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Contraction-Elimination Theorem

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0 Introduction

LK and **LJ** are the sequent calculi for, respectively, classical logic and intuitionistic logic. (See, e.g., [4] for the standard treatment of the sequent calculi.) They were introduced by Gentzen to prove certain meta-theorems about the logics, and many researchers, since Gentzen, have studied **LK**, **LJ**, and their variations to investigate various problems in logic. This paper is one of such studies.

LJ is obtained from **LK** by imposing the restriction that the right-hand side of a sequent consists of at most one formula. On the other hand, **LJ** is equivalent to the sequent calculus obtained by removing the right contraction rule from **LK** (see Theorem 1.2). Moreover, the sequent calculus for BCK-logic is obtained by removing the left contraction rule from **LJ** (see [3]). Thus the existence/non-existence of the contraction rules determines the logics of Gentzen's sequent calculi.

The purpose of this paper is to prove the "contraction-elimination theorem" (Theorem 1.1): If a sequent $\Gamma \Rightarrow A$ is provable in the implicational fragment of **LK** (resp. of **LJ**) and if no propositional variable has the PNN-occurrences (resp. PPN-occurrences) in it, then it is provable without the right (resp. left) contraction rule, where a propositional variable a is said to have the PNN-occurrences (resp. PPN-occurrences) in $\Gamma \Rightarrow A$ if a occurs at least once (resp. twice) in positive and at least twice (resp. once) in negative in $\Gamma \Rightarrow A$. (We wish to remove the restriction "the implicational fragment of," but this is somewhat problematic. See the discussion in [2].) As a corollary to the contraction-elimination theorem, we get the following: If an implicational formula A is a theorem of classical logic (resp. of intuitionistic logic) and is not a theorem of intuitionistic logic (resp. BCK-logic), then there is a propositional variable which has the PNN-occurrences (resp. PPN-occurrences) in A. (Corollary 1.3) This refines the Jaśkowski's result in [1]: If an implicational formula A is a theorem of classical logic and is not a theorem of BCK-logic, then there is a propositional variable which occurs at least three times in A.

To show the contraction-elimination theorem, we introduce some new notions and prove many lemmas. The author believes that the contraction-elimination theorem, together with the lemmas, will shed new light on the study of logic.

The contents of this paper are as follows. In Section 1, we present basic definitions and state the contraction-elimination theorem for our sequent calculi. In Section 2, we introduce novel modifications of sequent calculi, which we call "sequent calculi with $\star$." We show the relation between the sequent calculi and those with $\star$. Then our goal is reduced to proving the contraction-elimination theorem for the sequent calculi with $\star$. In Section 3, we prove the contraction-elimination theorem for $\mathbf{LJ}^{\star}_{\rightarrow}$ (i.e., the implicational fragment of **LJ** with $\star$). In Section 4, we prove the contraction-elimination theorem for $\mathbf{LK}^{\star}_{\rightarrow}$ (i.e., the implicational fragment of **LK** with $\star$). The proof for $\mathbf{LJ}^{\star}_{\rightarrow}$ is easy, but the proof for $\mathbf{LK}^{\star}_{\rightarrow}$ is somewhat laborious. The outline of the latter is as follows. We assign appropriate ordinal numbers to proofs in $\mathbf{LK}^{\star}_{\rightarrow}$ where 0 is assigned to the proofs containing no right contraction, and we give a transformation of proofs which decreases the ordinals. This has some analogy to the famous cut-elimination theorem by Gentzen.

1 Sequent Calculi

In this paper, we consider only the implicational fragments of propositional logics. Therefore our *formulas* are constructed from the propositional variables and $\rightarrow$ (implication). If Γ and Δ are (possibly empty) sequences of formulas, then an expression $\Gamma \Rightarrow \Delta$ is called a *sequent*. We will use letters $a, b, a_1, a_2, \ldots$ for propositional variables, letters $A, B, A_1, A_2, \ldots$ for formulas, and letters $\Gamma, \Delta, \Gamma_1, \Gamma_2, \ldots$ for sequences of formulas. Parentheses will be omitted in such a way that, for example, $A{\rightarrow}B{\rightarrow}C{\rightarrow}D$ denotes $A{\rightarrow}(B{\rightarrow}(C{\rightarrow}D))$.

We will use superscript to distinguish occurrences of sub-formulas. For example, in the sequent

$$A^1, A^2{\rightarrow}B \Rightarrow B, A^3, A^4$$

there are four occurrences of the formula A.

When we consider a sequent $\Gamma_1 \Rightarrow \Gamma_2$, the order of the formulas in Γ_i ($i = 1, 2$) is not important. Hence by $\Gamma \Rightarrow \Delta$, we will denote a sequent $\Gamma' \Rightarrow \Delta'$ where Γ' and Δ' are permutations of, respectively, Γ and Δ.

We define *positive* and *negative* occurrences of a propositional variable in a formula and in a sequent, as follows:

1. a^1 is a positive occurrence in the formula a^1.

2. A positive (resp. negative) occurrence in A^1 is a negative (resp. positive) occurrence in the formula $A^1 \to B$. A positive (resp. negative) occurrence in B^2 is a positive (resp. negative) occurrence in the formula $A \to B^2$.

3. A positive (resp. negative) occurrence in A^1 is a negative (resp. positive) occurrence in the sequent $A^1, \Gamma \Rightarrow \Delta$. A positive (resp. negative) occurrence in A^2 is a positive (resp. negative) occurrence in the sequent $\Gamma \Rightarrow \Delta, A^2$.

Now we define four sequent calculi $\mathbf{LK}_{\to}$, $\mathbf{LK}_{\to\text{-RC}}$, $\mathbf{LJ}_{\to}$, and $\mathbf{LJ}_{\to\text{-LC}}$.

The axioms in $\mathbf{LK}_{\to}$:

$$a \Rightarrow a \qquad (a \text{ is a propositional variable})$$

The inference rules in $\mathbf{LK}_{\to}$:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ Left Weakening} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \text{ Right Weakening}$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \text{ Left Contraction} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \text{ Right Contraction}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ Left} \to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \text{ Right} \to$$

$\mathbf{LK}_{\to\text{-RC}}$ is obtained by removing the right contraction rule from $\mathbf{LK}_{\to}$.

The axioms in $\mathbf{LJ}_{\to}$:

$$a \Rightarrow a \qquad (a \text{ is a propositional variable})$$

The inference rules in $\mathbf{LJ}_{\to}$:

$$\frac{\Gamma \Rightarrow B}{A, \Gamma \Rightarrow B} \text{ Left Weakening}$$

$$\frac{A, A, \Gamma \Rightarrow B}{A, \Gamma \Rightarrow B} \text{ Left Contraction}$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow C}{A \to B, \Gamma, \Delta \Rightarrow C} \text{ Left} \to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \text{ Right} \to$$

$\mathbf{LJ}_{\to\text{-LC}}$ is obtained by removing the left contraction rule from $\mathbf{LJ}_{\to}$.

Let L be a sequent calculus. *Proofs* in L are defined as usual by the tree-like figures of sequents. We will write "$L \vdash \Gamma \Rightarrow \Delta$" for "$\Gamma \Rightarrow \Delta$ is provable in L (i.e., there is a proof of $\Gamma \Rightarrow \Delta$ in L)."

Note that our sequent calculi are exchange-free and cut-free. But each rule contains the effect of the exchange rule (recall our notation of sequents), and the exchange rule is redundant. The cut rule is also redundant due to the cut-elimination theorem for $\mathbf{LK}_{\to}$, $\mathbf{LJ}_{\to}$, and $\mathbf{LJ}_{\to\text{-LC}}$ ([3], [4]).

