Classification of Small Spin Models

Eiichi Bannai        Attila Sali*
Kyushu University, Faculty of Sci.,
Department of Mathematics
Fukuoka, Higashi-ku, Hakozaki 6-10-1
Postal No. 812 Japan

François Jaeger
Laboratoire de Structures Discretes et de Didactique
Institut IMAG
BP 53X, 38041 Grenoble Cedex, France

Abstract

Symmetric spin models were introduced by Jones to provide invariants of links. In his paper he proposed to obtain the classification of models \((X, w_+, w_-)\) with \(|X| = 4, 5, 6\) and 7. In the present paper we complete this task by showing that the only spin models of these sizes are the Potts models and the ones coming from cyclic groups for \(5 \leq n \leq 7\). For \(n = 4\) we have some other models by the product construction of de la Harpe. Furthermore, we classify the non-symmetric Jones-type spin models introduced by Munemasa and Watatani, for \(n = 4, 5\), as well.
1 Introduction

Symmetric spin models were introduced by Jones[6] to obtain link invariants for non-oriented links. Later Jaeger[5] and de la Harpe[4] developed the connection between these models and association schemes. Recently several constructions were given; see Bannai and Bannai[1], Nomura[8] and Mune-masa and Watatani[7]. Even more recently, Bannai and Bannai have found far reaching generalization of the concept[2].

Definition 1.1 Let $a$ be a non-zero complex number, $n$ be a positive integer, and $D$ be one of its square roots. A spin model with loop variable $D$ and modulus $a$ is a triple $(X, w_+, w_-)$, where $X$ is a finite set of size $n = D^2$ and $w_+, w_-$ are complex-valued functions on $X \times X$ which satisfy the following properties for all $\alpha, \beta, \gamma$ in $X$:

1. $w_+(\alpha, \alpha) = a$, $w_- (\alpha, \alpha) = a^{-1}$.
2. $\sum_{x \in X} w_+(\alpha, x) = Da^{-1}$, $\sum_{x \in X} w_-(\alpha, x) = Da$.
3. $w_+(\alpha, \beta)w_-(\beta, \alpha) = 1$.
4. $\sum_{x \in X} w_+(\alpha, x)w_-(x, \beta) = n\delta_{\alpha, \beta}$ (where $\delta$ is the Kronecker symbol),
5. $\sum_{x \in X} w_+(\alpha, x)w_+(\beta, x)w_-(\gamma, x) = Dw_+(\alpha, \beta)w_-(\beta, \gamma)w_-(\gamma, \alpha)$.

The spin model is called symmetric if $w_+(\alpha, \beta) = w_+(\beta, \alpha)$, $w_-(\alpha, \beta) = w_-(\beta, \alpha)$ holds, as well.

The above definition can be reformulated using $n \times n$ matrices $W_+$ and $W_-$ see [5]. Let $W_{\pm} = (w_{\pm}(\alpha, \beta))_{\alpha \in X, \beta \in X}$ and let $\circ$ denote the Hadamard product of matrices (i.e., the entry-wise product of two matrices of the same size). Furthermore, let us define for $(\beta, \gamma) \in X \times X$ the column vector $Y_{\beta\gamma}$ indexed by $X$ as

$$Y_{\beta\gamma}(x) = w_+(\beta, x)w_-(\gamma, x) \quad \forall x \in X.$$ 

Proposition 1.2 $(X, w_+, w_-)$ is a spin model with loop variable $D$ and modulus $a$ if and only if the following properties hold:

1. $I \circ W_+ = aI$, $I \circ W_- = a^{-1}I$. 

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2. $JW_+ = Da^{-1}J$, $JW_- = DaJ$.

3. $W_+ \circ W_-^\top = nJ$.

4. $W_+ W_- = nI$.

5. For every $(\beta, \gamma) \in X \times X$, $W_+ Y_{\beta\gamma} = DW_- (\beta, \gamma) Y_{\beta\gamma}$.

Here $I$ denotes the identity matrix and $J$ denotes the matrix whose entries are all 1's.

For a symmetric spin model we have to require $W_+$ and $W_-$ to be symmetric. We shall use an other interpretation, too. Namely, we can write

$$W_+ = \sum_{i=0}^{t} a_i A_i,$$

where $a_0$ is the modulus of the model, $A_0 = I$, $a_i \neq a_j$ for $1 \leq i < j \leq t$ and $A_i$'s are adjacency matrices of edge-disjoint simple digraphs on vertex set $X$. Thus, $A_i \circ A_j = \delta_{ij} A_i$. We denote the graph whose adjacency matrix is $A_i$ by $G_i$. The above determined $t$ is called the degree of the model $(X, W_+, W_-)$.

Let us mention that the case when $(X, A_i; 0 \leq i \leq t)$ is an association scheme is interesting for its own sake.

The following is a fundamental result concerning the classification problem.

