) \leq 3 \) and if \( l(1, 1, k - 2) = 3 \), then \( c_{r+4} > 1 \) [4, 13].
It is worth mentioning that both results Theorem 1.4 and 1.5 are related to circuit chasing technique. See [26] for a result related to Theorem 1.4.

We use intersection diagrams as our tools. We refer those who are not familiar with them to [4, 13, 14, 16, 23, 25, 26] and [5, Section 5.10] for example.

For subsets $A$, $B$ of $\Gamma$ let $e(A, B)$ denote the number of edges between $A$ and $B$, and $e(x, A) = e(\{x\}, A)$.

$\Gamma^{(i)}$ will denote the distance-$i$-graph on $\Gamma$, i.e., the graph defined on the vertex set $V(\Gamma)$ of $\Gamma$ such that $\alpha$ and $\beta$ are adjacent if and only if $\partial_\Gamma(\alpha, \beta) = i$.

We write $\alpha \sim \beta$ when $\alpha \in \Gamma(\beta)$.

2 Strongly Closed Subgraphs

We shall prove Theorem 1.1 and related results in this section. The key of the proof is the determination of graphs such that $c_i$'s and $a_i$'s exist. Problems in similar settings are discussed in [12, 30, 20].

**Proposition 2.1** Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P$ exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. If $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$, with $r + 1 \leq m$, then the following hold.

1. If $c_i^P = c_{i-1}^L$ for some $i \leq m$, then $c_i^P$ exists and $c_{i-1}^P = c_i^L$. In particular, $c_1^L, \ldots, c_r^L$ exist and $c_1^P = \cdots = c_r^P = 1$.

2. If $c_1^L, \ldots, c_{2i}^L$ exist and $2i + 1 \leq m$, then $c_{2i+1}^L$ exists and $c_{2j}^P c_{2j+1}^P = c_{2j}^L c_{2j+1}^L$ for all $j \leq i$.

3. If $r$ is even, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover, $c_{r+1}^L$ exists and $c_{r+1}^P = c_{r+1}^L$.

4. If $r$ is odd, and $c_{r+1}^L$ exists, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover,

\[(c_{r+1}^P - 1)(b_0^L - 1) = (c_{r+1}^L - 1)(b_0^P - 1).\]

5. Suppose $\Gamma$ is biregular of valencies $k^P = b_0^P$ and $k^L = b_0^L$. Then $|P|^k_P = |L|^k_L$. Moreover, if $c_1^L, \ldots, c_{2i}^L$ exist with $2i \leq m$, then $b_s^P, b_t^L$ exist for $s \leq m, t \leq 2i$ and $b_{2j-1}^P b_{2j}^P = b_{2j-1}^L b_{2j}^L$, for all $j \leq i$.

We can obtain the following theorem as a direct corollary by applying Proposition 2.1 to $\Delta$.

**Theorem 2.2** Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P, c_i^L$ exist for $i = 1, \ldots, m$. Let $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$ with $r + 1 \leq m$. If $\Delta$ is a geodetically closed subgraph of $\Gamma$ of diameter $m$, then $\Delta$ is a distance-biregular graph.
Remark. For a distance-biregular graph $\Gamma = P \cup L$, let $d^P = \max \{ \partial(x, \alpha) | \alpha \in \Gamma \}$, where $x \in P$, and $d^L = \max \{ \partial(l, \alpha) | \alpha \in \Gamma \}$, where $l \in L$. In Theorem 2.2, if $d^{P \cap \Delta} \geq d^{L \cap \Delta}$, then $k^{P \cap \Delta} = c_m^P$. But we cannot determine the other valency when $d^{P \cap \Delta} > d^{L \cap \Delta}$.

Proposition 2.3 Let $\Gamma$ be a connected graph. Suppose $c_i$ exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. Suppose $c_1 = \cdots = c_r = 1$, $a_1, \ldots, a_r$ exist and $a_1 = \cdots = a_r$ and either $c_{r+1} > 1$ or $c_{r+1} = 1$ and $a_{r+1}$ exists with $a_{r+1} \neq a_1$, where $2 \leq r + 1 \leq m$. Then one of the following holds.

(i) $\Gamma$ is regular.

(ii) $\Gamma$ is a bipartite biregular graph such that $r \equiv 0 \pmod{2}$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$.

(iii) $\Gamma \simeq 3K_{k+1}$ or $3M_k$, where $k$ is the largest valency of a vertex in $\Gamma$. In particular, $r = 3$ or 6.

Lemma 2.4 Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_d$, $c_{d-1}$, $a_d$, $a_{d-1}$ exist. Then $\Gamma$ is regular of valency $c_d + a_d$ if and only if $(c_{d-1}, a_{d-1}) \neq (c_d, a_d)$.

Lemma 2.5 Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$ and $m < d$. Suppose $\Gamma$ has a strongly closed subgraph of diameter $m$ containing $\alpha$ and $\beta$ for every pair of vertices $\alpha$, $\beta$ with $\partial(\alpha, \beta) = m$. Then for all $\gamma$, $\delta \in \Gamma$ with $\partial(\gamma, \delta) \leq m + 1$, $C(\gamma, \delta)$ is a coclique.

Now we prove Theorem 1.1 under weaker conditions.

Theorem 2.6 Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_i$'s and $a_i$'s exist for all $i = 1, \ldots, m$, where $m \leq d$. Let

$$r = r(\Gamma) = \max \{ i | (c_1, a_1) = (c_2, a_2) = \cdots = (c_i, a_i) \}.$$ 

If $\Gamma$ contains a strongly closed subgraph $\Delta$ of diameter $m$, then one of the following holds.

(i) $\Delta$ is a distance-regular graph,

(ii) $2 \leq m \leq r$,

(iii) $\Delta$ is a distance-biregular graph and that $r \equiv m \equiv 0 \pmod{2}$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$, or

(iv) $\Delta \simeq 3K_{l+1}$ or $3M_l$ and $m = r + 2 = 5$ or 8, $a_1 = \cdots = a_r = 0$, $c_1 = \cdots = c_{r+2} = a_{r+1} = a_{r+2} = 1$. 


Proof. Since $\Delta$ is a strongly closed subgraph of $\Gamma$, we can apply Proposition 2.3 to the subgraph $\Delta$. If $r \geq m$, then (i) or (ii) holds.

Assume $r + 1 \leq m$. Then $\Delta$ is one of the types in Proposition 2.3. If $\Delta$ is regular, then $\Delta$ is distance-regular as $c_i$'s and $a_i$'s exist for $i \leq d(\Delta) = m$. Suppose $\Delta$ is not regular. Since $\Delta$ is strongly closed, $k(\alpha) = k(\beta)$ if $\alpha, \beta \in \Delta$ and $\partial(\alpha, \beta) = m$. So if $\Delta$ is a bipartite biregular graph, $\Delta$ is distance-biregular and $m \equiv 0 \pmod{2}$. Hence we have (iii). Suppose $\Delta \simeq aK_{l+1}$ or $3M_l$. Then $r = 3$ or $6$ and $m = r + 2$, $c_1 = \cdots = c_m = 1$, $a_1 = \cdots = a_r = 0$, $a_{r+1} = a_{r+2} = 1$ easily follow from the structure of $\Delta$.