We say a propositional variable a has the *PNN-occurrences* in a formula A (or in a sequent $\Gamma \Rightarrow \Delta$) if there are at least one positive and at least two negative occurrences of a in A (or in $\Gamma \Rightarrow \Delta$). The *PPN-occurrences* is defined similarly by "at least two positive and at least one negative occurrences." We say a formula or a sequent satisfies the *no-PNN-condition* (resp. *no-PPN-condition*) if no propositional variable has the PNN-occurrences (resp. PPN-occurrences) in it.

The purpose of this paper is to prove the following:

Theorem 1.1 (Contraction-Elimination Theorem)
(1) *If* $\Gamma \Rightarrow A$ *is provable in* $\mathbf{LK}_{\to}$ *and satisfies the no-PNN-condition, then it is provable in* $\mathbf{LK}_{\to\text{-RC}}$.
(2) *If* $\Gamma \Rightarrow A$ *is provable in* $\mathbf{LJ}_{\to}$ *and satisfies the no-PPN-condition, then it is provable in* $\mathbf{LJ}_{\to\text{-LC}}$.

We here remark that the statement of Theorem 1.1(1) cannot be generalized to "If $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{LK}_{\to}$ and ... ." Indeed, the sequent

$$(a \to b) \to (c \to d) \to e, (d \to a) \to (b \to c) \to f \Rightarrow e, f$$

is provable in $\mathbf{LK}_{\to}$ and satisfies the no-PNN-condition, but is not provable in $\mathbf{LK}_{\to\text{-RC}}$.

We show that $\mathbf{LK}_{\to\text{-RC}}$ and $\mathbf{LJ}_{\to}$ are equivalent:

Theorem 1.2 $\mathbf{LK}_{\rightarrow-\mathrm{RC}} \vdash \Gamma \Rightarrow A$ *if and only if* $\mathbf{LJ}_{\rightarrow} \vdash \Gamma \Rightarrow A$.

Proof If-part is trivial. Only-if-part is shown by induction on the length of the proof of $\Gamma \Rightarrow A$ in $\mathbf{LK}_{\rightarrow-\mathrm{RC}}$, using the fact that *if* $\mathbf{LK}_{\rightarrow-\mathrm{RC}} \vdash \Pi \Rightarrow \Sigma$ *then* Σ *is not empty* (which can be easily verified by induction). ∎

We know that $\mathbf{LK}_{\rightarrow}$, $\mathbf{LJ}_{\rightarrow}$ and $\mathbf{LJ}_{\rightarrow-\mathrm{LC}}$ are sequent calculi for, respectively, classical logic, intuitionistic logic and BCK-logic (see [3], [4]). Then Theorems 1.1 and 1.2 tell us an interesting property on the implicational fragments of propositional logics:

Corollary 1.3

(1) *If an implicational formula A is a theorem of classical logic and is not a theorem of intuitionistic logic, then there is a propositional variable which has the PNN-occurrences in A.*

(2) *If an implicational formula A is a theorem of intuitionistic logic and is not a theorem of BCK-logic, then there is a propositional variable which has the PPN-occurrences in A.*

2 Sequent Calculi with $\star$

To prove the contraction-elimination theorem, we will give a general way to transform, for example, the proof

$$
\begin{prooftree}
\AxiomC{$b \Rightarrow b$}
\UnaryInfC{$b \Rightarrow b, c$}
\UnaryInfC{$\Rightarrow b, b{\rightarrow}c$}
\AxiomC{$a \Rightarrow a$}
\BinaryInfC{$(b^1{\rightarrow}c){\rightarrow}a \Rightarrow b, a$}
\AxiomC{$c \Rightarrow c$}
\UnaryInfC{$b, c \Rightarrow c$}
\UnaryInfC{$c \Rightarrow b{\rightarrow}c$}
\AxiomC{$a \Rightarrow a$}
\BinaryInfC{$(b^2{\rightarrow}c){\rightarrow}a, c \Rightarrow a$}
\BinaryInfC{$b{\rightarrow}c, (b{\rightarrow}c){\rightarrow}a, (b{\rightarrow}c){\rightarrow}a \Rightarrow a, a$}
\RightLabel{right contraction}
\UnaryInfC{$b{\rightarrow}c, (b{\rightarrow}c){\rightarrow}a, (b{\rightarrow}c){\rightarrow}a \Rightarrow a$}
\UnaryInfC{$b{\rightarrow}c, (b{\rightarrow}c){\rightarrow}a \Rightarrow a$}
\end{prooftree}
$$

in $\mathbf{LK}_{\rightarrow}$ into a proof of $b{\rightarrow}c, (b{\rightarrow}c){\rightarrow}a \Rightarrow a$ in $\mathbf{LK}_{\rightarrow-\mathrm{RC}}$. To give such transformation, we need detailed argument about proofs in our sequent calculi; and for that argument, we must make a distinction between the occurrences of propositional variables which originate from the axioms and those which arise from the weakening rules in a proof. For example, b^1 and b^2 in the above proof have different natures. To substantiate this difference, we introduce "sequent calculi with $\star$" in this section.

First we introduce a new symbol $\star$, and extend our definition of formulas by admitting $\star$ as an atomic formula. $\star$ is not a propositional variable, and the no-PNN-condition/no-PPN-condition for sequents containing $\star$ is defined by considering only the number of occurrences of propositional variables.

We define a binary relation $\prec$ between formulas inductively as follows:

1. $\star \prec A$ for any formula A;
2. $a \prec A$ if and only if $A = a$ (a is a propositional variable);
3. $A_1 \rightarrow A_2 \prec B$ if and only if $((B = B_1 \rightarrow B_2)$ and $(A_i \prec B_i)$ $(i = 1, 2))$.

In other words, $A \prec B$ means that B is obtained from A by replacing some occurrences $\star^1, \star^2, ..., \star^n$ in A by some formulas $C_1, C_2, ..., C_n$, respectively.

Lemma 2.1

(1) *If $A \prec \star$, then $A = \star$.*

(2) *If $A \prec a$ (a is a propositional variable), then $(A = \star)$ or $(A = a)$.*

(3) *If $A \prec B_1 \rightarrow B_2$, then $(A = \star)$ or $((A = A_1 \rightarrow A_2)$ and $(A_i \prec B_i)$ $(i = 1, 2))$.*

Proof By the definition of $\prec$. ∎

Lemma 2.2 *$\prec$ is a partial order, i.e.,*

- $A \prec A$
- $A \prec B, B \prec C$ *implies* $A \prec C$
- $A \prec B, B \prec A$ *implies* $A = B$

hold for any formulas A, B, C.