**Lemma 1.3** ([6], Proposition 2.16.) *For each $z \in \mathbb{C}$ let $k_z$ be the number of ordered pairs $(\alpha, \beta)$ for which $w_+(\alpha, \beta) = z$. Then $k_z$ is a multiple of $n$.***

As an immediate consequence of this lemma we obtain that the degree of a spin model on a $n$ element set is at most $n - 1$. However, we shall need a stronger result in order to reduce the number of cases to be checked. This is Theorem 2.1 of the next section.

It is known [1], that there exist spin model coming from the cyclic group $C_n$ for any $n$. The aim of this paper is to prove the following theorems.
Theorem 1.4 Let \((X, W_+, W_-)\) be an \(n \times n\) symmetric spin model. Furthermore, let us assume that it is of degree \(t > 1\), i.e.,

\[
W_+ = a_0 I + a_1 A_1 + \ldots + a_t A_t.
\]

Then \((X, \{A_i\}_{0 \leq i \leq t})\) is the (symmetric) association scheme coming from the cyclic group \(C_n\) for \(5 \leq n \leq 7\). In particular, we have \(t = \lfloor \frac{n}{2} \rfloor\). If \(n = 4\) the model is either coming from the cyclic group or it is a product of two Potts models on 2 spins.

Theorem 1.5 Let \((X, W_+, W_-)\) be an \(n \times n\) non-symmetric spin model of Jones type. Then it can be written as

\[
W_+ = a_0 I + a_1 A_1 + \ldots + a_n A_n,
\]

where \((X, \{A_i\}_{0 \leq i \leq n})\) is the association scheme coming from the cyclic group \(C_n\) for \(n = 4, 5\).

In Theorem 1.5 we do not state that \(a_i\)'s are all different. All possible solutions will be given in a subsequent paper by Bannai, Bannai and Jaeger.

In Section 2 we formulate general results and in Section 3 we turn to the symmetric case, while Section 4 deals with non-symmetric models. We will use the three possible interpretations simultaneously, always switching to the one which is most convenient to formulate the statement in question.

We consider two spin models equivalent if one can be obtained from the other by simultaneous permutation of rows and columns (i.e. keeping diagonal elements in the diagonal). In the graph representation this means simultaneous renumbering of the vertices of each graph. Furthermore, we allow renumbering the \(G_i\)'s when we use the graph interpretation.

2 General results

In this section we prove a strengthening of Lemma 1.3 as follows.

Theorem 2.1 Let \((X, W_+, W_-)\) be a spin model of degree \(t\). Then each point in \(G_i\) has in-degree and out-degree \(k_i\) for \(i = 1, 2, \ldots, t\) where \(k_i \in \mathbb{N}\). In other words, \(J A_i = A_i J = k_i J\).
Proof of Theorem 2.1
Let \( X = \{1, 2, \ldots, n\} \) and let \( \Delta_{\gamma} = \text{diag}(w_{-}(1, \gamma), w_{-}(2, \gamma), \ldots, w_{-}(n, \gamma)) \) for \( \gamma \in X \). We claim that
\[
W_{+} \Delta_{\gamma} W_{+} = D \Delta_{\gamma} W_{+} \Delta_{\gamma}, \quad \forall \gamma \in X.
\] (1)
Indeed, the \((\alpha, \beta)\)-entry of the left hand side of (1) is
\[
\sum_{x \in X} w_{+}(\alpha, x) [w_{-}(x, \gamma) w_{+}(x, \beta)]
\] (2)
and the \((\alpha, \beta)\)-entry of the right hand side of (1) is
\[
D w_{-}(\alpha, \gamma) w_{+}(\alpha, \beta) w_{-}(\beta, \gamma).
\] (3)
The equality of (3) and (2) is equivalent to (5) of Definition 1.1. Note that \( \Delta_{\gamma} \) is invertible by (3) of Proposition 1.2. Now (1) is equivalent to
\[
\Delta_{\gamma}^{-1} W_{+} \Delta_{\gamma} W_{+} W_{-} = D W_{+} \Delta_{\gamma} W_{-}
\] (4)
i.e.,
\[
D \Delta_{\gamma}^{-1} W_{+} \Delta_{\gamma} = W_{+} \Delta_{\gamma} W_{-}.
\] (5)
Hence, \( \Delta_{\gamma} \) and \( D^{-1} W_{+} \) are conjugate.

It follows that the spectrum of \( \Delta_{\gamma} \) does not depend on the choice of \( \gamma \in X \). Equivalently, the columns of \( W_{-} \) are permutations of each other. Using (3) of Proposition 1.2 we obtain the same for \( W_{+} \).

Using (3JT) of [2] we obtain in a similar way that the rows of \( W_{+} \) are permutations of each other. Then simple counting shows that the in- and out-degrees of \( G_{i} \)'s must coincide.

