数理解析研究所講究録
第 846 巻 1993 年 1-5

On flag-transitive $C_3.c^*$-geometries

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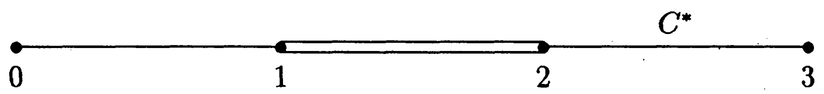
Flag-transitive $C_3.c^*$- and C_3-geometries.

The author has been recently investigating the flag-transitive incidence geometries belonging to the following diagram $C_3.c^*$ (see [2] for the fundamental terminologies on geometries):



Some sporadic groups including the Monster F_1, the Fischer group F_{24} and F_{22}, and the Conway group $Co.2$ are known to act flag-transitively on geometries belonging to this diagram. The main aim of my investigation is to characterize these sporadic groups via their geometries. In terms of group theory, this is an attempt to characterize them only by informations on their sections in some local subgroups.

We will set the notation. Let $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_3; *)$ be a *$C_3.c^*$-geometry of order* (x, y), that is, a residually connected incidence geometry in which the residue $\mathcal{G}_F$ of a flag F of type $\{i, j\}$ $(0 \leq i < j \leq 3)$ is the dual of a circle geometry for $(i, j) = (0, 1)$, a generalized quadrangle of order (x, y) for $(i, j) = (0, 3)$, a projective plane of order x for $(i, j) = (2, 3)$ and a generalized digon otherwise. We assume that $\mathcal{G}$ is (locally finite and) *thick*, that is, x and y are finite integers greater than 1. Further, we assume that $\mathcal{G}$ is *flag-transitive*, that is, there is a (not necessarily finite) subgroup G of $Aut(\mathcal{G})$ acting transitively on the set of maximal flags of $\mathcal{G}$. Note that the above geometries for sporadic groups are thick and flag-transitive.

What the author has started is, strictly speaking, the classification program of thick, flag-transitive $C_3.c^*$-geometries. For such geometry $\mathcal{G}$, the residue $\mathcal{G}_\pi$ of an element π of type 3 is a thick geometry belonging to the diagram C_3, admitting a flag-transitive action of the stabilizer G_π of π. Thus it is desirable to classify such geometries, before we tackle with the classification of thick flag-transitive $C_3.c^*$-geometries. However, as we will see below, this is one of the main open problems in diagram geometry. A thick geometry belonging to the

C_3-diagram admitting a flag-transitive group is called a thick, flag-transitive C_3-*geometry.* The order of such geometry is defined similarly to that of a $C_3.c^*$-geometry.

Examples of thick, flag-transitive C_3-geometries.

Any finite building of type C_3 (that is, the classical polar space consisting of the totally isotropic (or singular) points, lines and planes with respect to a non-degenerate symplectic, quadratic or hermitian form of Witt index 3) is an example of thick, flag-transitive C_3-geometries, except one associated with a non-singular quadratic form of plus type (as it has order $(x, 1)$). Note that the C_3-residues of the above-mentioned geometries associated with sporadic groups are classical polar spaces for $S_6(2)$, $U_6(2)$, $O_8^-(2)$, $O_7(3)$ or $O_8^-(3)$.

In the classification of thick, flag-transitive C_3-geometries, the difficulty arises in the existence of an exceptional example, *the (sporadic) A_7-geometry* $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2; *)$. The set $\mathcal{A}_0$ of *points* is the set of 7 letters, the set $\mathcal{A}_1$ of *lines* is the set of triples of points, and the set $\mathcal{A}_2$ of *planes* is defined as one of the two orbits under the action of the alternating group A_7 on the set of 30 different configurations of the projective planes of order $(2, 2)$ on $\mathcal{A}_0$. The plane-residues are clearly the projective plane of order 2, which is isomorphic to the Desarguesian projective plane associated with the 3-dimensional vector space over $GF(2)$. We may verify that the point-residues are the generalized quadrangle of order $(2, 2)$, which is isomorphic to the classical polar space consisting of the totally isotropic points and lines with respect to a non-degenerate symplectic form on the 4-dimensional space over $GF(2)$. Thus $\mathcal{A}$ is a C_3-geometry of order $(2, 2)$ on which A_7 acts flag-transitively. This geometry is not a building.

The existence of this exceptional geometry explains why the following celebrated theorem by Tits [11] (and Brouwer and Cohen) needs an assumption on the C_3-residues: Every geometry in which residues of corank 2 are generalized quadrangles are coverd by a building, if its C_3-residues are coverd by buildings.