Lemma 2.7 Let $\Gamma$ be a distance-biregular graph with a bipartition $P \cup L$. Suppose $k^P, k^L \geq 2$. Let $d = d(\Gamma)$,

$$d^P = \max\{\partial(x, \alpha)|x \in P, \alpha \in \Gamma\}, \quad d^L = \max\{\partial(l, \alpha)|l \in L, \alpha \in \Gamma\},$$

and $r(\Gamma) = \max\{i|c_i^P = 1\}$. Then $r(\Gamma) = \max\{i|c_i^L = 1\}$ and the following are equivalent.

(i) $r(\Gamma) + 2 = d^P + 1 = d^L = d$.

(ii) $d = d^L = r(\Gamma) + 2$, $c_{d-1}^d = c_d^d$ with $d$ even.

In this case $\Gamma$ is a Moore geometry and $d = 4$ or $6$. If $d = 4$, $\Gamma$ is nothing but a nonsymmetric $2-(|P|, k^L, 1)$ design. If $d = 6$, then the incidence graph on $P$ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (|P|, k^P(k^L - 1), k^L - 2, 1)$.

For the diameter bound of Moore geometries, see [8, 7, 10, 11] and [5, Section 6.8]

Remark. In the case Theorem 2.6.(iii), the smallest possible value for $m$ is $r + 2$ if the minimum valency is at least 2. By the previous lemma, we have $r = 2$ or 4. We treat these cases in the following sections. But it may be possible to give a bound of $r = r(\Gamma)$ of distance-regular graphs satisfying $a_1 = 0$, $c_{r+1} = c_{r+2}$ with $r$ even, by showing the existence of geodetically closed subgraphs of diameter $r + 2$, i.e., graphs discussed in the previous lemma.

3 A Refinement of a Theorem of Ray-Chaudhuri and Sprague

In [24], Ray-Chaudhuri and Sprague proved the following theorem in the context of incidence systems.

Theorem 3.1 Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. For some positive integer $q$, suppose $c_2(x) = 1$, $c_3(x) = c_4(x) = q + 1$ for every $x \in P$. Then $\Gamma$ is
biregular of valencies $k^P$ and $k^L$. If $k^P > q+1$ and $k^L \geq q^2 + q + 1$, then $\Gamma \simeq J_q(d, s, s - 1)$, where $s$ and $d$ are real numbers defined by

$$k^L = (q^s - 1)/(q - 1), \quad k^P = (q^{d - s + 1} - 1)/(q - 1).$$

In particular, $q$ is a power of a prime number and both $s$ and $d$ are integers.

The first part of this section is the following: By reviewing the proof of Ray-Chaudhuri and Sprague, we show that we can conclude either $d(\Gamma) \leq 4$ or $\Gamma \simeq J_q(d, s, s - 1)$ if we can construct a geodetically closed subgraph of diameter 4 having vertices of valency $q+1$ and that such a subgraph exists if one of the valencies $k^P$ or $k^L$ is at least 3$q$. Roughly speaking, we want to decrease the lower bound of the condition on the valencies in the hypothesis from $q^2 + q + 1$ to 3$q$.

Before we start, we prepare a proposition.

**Proposition 3.2** Let $\Gamma$ be a connected regular graph of valency $k$ and diameter $d$. Suppose the distance-2-graph $\Delta = \Gamma^{(2)}$ is distance-regular of diameter $\tilde{d}$. If each pair of vertices $\alpha, \beta$ at distance three in $\Gamma$ is contained in a shortest circuit of odd length $2m + 1$, then $\tilde{d} = m$ and a connected component of $\Delta_d(\alpha)$ is a clique of size $k$. Moreover, $\Delta_d(\alpha)$ is connected if and only if $d = \tilde{d}$ and $\Gamma$ is a generalized Odd graph, i.e., a distance-regular graph such that $a_i = 0, i = 1, \ldots, d - 1$ and $a_d \neq 0$.

**Proof.** Firstly, we have $a_1 = \cdots = a_{m-1} = 0, m \geq 3$. And we have the following.

$$\Delta_1(\alpha) = \Gamma_2(\alpha), \Delta_{m-1}(\alpha) \supset \Delta_3(\alpha), \Delta_m(\alpha) \supset \Gamma_1(\alpha).$$

Let $\beta \in \Gamma_1(\alpha)$. Then $\Delta_{m+1}(\alpha) \cap \Delta_1(\beta) = \emptyset, \tilde{d} = m$. Moreover,

$$\Gamma_1(\alpha) \setminus \{\beta\} \subset \Delta_1(\beta) \cap \Delta_d(\alpha) \subset \Gamma_1(\alpha) \setminus \{\beta\}.$$

Hence $\tilde{d} = k - 1$ and a connected component of $\Delta_d(\alpha)$ containing $\beta$ is a clique of size $k$.

If $\Delta_d(\alpha)$ is connected, as $\Delta$ is distance-regular, $\Delta_d(\gamma) = \Gamma_1(\gamma)$ is a clique of size $k$ in $\Delta$ for every $\gamma \in \Gamma$. Hence $\Gamma$ is a generalized Odd graph. See [1], [2, Section III.4], and [5, Section 4.2].

In the following we also treat the case when $\Gamma$ is a $k$-regular with the same conditions on $c_i$’s as those in Theorem 3.1.

Let $q$ be a positive integer and $r$ a positive even integer. A connected graph $\Gamma$ is said to be a $P(r, q)$-graph if $c_i, a_j$ exist for $1 \leq i \leq r + 2, 1 \leq j \leq r + 1$ and they satisfy

$$c_1 = \cdots = c_r = 1, \ a_1 = \cdots = a_{r+1} = 0, \ c_{r+1} = c_{r+2} = q + 1.$$

**Lemma 3.3** Let $q$ be a positive integer and $r$ an even positive integer. The following hold.
(1) Let $\Gamma$ be a connected bipartite graph of diameter at least $r+1$ with a bipartition $P \cup L$. If $c^P_i$ exists for $1 \leq i \leq r+2$, and $c^P_1 = \cdots c^P_r = 1$, $c^P_{r+1} = c^P_{r+2} = q+1$, then $\Gamma$ is a $P(r, q)$-graph.

(2) Let $\Gamma$ be a $P(r, q)$-graph. Then one of the following holds.

(i) $\Gamma$ is a bipartite biregular (possibly regular) graph; or

(ii) $\Gamma$ is a nonbipartite regular graph, i.e., a regular graph containing a circuit of odd length.

Proof. (1) This follows from Proposition 2.1.(1), (2).

(2) This follows from Proposition 2.3.

Let $\Gamma$ be a $P(r, q)$-graph of diameter at least $r+1$. According to the previous lemma, there are two possibilities.

(i) $\Gamma$ is a bipartite graph with a bipartition $P \cup L$ and biregular of valencies $k^P$ and $k^L$.

(ii) $\Gamma$ is a nonbipartite graph and regular of valency $k$. In this case, let $\Gamma = P = L$.

We give a list of known $P(r, q)$-graphs, which is not a polygon. $r = 2$ for the first three examples and $r = 4$ for the rest.

1. $J_q(d, s, s-1)$.
2. $O_k$, the Odd graph of valency $k$, (nonbipartite).
3. $2M_7$, the doubled Hoffman-Singleton graph, $(d = 5, q = 5)$.
4. $2M_k$, $k = 3, 7$, $(d = 6, q = 1)$.
5. Foster graph, that is the three fold cover of the incidence graph of $GQ(2, 2)$, the generalized quadrangle of order $(2, 2)$, $(d = 8, q = 1)$.