The following theorem was independently proved by Jaeger [5] and de la Harpe [4], but this proof is simpler.

**Theorem 2.2** Let \( (X, W_{+}, W_{-}) \) be a symmetric spin model of degree 2. Then \( G_{1} \) (and consequently \( G_{2} \)) is a strongly regular graph.

**Proof of Theorem 2.2**
We have to establish the existence of \( k, \lambda \) and \( \mu \), that is we have to prove that \( G_{1} \) is \( k \)-regular and the number of common neighbors of any pair of connected (non-connected) vertices is \( \lambda \) (\( \mu \)).
The regularity follows from Theorem 2.1. Now we prove the existence of $\lambda$ and $\mu$. Let $\alpha_{ij}$ be the number of common $a_i$ entries of rows $i$ and $j$. Furthermore, suppose that $(i, j)$ and $(r, s)$ are edges of graph $G_1$. We have to prove that $\alpha_{ij} = \alpha_{rs}$. Using that the $(i, j)$ (resp. $(r, s)$) entry of $W_+ W_-$ is 0, we obtain
\[
a_0a_1^{-1} + a_0^{-1}a_1 + n - 2k + 2\alpha_{ij} + (k - 1 - \alpha_{ij})(a_1a_2^{-1} + a_1^{-1}a_2) = 0
\]
and
\[
a_0a_1^{-1} + a_0^{-1}a_1 + n - 2k + 2\alpha_{rs} + (k - 1 - \alpha_{rs})(a_1a_2^{-1} + a_1^{-1}a_2) = 0.
\]
Taking the difference of these two equations we obtain
\[
(\alpha_{ij} - \alpha_{rs})(2 - a_1a_2^{-1} - a_1^{-1}a_2) = 0.
\]
If $\alpha_{ij} \neq \alpha_{rs}$, then $a_1a_2^{-1} + a_1^{-1}a_2 = 2$, which implies $a_1 = a_2$, a contradiction. The existence of $\mu$ can be proved exactly the same way.

Theorem 2.3 Let $(X, W_+, W_-)$ be a symmetric spin model of degree 3. Then $\{A_0, A_1, A_2, A_3\}$ are the adjacency matrices of a symmetric class 3 association scheme.

Proof of Theorem 2.3
Let $\mathcal{M}$ be the algebra generated by $\{J, W_+\}$ with respect to the ordinary matrix product, and let $\mathcal{H}$ be the algebra generated by $\{I, W_-\}$ with respect to the Hadamard product, as introduced in [5]. Furthermore, let $\mathcal{A}$ be the algebra generated by $\{A_0, A_1, A_2, A_3\}$ with respect to the Hadamard product. Because $A_i \circ A_j = \delta_{ij} A_i$, we have that $\{A_0, A_1, A_2, A_3\}$ is a basis of $\mathcal{A}$. It is clear that $\mathcal{H} \subseteq \mathcal{A}$. Now, $I, J, W_+$ and $W_-$ are in $\mathcal{H}$ by [5]. The transition matrix that takes $\{A_0, A_1, A_2, A_3\}$ into $\{I, J, W_+, W_-\}$ is of Vandermonde type. It's determinant is non-zero, because $t_i$'s are distinct for $i = 1, 2, 3$. Thus, $\{I, J, W_+, W_-\}$ is also a basis of $\mathcal{A}$, i.e. $\mathcal{H} = \mathcal{A}$. Consequently, $\mathcal{H}$ is of dimension 4. Using that $\mathcal{H} \cong \mathcal{M}$, we obtain that $\mathcal{M}$ is of dimension four, too. However, $\{I, J, W_+, W_-\} \subset \mathcal{M}$ that yields $\mathcal{M} = \mathcal{H}$. Now, applying Proposition 3 of [5], we obtain that $\mathcal{M}$ is the Bose-Mesner algebra of an association scheme.

For non-symmetric spin models we have the following analogous theorem to Theorem 2.3.
Theorem 2.4 Let \((X, W_+, W_-)\) be a non-symmetric spin model of degree 2. Furthermore, let us assume that \(A_1 A_1^T = 0\). Then \((X, A_0 = I, A_1, A_1^T, A_2 - A_1^T)\) is a non-symmetric class 3 association scheme provided \(A_1^T \neq A_2\). If \(A_2 = A_1^T\), then \((X, A_0, A_1, A_1^T)\) is a non-symmetric class two association scheme.

Proof of Theorem 2.4
Let us assume first \(A_1^T \neq A_2\). Let \(\mathcal{M}\) be the algebra generated by \((J, W_-, W_-^T)\) with respect to the ordinary matrix product and let \(\mathcal{H}\) be generated by \((I, W_+, W_+^T)\) with respect to Hadamard product. We shall prove that \(\mathcal{M} = \mathcal{H}\), which implies by Theorem 2.4 of [3] that \(\mathcal{M}\) is the Bose-mesner algebra of a self-dual association scheme.

