On flag-transitive $C_3.c^*$-geometries

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Flag-transitive $C_3.c^*$- and $C_3$-geometries.

The author has been recently investigating the flag-transitive incidence geometries belonging to the following diagram $C_3.c^*$ (see [2] for the fundamental terminologies on geometries):

![Diagram of $C_3.c^*$](image)

Some sporadic groups including the Monster $F_1$, the Fischer group $F_{24}$ and $F_{22}$, and the Conway group $Co.2$ are known to act flag-transitively on geometries belonging to this diagram. The main aim of my investigation is to characterize these sporadic groups via their geometries. In terms of group theory, this is an attempt to characterize them only by informations on their sections in some local subgroups.

We will set the notation. Let $G = (G_0, \ldots, G_3; *)$ be a $C_3.c^*$-geometry of order $(x, y)$, that is, a residually connected incidence geometry in which the residue $G_F$ of a flag $F$ of type $\{i,j\}$ ($0 \leq i < j \leq 3$) is the dual of a circle geometry for $(i,j) = (0,1)$, a generalized quadrangle of order $(x,y)$ for $(i,j) = (0,3)$, a projective plane of order $x$ for $(i,j) = (2,3)$ and a generalized digon otherwise. We assume that $G$ is (locally finite and) thick, that is, $x$ and $y$ are finite integers greater than 1. Further, we assume that $G$ is flag-transitive, that is, there is a (not necessarily finite) subgroup $G$ of $Aut(G)$ acting transitively on the set of maximal flags of $G$. Note that the above geometries for sporadic groups are thick and flag-transitive.

What the author has started is, strictly speaking, the classification program of thick, flag-transitive $C_3.c^*$-geometries. For such geometry $G$, the residue $G_*$ of an element $\pi$ of type 3 is a thick geometry belonging to the diagram $C_3$, admitting a flag-transitive action of the stabilizer $G_*$ of $\pi$. Thus it is desirable to classify such geometries, before we tackle with the classification of thick flag-transitive $C_3.c^*$-geometries. However, as we will see below, this is one of the main open problems in diagram geometry. A thick geometry belonging to the
$C_3$-diagram admitting a flag-transitive group is called a thick, flag-transitive $C_3$-geometry. The order of such geometry is defined similarly to that of a $C_3.c^*$-geometry.

**Examples of thick, flag-transitive $C_3$-geometries.**

Any finite building of type $C_3$ (that is, the classical polar space consisting of the totally isotropic (or singular) points, lines and planes with respect to a non-degenerate symplectic, quadratic or hermitian form of Witt index 3) is an example of thick, flag-transitive $C_3$-geometries, except one associated with a non-singular quadratic form of plus type (as it has order $(x,1)$). Note that the $C_3$-residues of the above-mentioned geometries associated with sporadic groups are classical polar spaces for $S_6(2)$, $U_6(2)$, $O^-_8(2)$, $O_7(3)$ or $O^-_8(3)$.

In the classification of thick, flag-transitive $C_3$-geometries, the difficulty arises in the existence of an exceptional example, the (sporadic) $A_7$-geometry $A = (A_0, A_1, A_2; *)$. The set $A_0$ of points is the set of 7 letters, the set $A_1$ of lines is the set of triples of points, and the set $A_2$ of planes is defined as one of the two orbits under the action of the alternating group $A_7$ on the set of 30 different configurations of the projective planes of order $(2,2)$ on $A_0$. The plane-residues are clearly the projective plane of order 2, which is isomorphic to the Desarguesian projective plane associated with the 3-dimensional vector space over $GF(2)$. We may verify that the point-residues are the generalized quadrangle of order $(2,2)$, which is isomorphic to the classical polar space consisting of the totally isotropic points and lines with respect to a non-degenerate symplectic form on the 4-dimensional space over $GF(2)$. Thus $A$ is a $C_3$-geometry of order $(2,2)$ on which $A_7$ acts flag-transitively. This geometry is not a building.

The existence of this exceptional geometry explains why the following celebrated theorem by Tits [11] (and Brouwer and Cohen) needs an assumption on the $C_3$-residues: Every geometry in which residues of corank 2 are generalized quadrangles are covered by a building, if its $C_3$-residues are covered by buildings.

**Classification of thick, flag-transitive $C_3$-geometries.**

It is now natural to ask whether there is a thick, flag-transitive $C_3$-geometries which is neither a building nor the sporadic $A_7$-geometry. We call such a $C_3$-geometry anomalous. It is conjectured that a flag-transitive thick $C_3$-geometry is either a finite building of type $C_3$ or the sporadic $A_7$-geometry.