In this section we study $P(2, q)$-graphs. Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five.

For $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = 2$ and $\gamma \in C(\alpha, \beta)$, let

$$T(\alpha, \beta) = \Gamma_2(\alpha) \cap \Gamma_2(\beta) \cap \Gamma_3(\gamma).$$

We say $\Gamma$ satisfies the condition $\#^L$ [resp. $\#^P$], if $\delta, \eta \in T(\alpha, \beta)$ implies $\partial(\delta, \eta) \leq 2$ for all $\alpha, \beta \in L$ [resp. $P$] with $\partial(\alpha, \beta) = 2$.

The condition above is called 'Pasch's axiom' in [24].

Lemma 3.4 (1) If $k^L \geq 3q$ or $q = 1$, then $\Gamma$ satisfies the condition $\#^L$. 
(2) If $k^P \geq 3q$ or $q = 1$, then $\Gamma$ satisfies the condition $\#^P$.

Proof. By symmetry it suffices to prove (1).
Let $m_1, m_2 \in L$ with $\partial(m_1, m_2) = 2$ and $\{x\} = C(m_1, m_2)$. Let $T = T(m_1, m_2)$. If $l \in T$, then $C(m_2, l) \subset \Gamma_3(m_1)$. Hence

$$|T| = |T(m_1, m_2)| = b_2^L(c_3^L - 1) = (k^L - 1)q.$$ 

Suppose the condition $\#^L$ fails. Then there exist $l, l' \in T$ with $\partial(l, l') = 4$. Let $\{x_i\} = C(l, m_i), \{x'_i\} = C(l', m_i), i = 1, 2$. Since $c_3 = c_4 = q + 1$, for $i, j = 1, 2$,

$$x'_j \in C(x_j, m_1) \setminus \{x, x_1\}, \ x'_i \in C(x_i, m_2) \setminus \{x, x_2\}.$$ 

Hence

$$|T \cap \Gamma_4(l)| \leq (q - 1)^2.$$ 

Similarly, $|T \cap \Gamma_4(l')| \leq (q - 1)^2$. In particular, $q \neq 1$. Thus

$$(q + 1)^2 = |\Gamma_2(l) \cap \Gamma_2(l')| \geq |T \cap \Gamma_2(l) \cap \Gamma_2(l')| + |\{m_1, m_2\}| \geq |T| + |\{m_1, m_2\}| - |T \cap \Gamma_4(l)| - |T \cap \Gamma_4(l')| \geq (k^L - 1)q + 2 - 2(q - 1)^2.$$ 

So $3q^2 - 2q + 1 \geq (k^L - 1)q$ or $k^L \leq 3q - 1 + \frac{1}{q}$. Since $q \neq 1$, $k^L \leq 3q - 1$, as desired.

For $m_1, m_2 \in \Gamma_2(l)$ with $m_1 \neq m_2$, we write $m_1 \approx m_2$ if $\partial(m_1, m_2) = 2$ and $C(m_1, m_2) \subset \Gamma_3(l)$, or equivalently if $m_2 \in T(l, m_1)$. Since the relation $\approx$ is symmetric, it defines a graph on $\Gamma_2(l)$.

Let $L_1(l, m)$ be a connected component in $\Gamma_2(l)$ containing $m$ with respect to $\approx$. Let

$$L(l, m) = \{l\} \cup L_1(l, m), \ P(l, m) = \bigcup_{n \in L(l, m)} \Gamma(n), \ \Delta(l, m) = P(l, m) \cup L(l, m).$$

Lemma 3.5 Suppose $\Gamma$ satisfies the condition $\#^L$. Then for $l, m \in L$ with $\partial(l, m) = 2$, $\Delta = \Delta(l, m)$ is a geodetically closed subgraph of $\Gamma$ of diameter 4.

Proof. Since $\Gamma$ satisfies the condition $\#^L$, we have $\partial(m_1, m_2) \leq 2$, if $m_1, m_2 \in T(l, m)$. Hence we can prove the assertion without difficulty.

Let $D = \{\Delta(l, m) \mid \partial(l, m) = 2, l, m \in L\}$.

Corollary 3.6 If $\Gamma$ satisfies the condition $\#^L$, then the following hold.

(1) $L(l, m)$ is a maximal clique in $\Gamma^{(2)}$. 

(2) If $l, m \in \Delta_1 \cap \Delta_2 \cap L$, then $\Delta_1 = \Delta_2$ or $l = m$.

(3) $\Delta$ is a bipartite biregular graph of valencies $q+1$ on $P(l, m)$ and $k^L$ on $L(l, m)$.

(4) $|L(l, m)| = qk^L + 1$.

(5) $|\{\Delta \in D | l \in \Delta\}| = (k^L - 1)/q$ for every $l \in L$.

Let $\Pi$ be a bipartite graph on $L \cup D$ with adjacency defined as follows: For $l \in L$, $\Delta \in D$, $l \in \Delta$ and the valency of $l$ in $\Delta$ is $k^L$. Note that $k^L > q + 1$ as $d(\Gamma) \geq 5$.

**Lemma 3.7** If $\Gamma$ satisfies the condition $\#^L$, then $\Pi$ is a $P(2, q)$-graph of valencies $(k^L - 1)/q$ on $L$ and $qk^L + 1$ on $D$.

**Proposition 3.8** Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five satisfying the condition $\#^L$. Then one of the following holds.

(i) $\Gamma \simeq J_q(d, s, s-1)$, where $k^L = (q^s - 1)/(q - 1)$, $k^P = (q^{d-s+1} - 1)/(q - 1)$, or

(ii) $\Gamma$ is a regular nonbipartite graph of valency $k$ and $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $J_q(2s - 3, s - 2, s - 3)$, where $k = (q^{s-1} - 1)/(q - 1)$. Moreover, if each pair of vertices of $\Gamma$ at distance three is contained in a shortest circuit of odd length, then $q = 1$ and $\Gamma$ is isomorphic to an Odd graph.

*Proof.* Firstly, note that $J_q(d, s, s-1) \simeq J_q(d, d-s+1, d-s)$, if we take the dual interchanging $P$ and $L$.

Suppose $\Gamma$ is bipartite. Since $d(\Gamma) \geq 5$, $k^P, k^L > q + 1$. By Theorem 3.1, (i) holds if $k^P \geq q^2 + q + 1$, using the first remark above.

Assume $k^P < q^2 + q + 1$. Since $\Gamma$ satisfies the condition $\#^L$, $\Pi$ is a $P(2, q)$-graph of valencies $(k^L - 1)/q$ on $L$. Since $(k^L - 1)/q < q + 1$, $\partial_\Pi(l, m) \leq 2$ for all $l, m \in L$. Hence $\partial_\Pi(l, m) \leq 2$ for all $l, m \in L$, which is not the case.

Suppose $\Gamma$ is not bipartite. By the previous lemma, $\Pi$ is a bipartite $P(2, q)$-graph of valencies $(k - 1)/q$ on $L$ and $qk + 1$ on $D$.