Let \(\mathcal{A}\) be the algebra generated by \((A_0 = I, A_1, A_1^T, A_2 - A_1^T)\) with respect to the Hadamard product. It is easy to see that \(\dim(\mathcal{A}) = 4\) and that \(\mathcal{H} \subseteq \mathcal{A}\). \(J\) is clearly in \(\mathcal{H}\). We claim, that \(I, J, W_+\) and \(W_+^T\) are linearly independent. Indeed, the transition matrix from the basis \((A_0 = I, A_1, A_1^T, A_2 - A_1^T)\) to \((I, J, W_+, W_+^T)\) is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
t_0 & t_1 & t_2 & t_2 \\
t_0 & t_2 & t_1 & t_2
\end{bmatrix}
\]

Its determinant is \(-(t_1 - t_2)^2 \neq 0\). Thus, \(\mathcal{H} = \mathcal{A}\). By Theorem 2.3 of [3] we have that \(\mathcal{M}\) and \(\mathcal{H}\) are isomorphic. Furthermore, \(I, J, W_+\) and \(W_+^T\) are all in \(\mathcal{M}\), thus \(\mathcal{M} = \mathcal{H}\).

The case of \(A_2 = A_1^T\) is similar and left to the reader.

3 Symmetric models
In this section we turn to the classification of small symmetric spin models. It is easy to see that for any \(n\), the only \(n \times n\) spin model of degree 1 is the Potts model [6]. So we shall always assume that the degree of the model is at least 2. For the sake of completeness we begin with the case \(n = 4\).

3.1 \(n = 4\)
There can be models of degree 2 and 3 besides the Potts model.
Degree 2  $G_1$ and $G_2$ are 1 and 2-regular graphs, respectively by Theorem 2.1. It is obvious that the two-regular graph must be the four-cycle, so we obtained the cyclic group case.

Degree 3  Now $G_1$, $G_2$ and $G_3$ are all perfect matchings. Because two of the matchings together form a 4-cycle, we may assume by renumbering the vertices and the $G_i$'s that $W_+$ is as follows

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$  

Writing $x = a_0/a_1$, $y = a_0/a_2$ and $z = a_0/a_3$ we obtain the following set of three equations from the condition $W_+W_-=nI$:

$$x + 1/x + y/z + z/y = 0$$
$$y + 1/y + x/z + z/x = 0$$
$$z + 1/z + x/y + y/x = 0.$$  

The only solution of this system is that one of the variables is equal to 1 and the other two are negatives of each other. Thus, we may assume that $W_+$ looks like

$$W_+ = \begin{bmatrix} a & a & b & -b \\ a & a & -b & b \\ b & -b & a & a \\ -b & b & a & a \end{bmatrix}.$$  

Now taking into account the various equations coming from the star-triangle equality we obtain that both $a$ and $b$ must be fourth roots of unity. All these cases are covered by the direct product construction of de la Harpe [4].

3.2  $n = 5$

By Theorem 2.1 the model is either of degree 1 or degree 2. Thus, if $(X, W_+, W_-)$ is not the Potts model, then we have that $G_1$ and $G_2$ are both 5-cycles so that their union is the complete graph $K_5$ by Theorem 2.2. In
this case $W_+$ looks like

$$W_+ = \begin{bmatrix}
  a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\
  a_1 & a_0 & a_2 & a_3 & a_4 & a_5 \\
  a_2 & a_1 & a_0 & a_2 & a_3 & a_4 \\
  a_3 & a_2 & a_1 & a_0 & a_2 & a_3 \\
  a_4 & a_3 & a_2 & a_1 & a_0 & a_2 \\
  a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\
\end{bmatrix}$$

This is the well-studied case of the pentagon $[4, 5]$.

### 3.3 $n = 6$

In this case a model could be of degree 2, 3, $\ldots$, 5.

#### 3.3.1 Degree 2

By Theorem 2.2 the graph $G_1$ has to be strongly regular. Furthermore, we may assume that it is 1- or 2-regular, otherwise we just have to switch between $G_1$ and $G_2$. However, in any case $G_1$ is disconnected that contradicts to Jaeger's conditions $[5, 4]$.