Classification of thick, flag-transitive C_3-geoemtries.

It is now natural to ask whether there is a thick, flag-transitive C_3-geometries which is neither a building nor the sporadic A_7-geometry. We call such a C_3-geometry *anomalous.* It is conjectured that a flag-transitive thick C_3-geometry is either a finite building of type C_3 or the sporadic A_7-geometry.

M. Aschbacher [1] proved this conjecture assuming that the residues of the elements of type 0 and 2 are classical. In the general case, A. Pasini derived several results, which are summarized in [7], but he did not yet succeed to prove the conjecture.

In particular, he showed that the conjecture can be proved if we weaken the hypotheses of Aschbacher [1], assuming only that the plane-residues are classical projective planes. This is sufficient to prove that anomalous C_3-geometries cannot appear as rank 3 residues in certain geometries of rank 4. For instance, there are no finite flag-transitive geometries for any of the following two diagrams where C_3-residues are thick and anomalous.

Indeed, the projective planes at the middle of the diagram are forced to be classical here (see [3]). This non-existence lemma is one of the first step in the classification of finite flag-transitive geometries belonging to the above diagrams (see [7],[4] and [5] for the first diagram; [10] and [6] for the second one).

Classification of thick, flag-transitive $C_3.c^*$-geometries.

Since $\mathcal{G}_\pi$ is a thick, flag-transitive C_3-geometry, the classification of thick, flag-transitive $C_3.c^*$-geoemtries can be formaly divided into the following three cases:

(1) The case where $\mathcal{G}_\pi$ is a building.

(2) The case where $\mathcal{G}_\pi$ is the sporadic A_7-building.

(3) The case where $\mathcal{G}_\pi$ is an anomalous C_3-geometry.

In [12] and [13], the author considered the case (1) (that is, *extended dual polar spaces*) and proved that $\mathcal{G}_\pi$ is a classical polar space for $S_6(2)$, $U_6(2)$, $O_8^-(2)$, $O_7(3)$ or $O_8^-(3)$. Also, those geometries with $\mathcal{G}_\pi$ isomorphic to the $U_6(2)$-polar space, in particular, the geometry for $Co.2$ are classified. (The geometries with $\mathcal{G}_\pi$ the $S_6(2)$-polar space was also investigated.) The method used there was based on generators and relations, which may not be effective for larger sporadic groups.

In [9], the author and A. Pasini proved that there is a unique thick flag-transitive $C_3.c^*$-geometry with the C_3-residues the sporadic A_7-geometry. (We actually classified all possibel towers of circular or dual-circualr extensions of the spoardic A_7-geometries.) This settled the case (2).

The case (3) can be eliminated by the following result. Of course, this immediately follows if the above-mentioned conjecture of non-existence of thick, flag-transitive C_3-geometry is proved. However, in view of the difficulty in proving this conjecture, it may be reasonable to establish this result independently.

Theorem.[14] *There is no flag-transitive C_3c^*-geometry, in which the residue at each element of type 3 is a thick anomalous C_3-geometry.*

Outline of the proof of the main theorem.

The outline of the original proof is as follows[1]: Consider a thick, flag-transitive C_3c^*-geometry $\mathcal{G}$ of order (x, y) in which the C_3-residue $\mathcal{G}_\pi$ ($\pi \in \mathcal{G}_3$) is an anomalous C_3-geometry. We first establish that the full automorphism group $Aut(\mathcal{G}) = G$ acts regularly on the set of maximal flags. Then using the classical results on doubly transitive permutation groups (not depending on the classification of finite simple groups), we almost determine the explicit structure of the stabilizer G_l of a line $l \in \mathcal{G}_1$, which acting doubly transitively on the set of elements of type 3 on l. Note that these strong informations are never obtained if we simply start from a C_3-geometry.

We turn to the investigation of the action of G_π on the set $\mathcal{G}_0(\pi)/O_{p'}(G_\pi)$. We can establish that G_π is a Frobenius group on the set of p blocks of imprimitivity, where $p = x^2 + x + 1$ is a prime. This strongly restricts the structure of the stabilizer $G_{P,\pi}$ acting on the generalized quadrangle $\mathcal{G}_{P,\pi}$. The final contradiction can be obtained by observing the action of $O_{p'}(G_\pi) \cap G_P$ on the set of lines through P and a "special point" related with so called the *Ott-Liebler number.* The importance of choosing these lines was suggested by A. Pasini.

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[2]This will be generalized to the paper "On flag-transitive anomalous C_3-geometries".