Suppose $(k - 1)/q \leq q + 1$. Since $d(\Gamma) \geq 5$, there are vertices $l_0, l_1, l_2, l_3$ such that

$$\partial(l_0, l_1) = \partial(l_1, l_2) = \partial(l_2, l_3) = 2, \quad \partial(l_0, l_2) = 4.$$ 

Since $|\Pi_3(l_0) \cap \Pi_3(l_2)| = q + 1$, $(k - 1)/q = q + 1$ and $\Delta(l_2, l_3) \in \Pi_3(l_0) \cap \Pi_3(l_2)$. So there is a vertex $l \in \Delta(l_2, l_3)$ such that $\partial(l, l_3) = \partial(l_0, l) = 2$. Hence $\partial(l_3, l_0) \leq 4$. In particular $d(\Gamma) = 5$, a5 exists and $a_5 = 0$. Since $\Gamma$ is not bipartite, we may assume that $\partial(l_0, l_3) = 3$. Then $|\Gamma_2(l_2) \cap \Gamma_2(l_0)| = 0$. This is a contradiction.

Thus $(k - 1)/q > q + 1$, $qk + 1 > q^2 + q + 1$. Hence by Theorem 3.1, $\Pi \simeq J_q(d, s, s-1)$, where $qk + 1 = (q^s - 1)/(q - 1)$, $(k - 1)/q = (q^{d-s+1} - 1)/(q - 1)$. 


Therefore \( k = (q^{s-1} - 1)/(q - 1) \) and \( d = 2s - 3 \). Since \( \partial_{\Gamma}(l, m) = 2 \) if and only if \( \partial_{\Pi}(l, m) = 2 \), \( \Gamma^{(2)} \) is isomorphic to a connected component of the distance-2-graph of \( \Pi \) on \( L \).

If \( \Gamma \) satisfies the additional condition in (ii), we can apply Proposition 2.2. If \( q \neq 1 \), then \( \Gamma^{(2)} \) is a Grassman graph, which is also called a q-analogue of Johnson graph. But in this case it is easy to check that the antipode is connected, while it is not a clique. Hence \( q = 1 \) and \( \Gamma^{(2)} \simeq J(2s - 3, s - 2) \). Thus \( \Gamma \) is an Odd graph.

In the following, we investigate the case when \( \Gamma \) does not satisfy \( \#^L \). By symmetry proved in Lemma 3.3, we may assume that \( \Gamma \) does not satisfy \( \#^P \) either. Hence by Lemma 3.4, we need only to consider the case \( k^P, k^L \leq 3q - 1 \).

The key to analyze this case is the following proposition proved by Terwilliger. We kept the notations in [30], where \( M_i \) is no longer a Moore graph.

**Proposition 3.9 ([30])** Let integers \( c, p \) and \( s \) all be at least 2. Suppose the vertices of some graph \( \Gamma \) can be partitioned into \( s + 1 \) disjoint sets \( V\Gamma = \bigcup_{i=0}^{s} M_i \), where for any \( u, v \in V\Gamma, u \in M_i, v \in M_j \) and \( (u, v) \in E\Gamma \) implies \( |i-j| \leq 1 \). For \( i = 1 \) or \( s \), let \( I_i \) and \( L_i \) denote the minimum and maximum number of vertices in \( M_{i-1} \) any vertex in \( M_i \) is adjacent to, and for \( i = 0 \) or \( s - 1 \), let \( R_i \) and \( R_{s-1} \) denote the minimum and maximum number of vertices in \( M_{i+1} \) any vertex in \( M_i \) is adjacent to. Also assume

(i) \( \partial(u, v) = s \) for some \( u \in M_0 \) and \( v \in M_s \),

(ii) for integers \( 0 \leq i, j \leq s \) and for any \( u \in M_i \) and \( v \in M_j \), there are either \( c \) or 0 paths of length \( s \) connecting them if \( |j - i| = s \), and either 0 or 1 paths of length \( |j - i| \) connecting them if \( 1 \leq |j - i| \leq s - 1 \), and

(iii) for any \( u, v \in V\Gamma \) with \( u \in M_1, v \in M_{s-1} \), and \( \partial(u, v) > s - 2 \), there are at most \( p \) paths \( \{u = v_0, v_1, \ldots, v_{s-1}, v_s = v\} \), where either \( v_1 \in M_0 \) or \( v_{s-1} \in M_s \).

Then

\[
\frac{p}{c - 1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1}.
\]

**Proposition 3.10** Let \( \Gamma \) be a \( P(2, q) \)-graph of diameter at least five. If \( c_5^P \) exists, then \( c_5 \) exists, i.e., \( c_5^P = c_5^L \), \( c_5 > q + 1 \) and the following hold.

(1) If \( d(\Gamma) \geq 7 \), then \( c_5 \geq 2q + 1 \).

(2) If \( \alpha, \beta, \gamma \in \Gamma \) with \( \partial(\alpha, \beta) = 8, \partial(\alpha, \gamma) = 3, \partial(\gamma, \beta) = 5 \), then \( k(\gamma) \geq 3q + 2 \).

(3) For \( \alpha \in \Gamma \) let \( j = k(\alpha) - c_5 \). If \( a_4 = 0 \), then

\[
k(\alpha) \geq \frac{2q + j + 3 + \sqrt{4jq^2 + (j-1)^2}}{2}.
\]

In particular, if \( j \geq 4 \), then \( k(\alpha) \geq 3q + 4 \).
Proof. It follows from Proposition 2.1.(2) that $c_5$ exists.
(1) Let $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = 7$. Let

$$M_i = \Gamma_{2+i}(\alpha) \cap \Gamma_{5-i}(\beta), \ i = 0, 1, 2, 3.$$ 

Apply Proposition 3.9.
(2) Since $d \geq 8$, we can apply (1). We have

$$k(\gamma) \geq c_3(\alpha, \gamma) + c_5(\beta, \gamma) \geq 3q + 2.$$ 

(3) Let $\alpha \in \Gamma$ and $M_i = \Gamma_{i+2}(\alpha), \ i = 0, 1, 2, 3$. Apply Proposition 3.8.

We now summarize our results in this section, from which we have Theorem 1.3 as a corollary.

Theorem 3.11 Let $\Gamma$ be a $P(2, q)$-graph of diameter at least five. Suppose $c_5$ exists. Then $\Gamma$ is a bipartite biregular graph of valencies $k^P$ and $k^L$, or a regular graph of valency $k = k^P = k^L$ and one of the following holds.

(i) $\Gamma \simeq J_q(d, s, s-1)$, where $k^L = (q^s - 1)/(q - 1)$, $k^P = (q^{d-s+1}-1)/(q-1)$,

(ii) $\Gamma$ is a regular nonbipartite graph of valency $k$ and the distance-2-graph $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $J_q(2s-3, s-2, s-3)$, where $k = (q^{s-1}-1)/(q-1)$. Moreover, if each pair of vertices of $\Gamma$ at distance three is contained in a shortest circuit of odd length, then $q = 1$ and $\Gamma$ is isomorphic to an Odd graph; or

(iii) $d(\Gamma) \leq 7$ and $k^P, k^L \leq 3q - 1, q \neq 1$. Moreover if $a_4 = 0$, then $\Gamma$ is bipartite and $k^P - c_5, k^L - c_5 \leq 3$. In particular, if $\Gamma$ is not bipartite and $a_4$ exists, then $d(\Gamma) \leq 6$.