M. Aschbacher [1] proved this conjecture assuming that the residues of the elements of type 0 and 2 are classical. In the general case, A. Pasini derived several results, which are summarized in [7], but he did not yet succeed to prove the conjecture.

In particular, he showed that the conjecture can be proved if we weaken the hypotheses of Aschbacher [1], assuming only that the plane-residues are classical projective planes. This is sufficient to prove that anomalous $C_3$-geometries cannot appear as rank 3 residues.
in certain geometries of rank 4. For instance, there are no finite flag-transitive geometries for any of the following two diagrams where $C_3$-residues are thick and anomalous.

Indeed, the projective planes at the middle of the diagram are forced to be classical here (see [3]). This non-existence lemma is one of the first step in the classification of finite flag-transitive geometries belonging to the above diagrams (see [7],[4] and [5] for the first diagram; [10] and [6] for the second one).

**Classification of thick, flag-transitive $C_3.c^*$-geometries.**

Since $\mathcal{G}_\pi$ is a thick, flag-transitive $C_3$-geometry, the classification of thick, flag-transitive $C_3.c^*$-geometries can be formally divided into the following three cases:

1. The case where $\mathcal{G}_\pi$ is a building.
2. The case where $\mathcal{G}_\pi$ is the sporadic $A_7$-building.
3. The case where $\mathcal{G}_\pi$ is an anomalous $C_3$-geometry.

In [12] and [13], the author considered the case (1) (that is, *extended dual polar spaces*) and proved that $\mathcal{G}_\pi$ is a classical polar space for $S_6(2)$, $U_6(2)$, $O_6^-(2)$, $O_7(3)$ or $O_8^-(3)$. Also, those geometries with $\mathcal{G}_\pi$ isomorphic to the $U_6(2)$-polar space, in particular, the geometry for $C_{0.2}$ are classified. (The geometries with $\mathcal{G}_\pi$ the $S_6(2)$-polar space was also investigated.) The method used there was based on generators and relations, which may not be effective for larger sporadic groups.

In [9], the author and A. Pasini proved that there is a unique thick flag-transitive $C_3.c^*$-geometry with the $C_3$-residues the sporadic $A_7$-geometry. (We actually classified all possible towers of circular or dual-circular extensions of the sporadic $A_7$-geometries.) This settled the case (2).

The case (3) can be eliminated by the following result. Of course, this immediately follows if the above-mentioned conjecture of non-existence of thick, flag-transitive $C_3$-geometry is proved. However, in view of the difficulty in proving this conjecture, it may be reasonable to establish this result independently.

**Theorem.[14] There is no flag-transitive $C_3c^*$-geometry, in which the residue at each element of type 3 is a thick anomalous $C_3$-geometry.**
Outline of the proof of the main theorem.

The outline of the original proof is as follows\(^1\): Consider a thick, flag-transitive $C_3c^*$-geometry $\mathcal{G}$ of order $(x, y)$ in which the $C_3$-residue $\mathcal{G}_x (\pi \in G_3)$ is an anomalous $C_3$-geometry. We first establish that the full automorphism group $Aut(\mathcal{G}) = G$ acts regularly on the set of maximal flags. Then using the classical results on doubly transitive permutation groups (not depending on the classification of finite simple groups), we almost determine the explicit structure of the stabilizer $G_i$ of a line $l \in \mathcal{G}_1$, which acting doubly transitively on the set of elements of type 3 on $l$. Note that these strong informations are never obtained if we simply start from a $C_3$-geometry.

We turn to the investigation of the action of $G_\pi$ on the set $\mathcal{G}_0(\pi)/O_p'(G_\pi)$. We can establish that $G_\pi$ is a Frobenius group on the set of $p$ blocks of imprimitivity, where $p = x^2 + x + 1$ is a prime. This strongly restricts the structure of the stabilizer $G_{P, \pi}$ acting on the generalized quadrangle $\mathcal{G}_{P, \pi}$. The final contradiction can be obtained by observing the action of $O_p'(G_\pi) \cap G_P$ on the set of lines through $P$ and a “special point” related with so called the Ott-Liebler number. The importance of choosing these lines was suggested by A. Pasini.

参考文献


\(^1\)After the conference, I found that we can prove a strong result about thick, flag-transitive anomalous $C_3$-geometries by carefully generalizing some arguments used in this proof.


\(^2\)This will be generalized to the paper "On flag-transitive anomalous $C_3$-geometries"