Proof By induction on the length of A. ∎

Let A and B be formulas. When $\{A, B\}$ has an upper bound with respect to $\prec$, i.e., there is a formula C such that $A \prec C$ and $B \prec C$, then we write

$$A \frown B.$$

Lemma 2.3

(1) *$\star \frown A$ for any formula A.*

(2) *$a \frown A$ (a is a propositional variable) if and only if ($A = \star$) or ($A = a$).*

(3) *$A_1 \to A_2 \frown B$ if and only if ($B = \star$) or (($B = B_1 \to B_2$) and ($A_i \frown B_i$) ($i = 1, 2$)).*

Proof Easy. ∎

Lemma 2.4 *Let A and B be formulas such that $A \frown B$. Then $\{A, B\}$ has a supremum with respect to $\prec$, i.e., there is a formula C such that*

- *$A \prec C$, $B \prec C$*
- *for any formula D, if $A \prec D$, $B \prec D$, then $C \prec D$.*

Proof For any formulas F and G such that $F \frown G$, we define a formula $\mathcal{S}(F, G)$ by induction on the length of F as follows:

1. $\mathcal{S}(\star, G) = G$;
2. $\mathcal{S}(a, G) = a$ if a is a propositional variable;
3. $\mathcal{S}(F_1 \to F_2, G) = \begin{cases} F_1 \to F_2 & \text{if } G = \star \\ \mathcal{S}(F_1, G_1) \to \mathcal{S}(F_2, G_2) & \text{if } G = G_1 \to G_2. \end{cases}$

Note that $\mathcal{S}(F_1 \to F_2, G)$ is well defined by this equation due to Lemma 2.3 (3). We can verify that $\mathcal{S}(A, B)$ is the supremum of $\{A, B\}$. ∎

Now, we define $\mathbf{LK}^{\star}_{\to}$, $\mathbf{LK}^{\star}_{\to\text{-RC}}$, $\mathbf{LJ}^{\star}_{\to}$, and $\mathbf{LJ}^{\star}_{\to\text{-LC}}$, which we call "sequent calculi with $\star$".

The axioms in $\mathbf{LK}^{\star}_{\to}$:

$$a \Rightarrow a \qquad (a \text{ is a propositional variable.})$$

The inference rules in $\mathbf{LK}^{\star}_{\to}$:

$$\frac{\Gamma \Rightarrow \Delta}{\star, \Gamma \Rightarrow \Delta}\ \text{Left Weakening} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \star}\ \text{Right Weakening}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{\mathcal{S}(A, B), \Gamma \Rightarrow \Delta}\ \text{Left Contraction } \dagger \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, \mathcal{S}(A, B)}\ \text{Right Contraction } \dagger$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \text{Left} \to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ \text{Right} \to$$

† Contraction is admitted when $A \frown B$. ($\mathcal{S}(A,B)$ is the supremum of $\{A,B\}$, defined in the proof of Lemma 2.4.)

$\mathbf{LK}^{\star}_{\rightarrow\text{-RC}}$ is obtained by removing the right contraction rule from $\mathbf{LK}^{\star}_{\rightarrow}$.

The axioms in $\mathbf{LJ}^{\star}_{\rightarrow}$:

$$a \Rightarrow a \qquad (a \text{ is a propositional variable.})$$

The inference rules in $\mathbf{LJ}^{\star}_{\rightarrow}$:

$$\frac{\Gamma \Rightarrow B}{\star, \Gamma \Rightarrow B} \text{ Left Weakening}$$

$$\frac{A, B, \Gamma \Rightarrow C}{\mathcal{S}(A,B), \Gamma \Rightarrow C} \text{ Left Contraction } \dagger$$

$$\frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow C}{A \rightarrow B, \Gamma, \Delta \Rightarrow C} \text{ Left} \rightarrow \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ Right} \rightarrow$$

† Contraction is admitted when $A \frown B$.

$\mathbf{LJ}^{\star}_{\rightarrow\text{-LC}}$ is obtained by removing the left contraction rule from $\mathbf{LJ}^{\star}_{\rightarrow}$.

Example of a proof in $\mathbf{LK}^{\star}_{\rightarrow}$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{b \Rightarrow b}{b \Rightarrow b, \star}}{\Rightarrow b, b \rightarrow \star} \quad a \Rightarrow a}{(b \rightarrow \star) \rightarrow a \Rightarrow b, a} \quad \cfrac{\cfrac{\cfrac{c \Rightarrow c}{\star, c \Rightarrow c}}{c \Rightarrow \star \rightarrow c} \quad a \Rightarrow a}{(\star \rightarrow c) \rightarrow a, c \Rightarrow a}}{b \rightarrow c, (b \rightarrow \star) \rightarrow a, (\star \rightarrow c) \rightarrow a \Rightarrow a, a}}{b \rightarrow c, (b \rightarrow \star) \rightarrow a, (\star \rightarrow c) \rightarrow a \Rightarrow a}}{b \rightarrow c, (b \rightarrow c) \rightarrow a \Rightarrow a}$$

(Compare this with the proof at the beginning of this section.)

We introduce notation convenient for our argument. By $A^{\star}$, we denote some formula B such that $B \prec A$. In other words, $A^{\star}$ is a formula which is obtained from A by replacing some sub-formulas by $\star$'s. $A^{\star}$ is not uniquely determined for any fixed A except $\star$, and we will use this notation as follows. For example, by

$$\cfrac{\begin{array}{c}\vdots\\ A^\star, A^\star, \Gamma \Rightarrow A^\star\end{array}}{A^\star, \Gamma \Rightarrow A^\star}\ \text{left contraction}$$

we mean

$$\cfrac{\begin{array}{c}\vdots\\ A_1, A_2, \Gamma \Rightarrow A_3\end{array}}{\mathcal{S}(A_1, A_2), \Gamma \Rightarrow A_3}\ \text{left contraction}$$

for some formulas A_1, A_2, and A_3 such that $A_i \prec A$ $(i = 1, 2, 3)$. (Note that $\mathcal{S}(A_1, A_2) \prec A$.) If $\Delta = B_1, B_2, ..., B_n$, then $\Delta^\star$ means $B_1^\star, B_2^\star, ..., B_n^\star$.

The following theorem shows the relation between sequent calculi and those with $\star$.

Theorem 2.5 *Let L be one of* $\mathbf{LK}_{\rightarrow}$, $\mathbf{LK}_{\rightarrow\text{-RC}}$, $\mathbf{LJ}_{\rightarrow}$, *and* $\mathbf{LJ}_{\rightarrow\text{-LC}}$, *and let $\Gamma \Rightarrow \Delta$ be a sequent containing no $\star$. Then, $L \vdash \Gamma \Rightarrow \Delta$ if and only if ($L^\star \vdash \Gamma^\star \Rightarrow \Delta^\star$ for some $\Gamma^\star \Rightarrow \Delta^\star$).*

Proof By induction on the length of the proofs. (See [2] for the detail.) ∎

Let A and B be formulas such that $A \prec B$. We define the *natural mapping θ by $A \prec B$* inductively as follows:

1. θ is a mapping from the set of all occurrences of sub-formulas in A to the set of all occurrences of sub-formulas in B.

2. If $A = \star$ or a (propositional variable), then $\theta(A) = B$.

3. If $A = A_1 \rightarrow A_2$, then
$$\begin{cases} \theta(A) &= B \\ \theta(A_1') &= \theta_1(A_1') \quad \text{if } A_1' \text{ is a sub-occurrence in } A_1 \\ \theta(A_2') &= \theta_2(A_2') \quad \text{if } A_2' \text{ is a sub-occurrence in } A_2 \end{cases}$$
where $B = B_1 \rightarrow B_2$ and θ_i is the natural mapping by $A_i \prec B_i$ $(i = 1, 2)$.

For example, when θ is the natural mapping by $(a^1 \rightarrow \star^2)^3 \prec (a^4 \rightarrow (\star \rightarrow b)^5)^6$, then $\theta(a^1) = a^4$, $\theta(\star^2) = (\star \rightarrow b)^5$, and $\theta((a \rightarrow \star)^3) = (a \rightarrow (\star \rightarrow b))^6$.

Let A be a formula. The right-most occurrence of the atomic formula in A is called the *core* of A.