Corollary 3.12 Let $\Gamma$ be a distance-regular graph of valency $k$. Suppose $c_2 = 1$, $c_3 = c_4 = q + 1$ and $a_1 = a_2 = a_3 = 0$ for some positive integer $q$. Then one of the following holds.

(i) $\Gamma \simeq J_q(2s-1, s-2, s-3)$, where $k = (q^s - 1)/(q - 1)$.

(ii) $\Gamma \simeq O_k$, an Odd graph of valency $k$; or

(iii) $d(\Gamma) \leq 7$, and the equality holds only if $\Gamma$ is bipartite.

Koolen [20] conjectured the following:

If $\Gamma$ is a distance-biregular graph of diameter at least 5 such that $c_i$ exists for all $i$, and $c_2 = 1, c_3 = c_4 > 2$, then $\Gamma \simeq J_q(d, s, s - 1)$. 

Our results asserts that \( d(\Gamma) \leq 7 \) and the parameters are restricted very much. It is known that if \( d(\Gamma) = 5 \) or 7, then \( \Gamma \) is distance-regular, under the assumption of the conjecture above. See [9, 20].

We also note that for \( d(\Gamma) = 5 \), the doubled Moore graph satisfy the hypothesis with \( c_5 = q+2 \). Moreover if it’s valency is not 3, say 7, then it does not come from \( J_q(d, s, s-1) \). So this gives a counter example to the conjecture above.

4 \( P(r, 1) \)-graphs

According to the remark following Lemma 3.3, a \( P(r, 1) \)-graph is a connected graph \( \Gamma \), which is either a bipartite biregular graph with a bipartition \( P \cup L \) or a nonbipartite regular graph such that

\[
c_1 = \cdots = c_r = 1, \ a_1 = \cdots = a_{r+1} = 0, \ c_{r+1} = c_{r+2} = 2,
\]

where \( r \) is an even positive integer. In this section we study \( P(r, 1) \)-graphs and we show the following when \( r = 4 \). We do not know any \( P(r, 1) \)-graphs with \( r > 4 \).

Theorem 4.1 Let \( \Gamma \) be a \( P(4, 1) \)-graph of diameter at least four and \( \alpha, \gamma \in \Gamma \) with \( \partial(\alpha, \gamma) = 4 \). Then there is a geodetically closed subgraph \( \Delta \) containing \( \alpha, \gamma \) isomorphic to \( 2M_k(\alpha) \). Here \( k(\alpha) \) denotes the valency of \( \alpha \) in \( \Gamma \). In particular, \( k(\alpha) \in \{2, 3, 7, 57\} \).

Let \( \Gamma \) be a \( P(r, 1) \)-graph with \( r \geq 4 \).

Fix a vertex \( \alpha \in \Gamma \). For \( \gamma, \delta \in \Gamma_r(\alpha) \), we write \( \gamma \approx \delta \) if \( \partial(\gamma, \delta) = 2 \) and \( C(\gamma, \delta) \subset \Gamma_{r+1}(\alpha) \). For \( \gamma \in \Gamma_r(\alpha) \), let \( C = C_\gamma \) be the connected component in \( \Gamma_r(\alpha) \) containing \( \gamma \) with respect to the relation \( \approx \). Let \( \Pi = \Pi_\gamma \) be a graph on \( C_\gamma \) defined by the relation \( \approx \). For \( \gamma, \delta \in \Gamma \) with \( \partial(\gamma, \delta) = r \), and \( 0 \leq i \leq r \), let

\[
\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).
\]

For \( \delta \in \Gamma_r(\alpha) \), let

\[
\alpha(\delta) = g_1(\delta, \alpha), \ \beta(\delta) = g_2(\delta, \alpha), \ \text{and} \ \gamma(\delta) = g_4(\delta, \alpha).
\]

Firstly we note that the intersection diagram with respect to \( x, l \) with \( \partial(x, l) = 1 \) has the following shape, where \( D_j^i = \Gamma_i(x) \cap \Gamma_j(l) \). See the properties (a) \( \sim \) (e) below.

\[
\begin{align*}
\{x\} &= D^0_1 - \cdots - D^r_1 - D^r_{r+1} - D^{r+1}_{r+2} - \\
\{l\} &= D^1_0 - \cdots - D^r_{r-1} - D^r_{r+1} - D^{r+2}_{r+1} - \\
\end{align*}
\]

Figure 2.
(a) $D^i_i = \emptyset$, for $1 \leq i \leq r + 1$.

(b) For $y \in D^1_1$, $z \in D^2_1$, $e(y, D^1_1) = e(z, D^1_1) = 2$.

(c) For $y \in D^2_1$, $z \in D^3_1$, $e(y, D^2_1) = e(z, D^2_1) = 1$.

(d) For $y \in D^3_1$, $z \in D^4_1$, $e(y, D^3_1) = e(z, D^3_1) = 1$.

(e) $e(D^i_i, D^i_{i+1}) = 0$, $1 \leq i \leq r - 1$ and $i = r + 1$.

The following two lemmas are related to circuit chasing technique. See [4, 13, 14] and [5, Section 5.10].

**Lemma 4.2** Let $x_0 \sim x_1 \sim \cdots \sim x_{2r+2t} = x_0$ be a circuit of length $2r + 2t$. i.e., a closed path and $x_{i-1} \neq x_{i+1}$, $i = 1, \ldots, 2r + 2t - 1$ and $x_{2r+2t-1} \neq x_1$. Suppose

$x_r, x_{r+2}, \ldots, x_{r+2t} \in \Gamma_r(x_0), x_{r+1}, x_{r+3}, \ldots, x_{r+2t-1} \in \Gamma_{r+1}(x_0)$.

Set $D^i_i = \Gamma_i(x_0) \cap \Gamma_j(x_1)$. Then the following hold.

1. $t \geq 1$ and $x_r \in D^r_{r-1}, x_{r+1} \in D^r_1, x_{r+2} \in D^r_{r+1}$.
2. If $t \geq 2$, then $x_{r+3} \in D^{r+1}_{r+2}$ and $x_{r+4} \in D^{r+1}_{r+1}$.
3. If $t = 2$, then the mutual distance of the vertices in the circuit is uniquely determined.
   In particular,
   $$\partial(x_2, x_{r+1}) = \partial(x_2, x_{r+4}) = r, \partial(x_2, x_{r+5}) = r + 1.$$
4. If $t = 3$, then $x_{r+5} \in D^{r+1}_{r+2}$, $x_{r+6} \in D^{r+1}_{r+1}$ and
   $$\partial(x_2, x_{r+1}) = \partial(x_2, x_{r+4}) = \partial(x_2, x_{r+6}) = r, \partial(x_4, x_{r+5}) = \partial(x_4, x_{r+7}) = r + 1.$$

**Proof.** In the following, we use (a) $\sim$ (e) to determine the locations of $x_j$'s in the diagram with respect to an edge $x_{i-1} \sim x_i$, using the information on the distances from $x_{i-1}$.