#### 3.3.2 Degree 3

$G_1$, $G_2$, and $G_3$ are regular graphs by Theorem 2.1. There are two possibilities, namely two matching and a three regular graph, or one matching and two 2-regular graphs. In the first case the two matchings together form a 6-cycle, so $W_+$ is as follows.

$$W_+ = \begin{bmatrix}
  a_0 & a_1 & a_3 & a_3 & a_2 & a_2 \\
  a_1 & a_0 & a_2 & a_3 & a_3 & a_3 \\
  a_3 & a_2 & a_0 & a_3 & a_3 & a_3 \\
  a_3 & a_2 & a_0 & a_3 & a_3 & a_3 \\
  a_2 & a_3 & a_3 & a_1 & a_0 & a_2 \\
  a_2 & a_3 & a_3 & a_1 & a_0 & a_2 \\
\end{bmatrix}$$

Taking the $(1, 3)$ and $(3, 1)$ entries of $W_+W_-$, we obtain the following two equations:

$$a_0a_3^{-1} + a_0^{-1}a_3 + a_1a_2^{-1} + a_3a_1^{-1} + a_2a_3^{-1} + 1 = 0$$

$$a_0a_3^{-1} + a_0^{-1}a_3 + a_1^{-1}a_2 + a_3^{-1}a_1 + a_2^{-1}a_3 + 1 = 0$$
Subtracting the second one from the first and multiplying by $a_1a_2a_3$ we obtain

$$a_1^2a_3 - a_2^2a_3 + a_3^2a_2 - a_1^2a_2 + a_2^2a_1 - a_3^2a_1 = 0$$

that is equivalent to

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2) = 0.$$  

However, this contradicts to the assumption of degree 3. We call the above type equation pair the cyclic permutation equation.

In the second case, we still have two choices. The first one is that $G_1$ is a matching and $G_2$ and $G_3$ are 6-cycles. There is only one way to decompose $K_6$ into one matching and two 6-cycles (see Figure 1). Indeed, we may assume that one of the 6-cycles is $(123456)$. Then in the other cycle we have to have a pair of adjacent edges so that the difference of the end points of one of them is 2, and the same for the other one is 3. By renumbering the vertices cyclically and possibly reversing the cyclic order we may assume that $(1, 4)$ and $(2, 4)$ are edges of the other cycle. Now if $(2, 5)$ were an edge of the second cycle then the other edge from vertex 5 should go to vertex 3, but that would imply $(3, 6)$ is an edge of the second cycle and $(6, 1)$, too. But that is a contradiction. Thus, the other edge from vertex 2 in the second cycle must be $(2, 6)$. The resulting decomposition is shown on Figure 1. Taking the difference of $(1, 2)$ and $(2, 1)$ entries of $W_+W_-$ we obtain the cyclic permutation equation as before, hence a contradiction.

The other possibility is the decomposition $K_6 = M_6 \cup 2C_3 \cup C_6$, where $M_6$ is a matching on six points (see Figure 2). This corresponds to the Cyclic Group $C_6$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Decomposition of $K_6$ into two $C_6$'s and a matching.}
\end{figure}
3.3.3 Degree 4

According to Theorem 2.1 there are two possible cases.

**Case 1.** $G_1, G_2$ and $G_3$ are matchings and $G_4$ is a 6-cycle. There is only one such decomposition, because two of the matchings together form another 6-cycle and we can apply the argument of Section 3.2.2. The corresponding $W_+$ is

$$ W_+ = \begin{bmatrix}
    a_0 & a_1 & a_2 & a_3 & a_4 & a_1 \\
    a_1 & a_0 & a_4 & a_2 & a_3 & a_4 \\
    a_2 & a_1 & a_0 & a_1 & a_2 & a_2 \\
    a_3 & a_4 & a_1 & a_0 & a_4 & a_1 \\
    a_4 & a_2 & a_3 & a_1 & a_0 & a_1 \\
    a_1 & a_3 & a_4 & a_2 & a_1 & a_0 
\end{bmatrix} $$

Taking the (1, 5) and (5, 1) entries of $W_+ W_-$ we obtain the cyclic permutation equation as before, hence a contradiction.

**Case 2.** $G_1, G_2$ and $G_3$ are matchings and $G_4$ is a union of two triangles. It is easy to see that if we assume that the two triangles are on vertices $\{1,2,3\}$ and $\{4,5,6\}$, respectively, then taking the (1, 2) and (2, 1) entries of $W_+ W_-$ we obtain the cyclic permutation equation, hence a contradiction.

3.3.4 Degree 5

Now we have a decomposition of $K_6$ into five matchings. There is only one way to decompose $K_6$ into five matchings up to permutation of the matchings,
because two pairs of these matchings form two 6-cycles and we can apply the argument of Section 3.2.2. The resulting $W_+$ is as follows.

\[
W_+ = \begin{bmatrix}
  a_0 & a_1 & a_3 & a_5 & a_4 & a_2 \\
  a_1 & a_0 & a_2 & a_4 & a_3 & a_5 \\
  a_3 & a_2 & a_0 & a_1 & a_5 & a_4 \\
  a_5 & a_4 & a_1 & a_0 & a_2 & a_3 \\
  a_4 & a_3 & a_5 & a_2 & a_0 & a_1 \\
  a_2 & a_5 & a_4 & a_3 & a_1 & a_0 
\end{bmatrix}
\]