Lemma 2.6 *Let θ be the natural mapping by $A \prec B$. Then we have the following:*

- $A' \prec \theta(A')$ *for any sub-occurrence A' in A.*
- *If c is a propositional variable in A, then $\theta(c)$ is the same propositional variable c in B. Moreover, if c is a positive (resp. negative) occurrence in A, then $\theta(c)$ is also a positive (resp. negative) occurrence in B; if c is the core of A, then $\theta(c)$ is also the core of B.*
- *θ is one-one.*
- *Let B' be a sub-occurrence in B. If there is no sub-occurrence A' in A such that $\theta(A') = B'$, then there is $\star$ in A such that B' is a sub-occurrence in $\theta(\star)$.*

Proof By the definition of the natural mappings. ∎

Lemma 2.7 *If $\Gamma \Rightarrow \Delta$ satisfies the no-PNN-condition/no-PPN-condition, then $\Gamma^\star \Rightarrow \Delta^\star$ also satisfies the condition, for any $\Gamma^\star \Rightarrow \Delta^\star$.*

Proof By Lemma 2.6. (The natural mappings preserve the occurrences of propositional variables.) ∎

Now Theorem 1.1 is reduced, by Theorem 2.5, Lemma 2.7, and transitivity of $\prec$, to the following:

Theorem 2.8 (Contraction-Elimination Theorem with $\star$)
(1) *If $\Gamma \Rightarrow A$ is provable in $\mathbf{LK}^\star_\rightarrow$ and satisfies the no-PNN-condition, then $\Gamma^\star \Rightarrow A^\star$ is provable in $\mathbf{LK}^\star_{\rightarrow-\mathrm{RC}}$ for some $\Gamma^\star \Rightarrow A^\star$.*
(2) *If $\Gamma \Rightarrow A$ is provable in $\mathbf{LJ}^\star_\rightarrow$ and satisfies the no-PPN-condition, then $\Gamma^\star \Rightarrow A^\star$ is provable in $\mathbf{LJ}^\star_{\rightarrow-\mathrm{LC}}$ for some $\Gamma^\star \Rightarrow A^\star$.*

In the rest of this section, we give some definitions and lemmas which will be used for proving Theorem 2.8.

Let R be an inference rule in sequent calculi with $\star$. We define the *child* of an occurrence of sub-formula in the upper sequent of R, as follows:

1. When R is the left/right weakening rule or the left/right $\rightarrow$ rule, then for any occurrence A^1 of sub-formula in the upper sequent, there is the uniquely corresponding occurrence A^2 in the lower sequent. We define that A^2 is the child of A^1.

2. When R is the left/right contraction rule, for example

$$\frac{B_1, B_2, \Gamma \Rightarrow \Delta}{\mathcal{S}(B_1, B_2), \Gamma \Rightarrow \Delta}$$

then the child of an occurrence in $\{\Gamma; \Delta\}$ is defined similarly to the case of the other inference rules. If A is an occurrence of sub-formula in B_i ($i = 1$ or 2) in the upper sequent, then the child of A is the occurrence $\theta(A)$ in the lower sequent where θ is the natural mapping by $B_i \prec \mathcal{S}(B_1, B_2)$.

Let P be a proof in sequent calculi with $\star$, and A and B be occurrences of sub-formulas in P. We say A is an *ancestor* of B, and B is a *descendant* of A, if there are occurrences $C_1, C_2, ..., C_n$ ($n \geq 1$) in P such that

- C_1 is A, and C_n is B;
- C_{i+1} is the child of C_i ($1 \leq i \leq n-1$).

Lemma 2.9

(1) *If A is an ancestor of B, then $A \prec B$.*

(2) *If a^1 in $\Gamma \Rightarrow \Delta$ is an ancestor of a^2 in $\Pi \Rightarrow \Sigma$, and a^1 is a positive (resp. negative) occurrence in $\Gamma \Rightarrow \Delta$, then a^2 is also a positive (resp. negative) occurrence in $\Pi \Rightarrow \Sigma$.*

Proof By Lemma 2.6. ∎

If $a^1 \Rightarrow a^2$ is an occurrence of an axiom in a proof, and a^3 is a descendant of either a^1 or a^2, then $a^1 \Rightarrow a^2$ is said to be an *ancestor axiom* of a^3.

Let $\Gamma \Rightarrow \Delta$ be an occurrence of a sequent in a proof, and a^1 and a^2 be, respectively, a positive and a negative occurrences of a propositional variable in $\Gamma \Rightarrow \Delta$ such that there is an ancestor axiom $a^4 \Rightarrow a^3$ of both a^1 and a^2 (i.e., a^3 and a^4 are the ancestors of, respectively, a^1 and a^2). Then a^i is said to be the *partner* of a^j ($(i,j) = (1,2), (2,1)$).

Lemma 2.10 *Let A_1 and A_2 be formulas such that $A_1 \frown A_2$, and θ_i be the natural mapping by $A_i \prec \mathcal{S}(A_1, A_2)$ $(i = 1, 2)$. Then, for any occurrence a^1 of a propositional variable in $\mathcal{S}(A_1, A_2)$, there is an occurrence a^2 in A_j such that $\theta_j(a^2) = a^1$ for some j $(j = 1$ or $2)$.*

Proof By induction on the length of $\mathcal{S}(A, B)$, using Lemma 2.1 and the definition of $\mathcal{S}(A, B)$. ∎

Lemma 2.11 *Let P be a proof of $\Gamma \Rightarrow \Delta$ in sequent calculi with $\star$. Then, for any axiom $a^1 \Rightarrow a^2$ in P, there are two occurrences a^3 and a^4 in $\Gamma \Rightarrow \Delta$ which are the descendants of, respectively, a^1 and a^2; and for any occurrence of propositional variable in $\Gamma \Rightarrow \Delta$, there is at least one ancestor axiom of it in P. Therefore, for any occurrence of propositional variable in $\Gamma \Rightarrow \Delta$, there is at least one partner of it.*

Proof By induction on the length of P, using Lemma 2.10. ∎

Lemma 2.11 shows an important property of sequent calculi with $\star$, and we will tacitly use this henceforth.

Lemma 2.12 *Let L be a sequent calculus with $\star$. Suppose that*

- *P is a proof of $\Gamma \Rightarrow \Delta$ in L, and Q is a sub-proof in P whose last sequent is $A_1, A_2, ..., A_m \Rightarrow B_1, B_2, ..., B_n$; (Fig. 1)*
- *sub-formulas $C_1, C_2, ..., C_m, D_1, D_2, ..., D_n$ in $\Gamma \Rightarrow \Delta$ are the descendants of, respectively, $A_1, A_2, ..., A_m$, B_1, B_2, $...,B_n$;*
- *R is a proof of $C_1^\star, C_2^\star, ..., C_m^\star \Rightarrow D_1^\star, D_2^\star, ..., D_n^\star$ in L.*

Then we get a proof P' in L such that

- *P' is a proof of $\Gamma^\star \Rightarrow \Delta^\star$ for some $\Gamma^\star \Rightarrow \Delta^\star$, and R is a sub-proof of P'; (Fig. 2)*
- *the part of proof which is obtained from P' by eliminating*

$$R, (C_1^\star, ..., C_m^\star \Rightarrow D_1^\star, ..., D_n^\star), \cdots, (\Gamma^\star \Rightarrow \Delta^\star)$$

is exactly the same as that obtained from P by eliminating

$$Q, (A_1, ..., A_m \Rightarrow B_1, ..., B_n), \cdots, (\Gamma \Rightarrow \Delta)$$

- *the sequence of inference rules between* $(C_1^\star, ..., C_m^\star \Rightarrow D_1^\star, ..., D_n^\star)$ *and* $(\Gamma^\star \Rightarrow \Delta^\star)$ *in* P' *is the same as that between* $(A_1, ..., A_m \Rightarrow B_1, ..., B_n)$ *and* $(\Gamma \Rightarrow \Delta)$ *in* P.