1. Since $x_{i-1} \neq x_{i+1}$, for all $i$, and $c_1 = \cdots = c_r = 1$, $t \geq 1$. It is clear that $x_r \in D^r_{r-1}$. Since $x_{r+1} \in \Gamma_{r+1}(x_0) \cap \Gamma(x_r)$, $x_{r+1} \in D^{r+1}_r$. $x_r \neq x_{r+2} \in \Gamma_r(x_0) \cap \Gamma(x_{r+1})$ implies that $x_{r+2} \in D^{r+1}_{r+1}$.
2. Since $x_{r+2} \in D^{r+1}_r$ and $e(x_{r+2}, D^{r+1}_r) = 1$ with $x_{r+1} \in D^{r+1}_r \cap \Gamma(x_{r+2})$, $x_{r+3} \in D^{r+1}_{r+2}$, $x_{r+4} \in D^{r+1}_{r+1}$.
3. It is easy to determine the mutual distances as follows.

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<tr>
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<th>$x_r$</th>
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<th>$x_{r+2}$</th>
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<td>$r + 1$</td>
<td>$r$</td>
<td>$r + 1$</td>
<td>$r$</td>
<td>$r - 1$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$r - 1$</td>
<td>$r$</td>
<td>$r + 1$</td>
<td>$r + 2$</td>
<td>$r + 1$</td>
<td>$r$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$r - 2$</td>
<td>$r - 1$</td>
<td>$r$</td>
<td>$r + 1$</td>
<td>$r$</td>
<td>$r + 1$</td>
</tr>
</tbody>
</table>
Now the distance pattern with respect to \( x_2 \) is the same as that with respect to \( x_0 \), the mutual distance of the vertices in the circuit is uniquely determined and the assertion follows.

(4) We do the same as in (3).

\[
\begin{array}{cccccccccccc}
\delta & x_r & x_{r+1} & x_{r+2} & x_{r+3} & x_{r+4} & x_{r+5} & x_{r+6} & x_{r+7} & x_{r+8} & x_{r+9} \\
x_0 & r & r+1 & r & r+1 & r & r+1 & r & r-1 & r-2 & r-3 \\
x_1 & r-1 & r & r+1 & r+2 & r+1 & r & r-1 & r-2 \\
x_2 & r-2 & r-1 & r & r+1 & r & r+1 & r & r-1 \\
x_3 & r-3 & r-2 & r-1 & r & r+1 & r+2 & r+1 & r \\
x_4 & r-4 & r-3 & r-2 & r-1 & r & r+1 & r & r+1 \\
\end{array}
\]

Note that since \( x_{r+7} \in D_r^{r-1}, x_{r+5} \) cannot be in \( D_r^{r+1} \).

Lemma 4.3 Let \( y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4 \) be a path of length four such that \( y_{i-1} \neq y_{i+1}, i = 1, \ldots, 3 \). Suppose \( y_0, y_4 \in \Gamma_r(\alpha) \). Then one of the following holds.

(i) \( y_2 \in \Gamma_{r-2}(\alpha) \),

(ii) \( y_1 \in \Gamma_{r-1}(\alpha) \) or \( y_3 \in \Gamma_{r-1}(\alpha) \) and \( \alpha(y_0) \neq \alpha(y_4) \),

(iii) \( y_1, y_3 \in \Gamma_{r+1}(\alpha), y_2 \in \Gamma_r(\alpha) \) and \( \alpha(y_0) \neq \alpha(y_4) \),

(iv) \( y_2 \in \Gamma_{r+2}(\alpha) \) and \( \alpha(y_0) = \alpha(y_4) \), while \( \beta(y_0) \neq \beta(y_4) \), or

(v) \( y_2 \in \Gamma_{r+2}(\alpha) \) and \( \alpha(y_0) \neq \alpha(y_4) \), \( \partial(\beta(y_0), y_4) = r+2 \).

By Lemma 4.2 and 4.3, we can prove the following concerning the connected component in \( \Gamma_r(\alpha) \) with respect to \( \sim \).

Lemma 4.4 Let \( \{\alpha_1, \ldots, \alpha_{k(\alpha)}\} = \Gamma(\alpha), \gamma \in \Gamma_r(\alpha), C = C_\gamma \). Let \( S_i = \{\delta \in C|\alpha(\delta) = \alpha_i\} \). Then the following hold.

(1) For \( \delta \in S_i, |\Pi(\delta) \cap S_j| = 1 - \delta_{i,j} \) and \( S_i \subset \Gamma_{r-2}(\beta(\delta)) \). In particular, \( \Pi \) is a \( k(\alpha) \)-partite \( (k(\alpha) - 1) \)-regular graph.

(2) Let \( \delta_0 \sim \delta_1 \sim \delta_2 \sim \delta_3 \) be a path in \( \Pi \). If \( \alpha(\delta_0) \neq \alpha(\delta_3) \), then there exists \( \delta_4 \in \Pi(\delta_3), \delta_5 \in \Pi(\delta_4) \) such that \( \gamma(\delta_0) = \gamma(\delta_5) \).

If \( r = 4 \), \( \gamma(\delta) = \delta \) for every \( \delta \in \Pi \). So by Lemma 4.4, we have the following.

Lemma 4.5 If \( r = 4 \), then the following holds.

(1) If \( \delta_0 \sim \delta_1 \sim \delta_2 \sim \delta_3 \) and \( \alpha(\delta_0) \neq \alpha(\delta_3) \), then there exists \( \delta_4 \) such that \( \delta_0 \sim \delta_4 \sim \delta_3 \).
(2) If \(\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3\) and \(\alpha(\delta_0) = \alpha(\delta_2)\), then \(\beta(\delta_0) = \beta(\delta_3)\).

(3) If \(\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4\) with \(\alpha(\delta_0) = \alpha(\delta_3), \alpha(\delta_1) = \alpha(\delta_4)\), then there exists \(\delta_5\) such that \(\delta_0 \approx \delta_5 \approx \delta_4\).

(4) \(d(\Pi) \leq 3\) and if \(\partial(\delta, \delta') = 3\), then \(\beta(\delta) = \beta(\delta')\).

Proof. (1) Since \(\gamma(\delta) = \delta\) for every \(\delta \in \Pi\), (1) is a direct consequence of Lemma 4.4.(2).

(2) This follows from Lemma 4.4.(1).

(3) By (2), \(\beta(\delta_0) = \beta(\delta_3) \neq \beta(\delta_1) = \beta(\delta_4)\). Now \(\delta_3, \beta(\delta_1) \in \Gamma_4(\delta_0)\), and there is a path of length 4,
\[y_0 = \delta_3 \sim y_1 \sim y_2 = \delta_4 \sim y_3 \sim y_4 = \beta(\delta_1),\]
where \(y_1 \in C(\delta_3, \delta_4), y_3 = g_1(\alpha, \delta_4)\).

It is easy to check that \(y_1, y_3 \in \Gamma_5(\delta_0)\) and that \(g_1(\delta_3, \delta_0) \neq g_1(\beta(\delta_1), \delta_0)\). Hence by Lemma 4.3.(iii) or (v) occurs.

If (v) occurs, \(\partial(\beta(\delta_0), \delta_4) = 6\), which is not the case. Hence \(\partial(\delta_0, \delta_4) = 4\).

Let \(\delta_0 = z_0 \sim z_1 \sim z_2 \sim z_3 \sim z_4 = \delta_4\) be a path connecting \(\delta_0\) and \(\delta_4\). Then by Lemma 4.3, we have (iii) as \(\partial(\beta(\delta_0), \delta_4) = 4\). Hence we can set \(z_2 = \delta_5\).

(4) This follows from (1), (2) and (3).