Taking the difference of the $(1, 2)$ entry of $W_+ W_-$ with the $(3, 4), (4, 3), (5, 6)$ and $(6, 5)$ entries, respectively, we obtain the following four equations.

\[
\begin{aligned}
& a_3a_2^{-1} + a_2a_5^{-1} + a_5a_4^{-1} - a_2a_4^{-1} - a_3a_5^{-1} - a_5a_2^{-1} = 0 \\
& a_3a_2^{-1} + a_4a_3^{-1} + a_5a_4^{-1} - a_3a_4^{-1} - a_4a_2^{-1} - a_5a_3^{-1} = 0 \\
& a_3a_2^{-1} + a_4a_3^{-1} + a_2a_5^{-1} - a_2a_3^{-1} - a_4a_2^{-1} - a_3a_5^{-1} = 0 \\
& a_4a_3^{-1} + a_2a_5^{-1} + a_5a_4^{-1} - a_2a_4^{-1} - a_5a_3^{-1} - a_4a_5^{-1} = 0
\end{aligned}
\]

Reducing we obtain

\[
\begin{aligned}
& (a_5 - a_2) (a_3a_4 - a_5a_4 - a_2a_4 + a_5a_2) = 0 \\
& (a_3 - a_4) (a_5a_2 + a_3a_4 - a_3a_2 - a_2a_4) = 0 \\
& (a_2 - a_3) (a_3a_2 + a_5a_4 - a_5a_2 - a_3a_5) = 0 \\
& (a_4 - a_5) (a_3a_2 + a_5a_4 - a_3a_4 - a_3a_5) = 0.
\end{aligned}
\]

Using that $a_i$'s are different, it yields $a_3 = -a_5$ and $a_4 = -a_2$ that easily leads to a contradiction. Thus, the case $n = 6$ is finished.

### 3.4 $n = 7$

Applying again Theorem 2.1 we obtain that the number of different off-diagonal entries of $W_+$ is at most three. If $(X, W_+, W_-)$ were of degree two, then $G_1$ should be a strongly regular graph. However, strongly regular graph on 7 vertices does not exist. So, we may assume that the model is of degree 3. Now $G_1, G_2$ and $G_3$ are all regular graphs. Thus, they are all 2-regular graphs, i.e. unions of cycles. Furthermore, by Theorem 2.3 we have that $A_i$'s are adjacency matrices of an association scheme. However, it is well known
folklore that the only association scheme on 7 points with \( k_1 = k_2 = k_3 = 2 \) is the scheme of the 7-gon. Thus the only case here is the cyclic group case (see Figure 3). The proof of Theorem 1.4 is now completed.

4 Non-symmetric models

In this section \( G_i \)'s are oriented graphs. Furthermore, we always assume that the models are really non-symmetric, i.e., there exists at least one \( G_i \) and an edge \((k, l) \in E(G_i)\) such that \((l, k) \not\in E(G_i)\). By Thorem 2.1 we have that in-degree=\( \)out-degree=\( k_i \) for every vertex in \( G_i \) \( i = 1, 2, \ldots \). If the model is of degree 1, then it is symmetric and it is the Potts model.

4.1 \( n = 4 \)

4.1.1 Degree 2

We may assume that \( k_1 = 1 \) and \( k_2 = 2 \). Now \( G_1 \) contains either two independent edges directed in both ways, or it is a directed four-cycle. In the first case we obtain a symmetric model. In the second case we may assume that the directed four cycle is \((1234)\), i.e., \( W_+ \) is

\[
W_+ = \begin{bmatrix}
  a_0 & a_1 & a_2 & a_2 \\
  a_2 & a_0 & a_1 & a_2 \\
  a_2 & a_2 & a_0 & a_1 \\
  a_1 & a_2 & a_2 & a_0 \\
\end{bmatrix}.
\]
Taking the $(1, 2)$ and $(2, 1)$ entries of $W_+ W_-$ we obtain a cyclic permutation equation in variables $a_0, a_1, a_2$, i.e., $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$. This implies that $a_0 = a_1$ or $a_0 = a_2$. In the first case we obtain

$$W_+ = \begin{bmatrix} a_0 & a_0 & a_2 & a_2 \\ a_2 & a_0 & a_0 & a_2 \\ a_0 & a_2 & a_0 & a_0 \\ a_0 & a_2 & a_2 & a_0 \end{bmatrix}.$$ 

Taking the $(1, 2)$ and $(1, 3)$ entries of $W_+ W_-$ we obtain

$$a_0 a_2^{-1} + a_0^{-1} a_2 + 2 = 0 = 2(a_0 a_2^{-1} + a_0^{-1} a_2),$$

a contradiction.

In the second case we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_0 & a_0 \\ a_0 & a_0 & a_1 & a_0 \\ a_0 & a_0 & a_0 & a_1 \\ a_1 & a_0 & a_0 & a_0 \end{bmatrix}.$$ 

This is an instance of the cyclic group case.