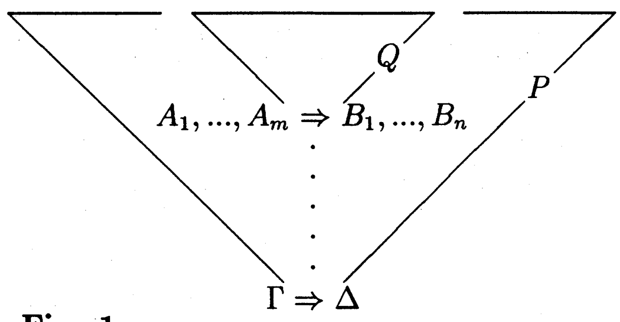

Fig. 1

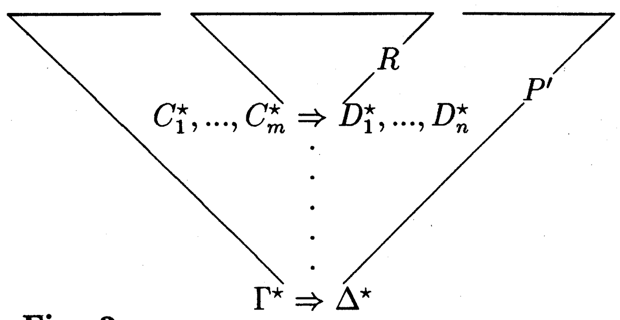

Fig. 2

Proof By induction on the number of sequents between $(A_1, ..., A_m \Rightarrow B_1, ..., B_n)$ and $(\Gamma \Rightarrow \Delta)$ in P. ∎

3 Contraction-Elimination for $\mathbf{LJ}^\star_\rightarrow$

In this section we prove Theorem 2.8 (2).

Lemma 3.1 *Suppose that* P *is a proof of* $A_1, A_2, ..., A_n, \Gamma \Rightarrow B$ $(n \geq 0)$ *in* $\mathbf{LJ}^\star_\rightarrow$, *and the core of* A_i *is* $\star$ *for all* i. *Then we can get a proof* Q *of* $\Gamma^\star \Rightarrow B^\star$ *in* $\mathbf{LJ}^\star_\rightarrow$, *for some* $\Gamma^\star \Rightarrow B^\star$, *such that the number of left contractions in* Q *is less than or equal to that in* P.

Proof By induction on the length of P. (See [2] for the detail.) ∎

We say that an instance of the left contraction rule

$$\frac{A, B, \Gamma \Rightarrow C}{\mathcal{S}(A, B), \Gamma \Rightarrow C}$$

is *essential* if both the core of A and the core of B are propositional variables, and is *nonessential* if it is not essential.

Lemma 3.2 *Suppose that*

- *P is a proof of $A^1, \Gamma \Rightarrow B$ in $\mathbf{LJ}^{\star}_{\rightarrow}$;*
- *there is no nonessential left contraction in P;*
- *a^2 is the core of A^1, and b^3 is another occurrence in A^1 than a^2 (both a and b are propositional variables).*

Then for any ancestor axiom $b^4 \Rightarrow b^5$ of b^3, there exists an ancestor axiom $a^6 \Rightarrow a^7$ of a^2 on the right of $b^4 \Rightarrow b^5$. (Fig. 3)

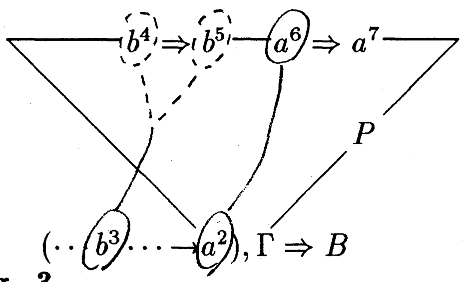

Fig. 3

Proof By induction on the length of P. We distinguish cases according to the form of P, and we show only the following case: P is of the form

$$\frac{\begin{array}{c}\vdots\\ A_1, A_2, \Gamma \Rightarrow B\end{array}}{\mathcal{S}(A_1, A_2), \Gamma \Rightarrow B}\text{ left contraction}$$

($A = \mathcal{S}(A_1, A_2)$). Both the core of A_1 and the core of A_2 are a since this is an essential left contraction. Then we have the following: (1) *Any ancestor axiom of the core of A_i ($i = 1, 2$) is an ancestor axiom of a^2.* On the other hand, by Lemma 2.10, we have the following: (2) *Any ancestor axiom of b^3 is an ancestor axiom of b^0 in A_j for some j ($j = 1$ or 2).* Hence by (1),(2), and the induction hypothesis, we can show that this Lemma holds. ∎

Lemma 3.3 (Contraction-Elimination Lemma for $\mathbf{LJ}^{\star}_{\rightarrow}$) *Let P be a proof of $\Gamma \Rightarrow A$ in $\mathbf{LJ}^{\star}_{\rightarrow}$ such that*

- *$\Gamma \Rightarrow A$ satisfies the no-PPN-condition;*
- *there is at least one left contraction in P.*

Then we can get a proof Q of $\Gamma^{\star} \Rightarrow A^{\star}$ in $\mathbf{LJ}^{\star}_{\rightarrow}$, for some $\Gamma^{\star} \Rightarrow A^{\star}$, such that the number of left contractions in Q is less than that in P.

Proof First we show that there is at least one nonessential left contraction in P. Assume that there is no nonessential left contraction in P. Then there is an essential left contraction

$$\begin{array}{c} \vdots \\ \dfrac{(\cdots \rightarrow a^1), (\cdots \rightarrow a^2), \Delta \Rightarrow B}{(\cdots \rightarrow a), \Delta \Rightarrow B} \\ \vdots \end{array}$$

in P. Let $a^3 \Rightarrow a^4$ and $a^5 \Rightarrow a^6$ be ancestor axioms of, respectively, a^1 and a^2. Then by the no-PPN-condition for $\Gamma \Rightarrow A$, the descendants of a^4 and the descendants of a^6 must be united by an essential left contraction in P. This means that P is of the form

$$\begin{array}{c} \vdots \\ \dfrac{(\cdots a^7 \cdots \rightarrow b^8), (\cdots a^9 \cdots \rightarrow b^{10}), \Pi \Rightarrow C}{(\cdots a \cdots \rightarrow b), \Pi \Rightarrow C} \text{ left contraction} \\ \vdots \end{array}$$

where a^7 and a^9 are descendants of, respectively, a^4 and a^6. Then by Lemma 3.2, there is an ancestor axiom $b^{11} \Rightarrow b^{12}$ of b^8 on the right of $a^3 \Rightarrow a^4$. By iteration of this argument, we have infinitely many axioms in P. Contradiction.

Hence, there is a nonessential left contraction in P, and we can eliminate it by Lemma 3.1. ∎

Now we can prove contraction-elimination theorem for $\mathbf{LJ}^{\star}_{\rightarrow}$:

Proof of Theorem 2.8 (2) By Lemma 3.3, Lemma 2.7, transitivity of $\prec$, and induction on the number of left contractions in the proof. ∎

4 Contraction-Elimination for $\mathbf{LK}^{\star}_{\to}$

In this section we prove Theorem 2.8 (1).