Proof of Theorem 4.1. Let \(r = 4\) and
\[
L(\alpha, \gamma) = \{\alpha\} \cup \bigcup_{\delta \in C_\gamma} (\Gamma_2(\alpha) \cap \Gamma_2(\delta)) \cup C_\gamma,
\]
\[
P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_1(\delta),
\]
\[
\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)
\]
In this definition we also write \(P(\Delta) = P(\alpha, \gamma)\), and \(L(\Delta) = L(\alpha, \gamma)\).

We shall show in the sequel that \(\Delta\) is a geodetically closed subgraph isomorphic to \(2M_{k(\alpha)}\).

Let \(\gamma = \gamma_1\) and \(\{\gamma_2, \ldots, \gamma_{k(\alpha)}\} = \Pi(\gamma)\). Thanks to Lemma 4.4,
\[
L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_{k(\alpha)})\} \cup C_\gamma.
\]

By Lemma 4.5, the distance-2-graph induced on \(L(\Delta)\) is of diameter 2 and geodetically closed.

If \(k(\alpha) = 2\), there is nothing to prove. Assume \(k(\alpha) > 2\).

\(\partial(\beta(\gamma), \gamma_2) = 4\) and
\[
\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k(\alpha)-1}\} \subset \Gamma_4(\beta(\gamma)),
\]
there is a vertex \( \delta_i' \in \Pi(\delta_i) \cap \Gamma_2(\beta(\gamma)) \) for each \( i \). Since the girth of \( \Gamma \) is 10, we can conclude that the valency of \( \beta(\gamma) \) in the distance-2-graph induced on \( L(\Delta) \) equals \( k(\alpha) \). By Lemma 4.5, this means that the valency of vertex in \( P(\Delta) \) is 2.

Now we can conclude that \( \Delta \) is geodetically closed subgraph of \( \Gamma \) isomorphic to \( 2M_{k(\alpha)} \) easily.

This completes the proof of Theorem 4.1.

We remark that in the final step, we can also apply [5, Theorem 1.17.1] to determine the regularity of the distance-2-graph induced on \( L(\Delta) \). See the proof of [5, Proposition 4.3.11].

5 Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. We can follow the proof in the previous section step by step, replacing each path of length 2 by a path of length 3.

Let \( \Gamma \) be a graph satisfying the hypothesis in Theorem 1.5.

Fix a vertex \( \alpha \in \Gamma \). For \( \gamma, \delta \in \Gamma_\alpha(\alpha) \), we write \( \gamma \approx \delta \) if \( \partial(\gamma, \delta) = 3 \). Then \( C(\gamma, \delta) \cup C(\delta, \gamma) \subset \Gamma_{r+1}(\alpha) \). For \( \gamma \in \Gamma_\alpha(\alpha) \), let \( C = C_\gamma \) be the connected component in \( \Gamma_\alpha(\alpha) \) containing \( \gamma \) with respect to the relation \( \approx \). Let \( \Pi = \Pi_\gamma \) be a graph on \( C_\gamma \) defined by the relation \( \approx \). Hence \( C \) is a connected component of the distance-3-graph of \( \Gamma \) induced on the set \( \Gamma_\alpha(\alpha) \).

For \( \gamma, \delta \in \Gamma \) with \( \partial(\gamma, \delta) = r \), and \( 0 \leq i \leq r \), let
\[
\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).
\]

For \( \delta \in \Gamma_\alpha(\alpha) \), let
\[
\alpha(\delta) = g_1(\delta, \alpha), \alpha'(\delta) = g_2(\delta, \alpha), \beta(\delta) = g_3(\delta, \alpha) \quad \text{and} \quad \gamma(\delta) = g_6(\delta, \alpha).
\]

Firstly we note that the intersection diagram with respect to \( x, y \) with \( \partial(x, y) = 1 \) has the following shape, where \( D_i^j = \Gamma_i(x) \cap \Gamma_j(y) \). See the properties (a) \( \sim (g) \) below.

(a) \( D_i^r = \emptyset \) for \( 1 \leq i \leq r \).
(b) For \( y \in D_i^{i+1}, z \in D_{i+1}^i, e(y, D_i^{i-1}) = e(z, D_i^{i-1}) = 1, 1 \leq i \leq r + 2. \)
(c) For \( y \in D_{i}^{i+1}, z \in D_{i+1}^{i}, e(y, D_{i}^{i+1}) = e(z, D_{i+1}^{i}) = 0, 1 \leq i \leq r \) and \( e(y, D_{i}^{i+1}) = e(z, D_{i+1}^{i}) = 1, i = r + 1, r + 2. \)

(d) For \( y \in D_{r+1}^{r+1}, e(y, D_{r}^{r+1}) = e(y, D_{r+1}^{r}) = 1 \) and \( e(y, D_{r+1}^{r+1}) = 1, i = r + 2. \)

(e) For \( y \in D_{r+1}^{r+1}, z \in D_{r+1}^{r+1}, e(y, D_{r+1}^{r+1}) = e(z, D_{r+1}^{r+1}) = 1. \)

(f) For \( y \in D_{r+2}^{r+2}, e(y, D_{r+1}^{r+2}) = e(y, D_{r+2}^{r+2}) = 1. \)

(g) \( e(D_{i}^{i+1}, D_{i+1}^{i}) = 0, 1 \leq i \leq r + 2. \)

We again apply circuit chasing technique.

**Lemma 5.1** Let \( x_{0} \sim x_{1} \sim \cdots \sim x_{2r+3t} = x_{0} \) be a circuit of length \( 2r + 3t \). i.e., a closed path and \( x_{i-1} \neq x_{i+1}, i = 1, \ldots, 2r + 3t - 1 \) and \( x_{2r+3t-1} \neq x_{1} \). Suppose \( x_{r}, x_{r+3}, \ldots, x_{r+3t} \in \Gamma_{r}(x_{0}), x_{r+1}, x_{r+2}, x_{r+4}, \ldots, x_{r+3t-2}, x_{r+3t-1} \in \Gamma_{r+1}(x_{0}). \)

Set \( D_{j}^{i} = \Gamma_{i}(x_{0}) \cap \Gamma_{j}(x_{1}) \). Then the following hold.

(1) \( t \geq 1 \) and \( x_{r} \in D_{r-1}^{r}, x_{r+1} \in D_{r}^{r+1}, x_{r+2} \in D_{r+1}^{r} \) and \( x_{r+3} \in D_{r+1}^{r+1}. \)

(2) If \( t \geq 2 \), then \( x_{r+4}, x_{r+5} \in D_{r+2}^{r+1} \) and \( x_{r+6} \in D_{r+1}^{r+1}. \)

(3) If \( t = 2 \), then the mutual distance of the vertices in the circuit is uniquely determined. In particular, \( r \equiv 0 \pmod{3} \), and

\[
\partial(x_{3}, x_{r+3}) = \partial(x_{3}, x_{r+6}) = r, \partial(x_{3}, x_{r+7}) = r + 1.
\]

(4) Suppose \( r \geq 6 \). If \( t = 3 \), then \( x_{r+7}, x_{r+8} \in D_{r+2}^{r+2}, x_{r+9} \in D_{r+1}^{r+1} \) and

\[
\partial(x_{3}, x_{r+6}) = \partial(x_{3}, x_{r+9}) = \partial(x_{6}, x_{r+9}) = r, \partial(x_{6}, x_{r+8}) = \partial(x_{6}, x_{r+10}) = r + 1.
\]

**Lemma 5.2** Let \( y_{0} \sim y_{1} \sim y_{2} \sim y_{3} \sim y_{4} \sim y_{5} \sim y_{6} \) be a path of length 6 such that \( y_{i-1} \neq y_{i+1}, i = 1, \ldots, 5 \). Suppose \( y_{0}, y_{6} \in \Gamma_{r}(\alpha) \). Then one of the following holds.