### 4.1.2 Degree 3

$G_1$, $G_2$ and $G_3$ all have both in-degree and out-degree 1. If the model is non-symmetric, then we may assume that $G_1$ is a directed four-cycle. This implies that the other two graphs must be the reverse four-cycle and the diagonals oriented in both ways. Thus,

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}.$$ 

This is another instance of the cyclic group case.

### 4.2 $n = 5$

#### 4.2.1 Degree 2

Now we have two cases to be distinguished: $k_1 = 1$ and $k_1 = 2$. 


\( k_1 = 1 \) In this case \( G_1 \) cannot be symmetric. There are two possibilities for \( G_1 \). One is that it is a union of a directed triangle and an edge oriented both ways, the other is the directed five-cycle. In the first case assuming that the directed triangle is (123) we have that

\[
W_+ = \begin{bmatrix}
    a_0 & a_1 & a_2 & a_2 & a_2 \\
    a_2 & a_0 & a_1 & a_2 & a_2 \\
    a_1 & a_2 & a_0 & a_2 & a_2 \\
    a_2 & a_2 & a_2 & a_0 & a_1 \\
    a_2 & a_2 & a_2 & a_1 & a_0
\end{bmatrix}
\]

From the (1, 2) and (2, 1) entries of \( W_+W_- \) we obtain the cyclic permutation equation \((a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0\). This implies that either \( a_0 = a_1 \) or \( a_0 = a_2 \). In the first case the (4, 5) entry of \( W_+W_- \) would be 5, a contradiction. In the second case we obtain the system of equations

\[
\begin{align*}
    a_0a_1^{-1} + a_0^{-1}a_1 + 3 &= 0 \\
    4a_0 + a_1 &= \pm\sqrt{5}a_0^{-1} \\
    4a_0^{-1} + a_1^{-1} &= \pm\sqrt{5}a_0
\end{align*}
\]

that has no solution.

If \( G_1 \) is a directed five-cycle, then we obtain an instance of the cyclic group model.

\( k_1 = 2 \) Because the model is non-symmetric, we may assume that \((1, 2) \in E(G_1)\) and \((2, 1) \in E(G_2)\). Now the first two rows of \( W_+ \) look like

\[
\begin{array}{cccc}
    a_0 & a_1 & a_2 & x \\
    a_2 & a_0 & a_1 & y \\
\end{array}
\]

after suitable rearrangement of the last three rows and columns, where the part \( \begin{array}{cc}
    x & y \\
    z & u
\end{array} \) stands for either \( \begin{array}{cc}
    a_1 & a_2 \\
    a_2 & a_1
\end{array} \) or \( \begin{array}{cc}
    a_1 & a_2 \\
    a_2 & a_1
\end{array} \). In any case, from the (1, 2) and (2, 1) entries of \( W_+W_- \) we obtain again the cyclic permutation equation \((a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0\), which implies that either \( a_0 = a_1 \) or \( a_0 = a_2 \). By symmetry reasons we may assume that \( a_0 = a_2 \). Now the product of row \( i \) of \( W_+ \) and column \( j \) of \( W_- \) for \( i \neq j \) is either \( 2(a_0a_1^{-1} + a_0^{-1}a_1) + 1 \) or \( a_0a_1^{-1} + a_0^{-1}a_1 + 3 \). However, both cannot occur at the same time. If the first
two rows of $W_+$ are $a_0$ $a_0$ $a_1$ $a_1$ $a_0$, then $a_0$'s should stand under the $a_1$'s of the first row, the second row, ... of $W_+$ otherwise we would get both types of products, a contradiction. However, that would imply four of the $a_0$'s in the third row, also a contradiction.

On the other hand, if the first two rows of $W_+$ look like $a_0$ $a_0$ $a_1$ $a_1$ $a_0$, then the $(3, 5)$ entry of $W_+$ must be $a_1$, otherwise again both types of product would occur. However, that would imply three $a_1$'s in the fifth column, a contradiction. To finish this case we have to note only that the first two rows of $W_+$ can be assumed of one of the above two forms via suitable rearrangement of the rows and columns.