Lemma 4.1

(1) *Let B and C be formulas such that $B \frown C$, and let θ be the natural mapping by $B \prec \mathcal{S}(B,C)$. Moreover, let a^0 be an occurrence of a propositional variable in B, and let $A_1, A_2, ..., A_n$ be formulas such that $\theta(a^0)$ occurs in the form of $(A_n \to A_{n-1} \to \cdots \to A_1 \to \theta(a^0))$ in a sub-formula in $\mathcal{S}(B,C)$. Then a^0 occurs in the form of $(A_n^\star \to A_{n-1}^\star \to \cdots \to A_1^\star \to a^0)$ in a sub-formula in B.*

(2) *Let P be a proof of $\Gamma \Rightarrow \Delta$ in $\mathbf{LK}^{\star}_{\to}$, $a^1 \Rightarrow a$ be an axiom in P, and a^0 be the descendant of a^1 in $\Gamma \Rightarrow \Delta$. If a^0 occurs in the form of $(A_n \to A_{n-1} \to \cdots \to A_1 \to a^0)$ in a sub-formula in $\Gamma \Rightarrow \Delta$, then P is of the form*

$$
\begin{array}{c}
a^1 \Rightarrow a \\
\vdots \\
\dfrac{\Pi_1 \Rightarrow \Sigma_1, A_1^\star \qquad a^2, \Gamma_1 \Rightarrow \Delta_1}{A_1^\star \to a, \Pi_1, \Gamma_1 \Rightarrow \Sigma_1, \Delta_1}\ \text{left} \to \\
\vdots \\
\dfrac{\Pi_2 \Rightarrow \Sigma_2, A_2^\star \qquad A_1^\star \to a^3, \Gamma_2 \Rightarrow \Delta_2}{A_2^\star \to A_1^\star \to a, \Pi_2, \Gamma_2 \Rightarrow \Sigma_2, \Delta_2}\ \text{left} \to \\
\vdots \\
\dfrac{\Pi_n \Rightarrow \Sigma_n, A_n^\star \qquad A_{n-1}^\star \to \cdots \to A_1^\star \to a^{n+1}, \Gamma_n \Rightarrow \Delta_n}{A_n^\star \to A_{n-1}^\star \to \cdots \to A_1^\star \to a, \Pi_n, \Gamma_n \Rightarrow \Sigma_n, \Delta_n}\ \text{left} \to \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
$$

where $a^2, a^3, .., a^{n+1}$ are the descendants of a^1.

Proof

(1) By Lemmas 2.1 and 2.6.

(2) By induction on the length of P, using (1) of this Lemma. ∎

Lemma 4.2 *Let P be a proof of $A_1, A_2, ..., A_n \Rightarrow a$ (a is a propositional variable, and $n \geq 1$) in $\mathbf{LK}^{\star}_{\to}$. Then the core of A_t is the propositional variable a, for some t ($1 \leq t \leq n$).*

Proof See [2]. ∎

We say that an instance of the right contraction rule

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, \mathcal{S}(A, B)}$$

is *essential* if $A = B = a$ (a is a propositional variable), and is *nonessential* if it is not essential.

In the following, we will use ordinals less than ω^ω. Let α and β be ordinals such that

$$\begin{aligned} \alpha &= \omega^{n_1} + \omega^{n_2} + \cdots + \omega^{n_i} + k \quad (n_1 \geq n_2 \geq \cdots \geq n_i > 0, \quad k < \omega) \\ \beta &= \omega^{m_1} + \omega^{m_2} + \cdots + \omega^{m_j} + l \quad (m_1 \geq m_2 \geq \cdots \geq m_j > 0, \quad l < \omega) \end{aligned}$$

Then $\alpha \sharp \beta$ denotes the ordinal

$$\omega^{p_1} + \omega^{p_2} + \cdots + \omega^{p_{i+j}} + k + l$$

where $(p_1, p_2, ..., p_{i+j})$ is a permutation of $(n_1, n_2, ..., n_i, m_1, m_2, ..., m_j)$ such that

$$p_1 \geq p_2 \geq \cdots \geq p_{i+j}.$$

We call $\alpha \sharp \beta$ the *natural sum* of α and β.

We define the *width* of a proof P as the number of all occurrences of axioms in P. We define the *length* of a formula A as the number of all occurrences of atomic formulas in A. Let P be a proof in $\mathbf{LK}^{\star}_{\rightarrow}$ of the form

$$\dfrac{\begin{array}{c} \vdots\, R \\ \Gamma \Rightarrow \Delta, A, B \end{array}}{\begin{array}{c}\Gamma \Rightarrow \Delta, \mathcal{S}(A, B) \\ \vdots \end{array}} \text{ right contraction}$$

and let the width of R be m and the length of $\mathcal{S}(A, B)$ be n. Then we define the *degree* of this right contraction as an ordinal $(\omega^m + n)$. Also we define the *degree* of P as the natural sum of the degrees of all right contractions in P.

Lemma 4.3 *Let P be a proof of $\Gamma \Rightarrow a$ (a is a propositional variable) in $\mathbf{LK}^{\star}_{\rightarrow}$ such that*

- *P is of the form*

$$\dfrac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow a, a \end{array}}{\Gamma \Rightarrow a} \text{ right contraction}$$

- *there is no nonessential right contraction in P.*

Then we can get a proof Q of $\Gamma \Rightarrow a$ *in* $\mathbf{LK}^{\star}_{\rightarrow}$ *such that*

- *Q is of the form*

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots \\ \Pi \Rightarrow a, A & & B, \Sigma \Rightarrow a\end{matrix}}{A{\to}B, \Pi, \Sigma \Rightarrow a, a}\ \text{left} \to}{\begin{matrix}A{\to}B, \Pi, \Sigma \Rightarrow a \\ \vdots\ (*) \\ \Gamma \Rightarrow a\end{matrix}}\ \text{right contraction}$$

- *there is no right contraction in* $(*)$*;*
- *degree of* $P \geq$ *degree of* Q*.*

Proof See [2]. ∎

Lemma 4.4 *Let P be a proof of* $(\Gamma \Rightarrow \Delta, a^1, a^2, ..., a^n)$ *in* $\mathbf{LK}^{\star}_{\rightarrow}$ *(a is a propositional variable, and* $n \geq 1$*) and A be a formula such that the following condition holds: For any partner* a^0 *of* a^i $(1 \leq i \leq n)$*,* a^0 *occurs in the form of* $(A^{\star}{\to} \cdots {\to} a^0)$ *in a sub-formula in* Γ, Δ*. (We call this condition the* partner condition.*) Then we can get a proof Q of* $(\Gamma^{\star} \Rightarrow \Delta^{\star}, \underbrace{A^{\star}, A^{\star}, ..., A^{\star}}_{n})$ *in* $\mathbf{LK}^{\star}_{\rightarrow}$*, for some* $(\Gamma^{\star} \Rightarrow \Delta^{\star}, A^{\star}, A^{\star}, ..., A^{\star})$*, such that*

- *width of* $P >$ *width of* Q
- *degree of* $P \geq$ *degree of* Q*.*

Proof By induction on the length of P. We distinguish cases according to the form of P, and we show only some critical cases.

(Case 1) P is of the form

$$\dfrac{\begin{matrix}\vdots\ P' \\ \Gamma \Rightarrow \Delta, a, a, ..., a, B_1, B_2\end{matrix}}{\Gamma \Rightarrow \Delta, a^1, a^2, ..., a^{n-1}, \mathcal{S}(B_1, B_2)}\ \text{right contraction}$$

where $a^n = \mathcal{S}(B_1, B_2)$. In this case, both B_1 and B_2 are a, or one of B_i is a and the other is $\star$; and the partner condition holds for this upper sequent. Then by the induction hypothesis, we get a proof Q' of

$$\Gamma^{\star} \Rightarrow \Delta^{\star}, \underbrace{A^{\star}, A^{\star}, ..., A^{\star}}_{n+1}$$

where the width is decreased and the degree is unchanged or decreased compared with P', and we get the required proof Q as follows:

$$\dfrac{\begin{array}{c}\vdots\ Q' \\ \Gamma^\star \Rightarrow \Delta^\star, A^\star, ..., A^\star\end{array}}{\Gamma^\star \Rightarrow \Delta^\star, \underbrace{A^\star, ..., A^\star}_{n}}\ \text{right contraction}$$

(Note that $\mathcal{S}(A^\star, A^\star)$ is $A^\star$.) In spite of the fact that

$$\text{length of } a \leq \text{ length of } A^\star,$$

the degree of this right contraction is less than that of the last right contraction in P due to the width of Q'.