(i) \( y_{3} \in \Gamma_{r-3}(\alpha), \)

(ii) \( y_{1}, y_{2}, y_{4}, y_{5} \in \Gamma_{r+1}(\alpha), y_{3} \in \Gamma_{r}(\alpha) \) and \( \alpha(y_{0}) \neq \alpha(y_{6}), \)

(iii) \( y_{3} \in \Gamma_{r+2}(\alpha) \) and \( y_{5} \in \Gamma_{r+1}(\alpha) \cap \Gamma_{r+1}(\alpha(y_{0})), \) while \( \partial(\beta(y_{0}), y_{5}) \geq r + 1. \)

**Lemma 5.3** Let \( \{\alpha_{1}, \ldots, \alpha_{k}\} = \Gamma(\alpha), \gamma \in \Gamma_{r}(\alpha), C = C_{\gamma}. \) Let \( S_{i} = \{\delta \in C|\alpha(\delta) = \alpha_{i}\}. \)

Then the following hold.

(1) For \( \delta \in S_{i}, |\Pi(\delta) \cap S_{j}| = 1 - \delta_{i,j} \) and \( S_{i} \subset \Gamma_{r-3}(\beta(\delta)). \) In particular, \( \Pi \) is a \( k \)-partite \((k - 1)\)-regular graph.
(2) Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in $\Pi$. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.

**Lemma 5.4** If $r = 6$, then the following holds.

1. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4$ such that $\delta_0 \approx \delta_4 \approx \delta_3$.

2. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) = \alpha(\delta_3)$, then $\beta(\delta_0) = \beta(\delta_3)$.

3. If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ with $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, then there exists $\delta_5$ such that $\delta_0 \approx \delta_5 \approx \delta_4$.

4. $d(\Pi) \leq 3$ and if $\partial_{\Pi}(\delta, \delta') = 3$, then $\beta(\delta) = \beta(\delta')$.

**Proof of Theorem 1.5.** Suppose $r = 3$. Let

\[
L(\alpha, \gamma) = \{\alpha\} \cup C_{\gamma},
\]
\[
P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_{1}(\delta),
\]
\[
\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)
\]

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$. Clearly $L(\Delta)$ is a maximal clique in the distance-3-graph of $\Gamma$, and the assertion follows easily from Lemma 5.3.

Let $r = 6$ and

\[
L(\alpha, \gamma) = \{\alpha\} \cup \bigcup_{\delta \in C_{\gamma}} (\Gamma_{3}(\alpha) \cap \Gamma_{3}(\delta)) \cup C_{\gamma},
\]
\[
P(\alpha, \gamma) = \bigcup_{\delta \in L(\alpha, \gamma)} \Gamma_{1}(\delta),
\]
\[
\Delta = \Delta(\alpha, \gamma) = P(\alpha, \gamma) \cup L(\alpha, \gamma)
\]

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$.

We shall show in the sequel that $\Delta$ is a geodetically closed subgraph isomorphic to $3M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_k\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

\[
L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_k)\} \cup C_{\gamma}.
\]

By Lemma 5.4, the distance-3-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

$\partial(\beta(\gamma), \gamma_2) = 6$ and

\[
\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k-1}\} \subset \Gamma_{6}(\beta(\gamma)),
\]
there is a vertex $\delta'_i \in \Pi(\delta_i) \cap \Gamma_3(\beta(\gamma))$ for each $i$. Since the girth of $\Gamma$ is 15, we can conclude that the valency of $\beta(\gamma)$ in the distance-3-graph induced on $L(\Delta)$ equals $k$. By Lemma 5.4, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that $\Delta$ is geodetically closed easily.

This completes the proof of Theorem 1.5.

6 Concluding Remarks

It may be too optimistic to expect a classification of $P(r, q)$-graphs or the graphs similar to those discussed in the previous section in the near future. But we believe that the investigation of such graphs plays a key role to give an absolute bound of the girth of distance-biregular graphs or distance-regular graphs.

We list several problems, which we want to see solved.

1. Study geodetically closed subgraphs of distance-regular graphs and prove results corresponding to Proposition 2.3 and Theorem 2.6, especially when $a_1 \neq 0$. See [20].

2. Classify $P(r, q)$-graphs.

a) For $r = 2$, it may be possible to improve Lemma 3.4 to have $2q$ as the lower bound. Then we have $d \leq 5$, by Proposition 3.10.

b) For $q = 1$, the classification implies a classification of distance-biregular graphs with vertices of valency three, [26]. Hence we can obtain an absolute diameter bound of distance-regular graphs of order $(s, 2)$, i.e., those with $\Gamma(x) \simeq 3 \cdot K_s$. See [17, 3, 15, 31].

3. Let $\Gamma$ be a bipartite biregular graph with a bipartition $P \cup L$, or a regular graph with $\Gamma = P = L$. For a positive integer $q$ and a positive odd integer $r$, we call $\Gamma$ a $P(r, q)$-graph, if it is a connected graph such that

$$c_1^P = \cdots = c_r^P = 1, \quad a_1 = \cdots = a_{r+1} = 0, \quad c_{r+1}^P = q + 1 \quad \text{and} \quad c_{r+1}^L = c_{r+2}^P.$$

Classify them. If $q = 1$, then $\Gamma$ is a thin generalized polygon by a result in [26].

4. Study a distance-regular graphs $\Gamma$ with $r = r(\Gamma)$, $c_{r+1} = c_{r+2} = 1$, and clarify the correspondence with $P(r, q)$-graphs. In particular, show $r \leq 6$ in Theorem 1.5.

5. Let $\Gamma$ be a connected graph of diameter $d$. For a subset $I \subset \{1, \ldots, d\}$, let $\Gamma^{(I)}$ denote the distance-$I$-graph, i.e., $V(\Gamma^{(I)}) = V(\Gamma)$, and $\alpha, \beta$ are adjacent in $\Gamma^{(I)}$ if and only if $\partial(\alpha, \beta) \in I$. Study $\Gamma$ such that at least one of the connected components of $\Gamma^{(I)}$ is distance-regular of diameter at least three. To start with, assume $\Gamma^{(I)}$
is connected. It is not hard to determine parametrical conditions if $\Gamma$ itself is a distance-regular graph. In particular, classify distance-regular graphs $\Gamma$ such that $\Gamma^{(2)}$ is distance-regular of diameter $d(\Gamma) \neq d(\Gamma^{(2)}) \geq 3$. See Proposition 3.2 and [27, 29].

6. Give a geometrical classification of Moore graphs. One of the reasons, we could not obtain the results for $P(r, 1)$-graphs with $r \geq 6$, is a lack of such classification. We believe that this is one of the keys when we develop structure theories of distance-regular graphs just as the group theoretical proof of Burnside's $p^aq^b$ theorem gave a breakthrough to the classification of finite simple groups.

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