4.2.2 Degree 3

We may assume that $k_1 = k_2 = 1$ and $k_3 = 2$. If $G_1$ is a union of a directed triangle and an edge directed in both ways, then we may assume that the triangle is $(123)$. By symmetry, and the regularity of the $G_i$'s we may assume that the last two rows of $W_+$ are as follows:

\[
\begin{array}{cccc}
a_3 & a_3 & a_2 & a_0 \\
a_2 & a_3 & a_1 & a_0 \\
\end{array}
\]

Now $G_1$ contains the triangle $(123)$ and we can apply the regularity, so $W_+$ is

\[
W_+ = \begin{bmatrix}
a_0 & a_1 & a_3 & a_3 & a_2 \\
a_3 & a_0 & a_1 & a_2 & a_3 \\
a_1 & a_2 & a_0 & a_3 & a_3 \\
a_3 & a_3 & a_2 & a_0 & a_1 \\
a_2 & a_3 & a_3 & a_1 & a_0 \\
\end{bmatrix}
\]

Taking the $(1, 2)$ and $(2, 1)$ entries of $W_+W_-$ we obtain a cyclic permutation equation in variables $a_0, a_1$ and $a_3$. On the other hand, $(3, 5)$ and $(5, 3)$ entries give the cyclic permutation equation in variables $a_0, a_2$ and $a_3$. This implies that $a_0 = a_3$. However, in this case the $(1, 3)$ and $(3, 1)$ entries give a cyclic permutation equation in $a_1, a_2$ and $a_3$, a contradiction. Thus, we may assume that both $G_1$ and $G_2$ are directed five-cycles. Let us denote for a digraph $G$ by $-G$ the graph with edges exactly the reverses of those of $G$. If $G_1 = -G_2$ or $G_1 \cap -G_2 = \emptyset$, then we have instances of the cyclic group case.
Thus, we may assume that $G_1$ is the five-cycle $(12345)$ and that $(2,1) \in G_3$ and $(3,2) \in G_2$. Now we have to consider two cases.

**Case 1.** $(2,5) \in G_2$. It is easy to see that this implies that $G_2$ is the cycle $(13254)$. So $W_+$ is

$$W_+ = \begin{bmatrix}
    a_0 & a_1 & a_2 & a_3 & a_3 \\
    a_3 & a_0 & a_1 & a_3 & a_2 \\
    a_3 & a_2 & a_0 & a_1 & a_3 \\
    a_2 & a_3 & a_0 & a_1 & a_1 \\
    a_1 & a_3 & a_2 & a_3 & a_0
\end{bmatrix}.$$  

Considering again several entries of the product $W_+W_-$, we have that $(3,4)$ and $(4,3)$ give cyclic permutation equation in variables $a_0, a_1$ and $a_3$, while $(4,5)$ and $(5,4)$ give one in variables $a_0, a_1$ and $a_2$. This implies $a_0 = a_1$. Substituting that value we get that $(1,2)$ and $(2,1)$ give the cyclic permutation in variables $a_1, a_2$ and $a_3$, a contradiction.

**Case 2.** $(2,4) \in G_2$. This results in that $G_2 = (15324)$, and

$$W_+ = \begin{bmatrix}
    a_0 & a_1 & a_3 & a_3 & a_2 \\
    a_3 & a_0 & a_1 & a_2 & a_3 \\
    a_3 & a_2 & a_0 & a_1 & a_3 \\
    a_2 & a_3 & a_0 & a_1 & a_1 \\
    a_1 & a_3 & a_2 & a_3 & a_0
\end{bmatrix}.$$  

In a similar way as in Case 1 we obtain from the cyclic permutation equations given by $(1,2)$ and $(2,1)$ moreover $(2,3)$ and $(3,2)$ that $a_0 = a_1$. However, with this value we obtain the cyclic permutation equation from $(4,5)$ and $(5,4)$ in variables $a_1, a_2$ and $a_3$, a contradiction.

### 4.2.3 Degree 4

First we must observe that if $G_i = -G_j$ for some $i$ and $j$, then we have an instance of the cyclic group case. Next, if $G_1$ is a union of a directed triangle and an edge directed in both ways, then assuming that the triangle is $(123)$, we have

$$W_+ = \begin{bmatrix}
    a_0 & a_1 & * & * & * \\
    * & a_0 & a_1 & * & * \\
    a_3 & * & a_0 & a_1 & * \\
    a_{i_1} & a_{i_2} & a_{i_3} & a_0 & a_1 \\
    a_{j_1} & a_{j_2} & a_{j_3} & a_3 & a_0
\end{bmatrix}.$$  


Here $i_1,i_2,i_3$ and $j_1,j_2,j_3$ are permutations of $\{2,3,4\}$ so that $i_k \neq j_k$ for $k = 1,2,3$. This immediately implies a cyclic permutation equation in variables $a_2,a_3$ and $a_4$, a contradiction.

Thus, we may assume that all four $G_i$'s are five-cycles. If $G_1 \cap G_2 = \emptyset$, then again we have an instance of the cyclic group case. So we assume that $G_1 = (12345), (2,1) \notin G_2$ and $(3,2) \in G_2$. Similarly to the degree 3 case we obtain that either $G_2 = (13254)$ or $G_2 = (15324)$. However, it is not hard to verify that in neither of these cases can we decompose the remaining edges into two directed five-cycles. This completes the proof of Theorem 1.5.

References

[1] E. Bannai and E. Bannai, Spin models on finite cyclic groups, preprint