(Case 2) P is of the form

$$\dfrac{\begin{array}{c}\vdots\ P' \\ \Gamma \Rightarrow \Pi, a, a, ..., a, B_1, B_2\end{array}}{\Gamma \Rightarrow \Pi, a^1, a^2, ..., a^n, \mathcal{S}(B_1, B_2)}\ \text{right contraction}$$

By Lemma 4.1 (1), the partner condition holds for this upper sequent. Then we get the required proof Q by using the induction hypothesis. (Note that $\mathcal{S}(B_1^\star, B_2^\star) \prec \mathcal{S}(B_1, B_2)$.)

(Case 3) P is of the form

$$\dfrac{\begin{array}{c}\vdots\ P_1 \\ \Pi \Rightarrow \Sigma, \overbrace{a, a, ..., a}^{k}, B\end{array} \quad \begin{array}{c}\vdots\ P_2 \\ C, \Theta \Rightarrow \Lambda, \overbrace{a, a, ..., a}^{l}\end{array}}{B \rightarrow C, \Pi, \Theta \Rightarrow \Sigma, \Lambda, a^1, a^2, ..., a^n}\ \text{left} \rightarrow$$

where $(n = k + l)$, $(l \geq 1)$, and the partner condition does not hold for this right-hand upper sequent. Since the partner condition holds for the lower sequent, the core of C is a, and B is $A^\star$. When $k = 0$, we get the required proof Q as follows:

$$\begin{array}{c}\vdots\ P_1 \\ \Pi \Rightarrow \Sigma, A^\star \\ \vdots\ \text{some left/right weakenings} \\ (B \rightarrow C)^\star, \Pi, \Theta^\star \Rightarrow \Sigma, \Lambda^\star, \underbrace{A^\star, A^\star, ..., A^\star}_{n}\end{array}$$

Obviously the width is decreased and the degree is unchanged or decreased. When $k > 0$, the partner condition holds for the last sequent of P_1. Then by the induction hypothesis for P_1, we have the proof

$$\begin{array}{c}\vdots \\ \Pi^\star \Rightarrow \Sigma^\star, \overbrace{A^\star, A^\star, ..., A^\star}^{k}, A^\star\end{array}$$

and we can get the required proof Q by applying some left/right weakenings. ∎

Lemma 4.5 *Let P be a proof of $(\Gamma \Rightarrow \Delta, \star^1, \star^2, ..., \star^n)$ in $\mathbf{LK}^\star_\rightarrow$ $(n \geq 0)$. Then we can get a proof Q of $\Gamma \Rightarrow \Delta$ in $\mathbf{LK}^\star_\rightarrow$ such that (width of P = width of Q) and (degree of P $\geq$ degree of Q).*

Proof By induction on the length of P. ∎

Lemma 4.6 *Let P be a proof of $(\Gamma \Rightarrow \Delta, A_1 \rightarrow B_1, A_2 \rightarrow B_2, ..., A_n \rightarrow B_n)$ in $\mathbf{LK}^\star_\rightarrow$ $(n \geq 0)$. Then we can get a proof Q of $(A_1, A_2, ..., A_n, \Gamma \Rightarrow \Delta, B_1, B_2, ..., B_n)$ in $\mathbf{LK}^\star_\rightarrow$ such that (width of P = width of Q) and (degree of $P \geq$ degree of Q).*

Proof By induction on the length of P. ∎

Lemma 4.7 (Contraction-Elimination Lemma for $\mathbf{LK}^\star_\rightarrow$) *Let P be a proof of $\Gamma \Rightarrow A$ in $\mathbf{LK}^\star_\rightarrow$ such that*

- *$\Gamma \Rightarrow A$ satisfies the no-PNN-condition;*
- *degree of $P > 0$.*

Then we can get a proof Q of $\Gamma^\star \Rightarrow A^\star$ in $\mathbf{LK}^\star_\rightarrow$, for some $\Gamma^\star \Rightarrow A^\star$, such that

- *degree of $P >$ degree of Q.*

Proof We distinguish cases according to the number and form of nonessential right contractions in P.

(Case 1) P contains a nonessential right contraction:

$$\frac{\Pi \Rightarrow \Sigma, B, \star}{\Pi \Rightarrow \Sigma, B}$$

Then using Lemma 4.5, we get the required proof Q.

(Case 2) P contains a nonessential right contraction:

$$\frac{\Pi \Rightarrow \Sigma, B \rightarrow C, D \rightarrow E}{\Pi \Rightarrow \Sigma, \mathcal{S}(B \rightarrow C, D \rightarrow E)}$$

Then using Lemma 4.6, we get the required proof Q.

(Case 3) There is no nonessential right contraction in P. Then consider the lower-most right contraction in P, say

$$\frac{\Delta \Rightarrow \Delta', a, a}{\Delta \Rightarrow \Delta', a}$$

(a is a propositional variable). The sequence Δ' is empty because other inference rules than right contraction do not decrease the number of the occurrences of the formulas in the right-hand side of a sequent. Then, by Lemma 4.3, we get a proof P' as

$$\dfrac{\dfrac{\begin{array}{c}\vdots P_1 \\ \Pi \Rightarrow a^1, B'\end{array} \quad \begin{array}{c}\vdots P_2 \\ C', \Sigma \Rightarrow a\end{array}}{B' \to C', \Pi, \Sigma \Rightarrow a, a}\ \text{left} \to}{\begin{array}{c} B' \to C', \Pi, \Sigma \Rightarrow a \\ \vdots \\ \Gamma \Rightarrow A\end{array}}\ \text{right contraction}$$

where there is no right contraction below $(B' \to C', \Pi, \Sigma \Rightarrow a)$, and (degree of $P \geq$ degree of P'). Let B and C be the descendants of, respectively, B' and C', in $\Gamma \Rightarrow A$. Now by applying Lemma 4.2 to P_2, we know that there is a formula F in $\{C', \Sigma\}$ such that the core of F is a. Then we consider the following cases.

(Sub-case 3-1) The core of C' is a. In this case, P' is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots P_1 \\ \Pi \Rightarrow a^1, B^\star\end{array} \quad \begin{array}{c}\vdots P_2 \\ C^\star_m \to C^\star_{m-1} \to \cdots \to C^\star_1 \to a^2, \Sigma \Rightarrow a\end{array}}{B^\star \to C^\star_m \to \cdots \to C^\star_1 \to a, \Pi, \Sigma \Rightarrow a, a}\ \text{left} \to}{\begin{array}{c} B^\star \to C^\star_m \to \cdots \to C^\star_1 \to a, \Pi, \Sigma \Rightarrow a \\ \vdots \\ \Gamma \Rightarrow A\end{array}}\ \text{right contraction}$$

where

$$C = C_m \to C_{m-1} \to \cdots \to C_1 \to a \quad (m \geq 0).$$

Let a^0 be a partner of a^1. By the no-PNN-condition for $\Gamma \Rightarrow A$, the descendant of a^0 in $\Gamma \Rightarrow A$ is also the descendant of a^2. Now we consider the following cases.

(Sub-sub-case 3-1-1) Any partner a^0 of a^1 occurs in the form of

$$(B^\star \to C^\star_m \to C^\star_{m-1} \to \cdots \to C^\star_1 \to a^0)$$

in a sub-formula in the last sequent of P_1. In this case, we apply Lemma 4.4 to P_1, and get a proof Q_1 of $(\Pi^\star \Rightarrow B^\star, B^\star)$ such that (width of $P_1 >$ width of Q_1) and (degree of P_1

$\geq$ degree of Q_1). Then we get the proof Q as follows:

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\; Q_1\\ \Pi^\star \Rightarrow B^\star, B^\star\end{array}}{\Pi^\star \Rightarrow B^\star}\text{ right contraction} \qquad \begin{array}{c}\vdots\; P_2\\ C^\star, \Sigma \Rightarrow a\end{array}}{B^\star \to C^\star, \Pi^\star, \Sigma \Rightarrow a}\text{ left} \to \\
\vdots \\
\Gamma^\star \Rightarrow A^\star
\end{array}$$

Lemma 2.12 guarantees that this is the required proof.

(Sub-sub-case 3-1-2) Not the case (3-1-1). In this case, Lemma 4.1 (2) tells us that P' is of the form

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\; P_3\\ \Theta \Rightarrow B^\star\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\; P_1\\ \Pi \Rightarrow a^1, B^\star\end{array} \quad \begin{array}{c}\vdots\; P_2\\ C^\star, \Sigma \Rightarrow a\end{array}}{B^\star \to C^\star, \Pi, \Sigma \Rightarrow a, a}\text{ left} \to}{B^\star \to C^\star, \Pi, \Sigma \Rightarrow a}\text{ right contraction}\\ \vdots\; (*)\\ C^\star_m \to \cdots \to C^\star_1 \to a^0, \Lambda \Rightarrow D\end{array}}{B^\star \to C^\star_m \to \cdots \to C^\star_1 \to a, \Theta, \Lambda \Rightarrow D}\text{ left} \to \\
\vdots\; (**) \\
\Gamma \Rightarrow A
\end{array}$$

where a^0 is a descendant of a partner of a^1. Then we get the proof Q as follows:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\; P_3\\ \Theta \Rightarrow B^\star\end{array} \qquad \dfrac{\begin{array}{c}\vdots\; P_2\\ C^\star, \Sigma \Rightarrow a\\ \vdots\; \text{some left weakenings}\\ C^\star, (B \to C)^\star, \Pi^\star, \Sigma \Rightarrow a\\ \vdots\; (*)(C^\star \text{ is untouched})\\ C^\star, (C_m \to \cdots \to C_1 \to a)^\star, \Lambda^\star \Rightarrow D^\star\end{array}}{C^\star_m \to \cdots \to C^\star_1 \to a, \Lambda^\star \Rightarrow D^\star}\text{ left contraction}}{B^\star \to C^\star_m \to \cdots \to C^\star_1 \to a, \Theta, \Lambda^\star \Rightarrow D^\star}\text{ left} \to \\
\vdots\; (**) \\
\Gamma^\star \Rightarrow A^\star
\end{array}$$

Lemma 2.12 guarantees that this is the required proof.

(Sub-case 3-2) Not the case (3-1). In this case, there is a formula E' in Σ such that the core of E' is a. Let E be the descendant of E' in $\Gamma \Rightarrow A$, and let

$$E = E_n \to E_{n-1} \to \cdots \to E_1 \to a \quad (n \geq 0),$$

$$\Sigma = E', \Sigma_0.$$

Then P' is of the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots P_1\\ \Pi \Rightarrow a^1, B^\star\end{array} \quad \begin{array}{c}\vdots P_2\\ C^\star, E_n^\star \to \cdots \to E_1^\star \to a^2, \Sigma_0 \Rightarrow a^3\end{array}}{B^\star \to C^\star, \Pi, E_n^\star \to \cdots \to E_1^\star \to a, \Sigma_0 \Rightarrow a, a}\text{ left }\to}{\begin{array}{c}B^\star \to C^\star, \Pi, E_n^\star \to \cdots \to E_1^\star \to a, \Sigma_0 \Rightarrow a\\ \vdots \\ \Gamma \Rightarrow A\end{array}}\text{ right contraction}$$

Let a^0 be a partner of a^1 or a partner of a^3. By the no-PNN-condition for $\Gamma \Rightarrow A$, the descendant of a^0 in $\Gamma \Rightarrow A$ is also the descendant of a^2. Now, let p $(1 \leq p \leq n)$ be the maximal number such that the following condition holds: For any occurrence a^0, if a^0 is a partner of a^1 or a partner of a^3, then a^0 occurs in the form of

$$(E_p^\star \to E_{p-1}^\star \to \cdots \to E_1^\star \to a^0)$$

in a sub-formula in the last sequent of P_1 or in that of P_2. If such p does not exist, we define $p = 0$. Then we distinguish cases according to p and n.

(Sub-sub-case 3-2-1) $p = n > 0$. In this case, we apply Lemma 4.4 to P_1 and P_2, and get the proofs $Q_{1,t}$ of $(\Pi^\star \Rightarrow E_t^\star, B^\star)$ and $Q_{2,t}$ of $(C^\star, \Sigma^\star \Rightarrow E_t^\star)$ for any t $(1 \leq t \leq p)$. Then we define proofs R_t $(1 \leq t \leq p)$ as

$$\dfrac{\dfrac{\begin{array}{c}\vdots Q_{1,t}\\ \Pi^\star \Rightarrow E_t^\star, B^\star\end{array} \quad \begin{array}{c}\vdots Q_{2,t}\\ C^\star, \Sigma^\star \Rightarrow E_t^\star\end{array}}{B^\star \to C^\star, \Pi^\star, \Sigma^\star \Rightarrow E_t^\star, E_t^\star}\text{ left }\to}{B^\star \to C^\star, \Pi^\star, \Sigma^\star \Rightarrow E_t^\star}\text{ right contraction}$$

Let α_t be the degree of R_t $(1 \leq t \leq p)$, and β be the degree of

$$\dfrac{\dfrac{\begin{array}{c}\vdots P_1\\ \Pi^\star \Rightarrow a, B^\star\end{array} \quad \begin{array}{c}\vdots P_2\\ C^\star, \Sigma^\star \Rightarrow a\end{array}}{B^\star \to C^\star, \Pi^\star, \Sigma^\star \Rightarrow a, a}\text{ left }\to}{B^\star \to C^\star, \Pi^\star, \Sigma^\star \Rightarrow a}\text{ right contraction}$$

Due to the width and degree of $Q_{i,t}$, we have

$$\beta > \alpha_1 \sharp \alpha_2 \sharp \cdots \sharp \alpha_p.$$

Now let $\Psi = (B{\to}C, \Pi, \Sigma)$, and let S be the proof

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots R_p\\ \Psi^\star \Rightarrow E_p^\star\end{array}\quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots R_2\\ \Psi^\star \Rightarrow E_2^\star\end{array}\quad \dfrac{\begin{array}{c}\vdots R_1\\ \Psi^\star \Rightarrow E_1^\star\end{array}\quad a \Rightarrow a}{E_1^\star{\to}a, \Psi^\star \Rightarrow a}\text{ left }\to}{E_2^\star{\to}E_1^\star{\to}a, \Psi^\star, \Psi^\star \Rightarrow a}\text{ left }\to\\ \vdots\\ E_{p-1}^\star{\to}\cdots{\to}E_1^\star{\to}a, \Psi^\star, \ldots, \Psi^\star \Rightarrow a\end{array}}{E_p^\star{\to}E_{p-1}^\star{\to}\cdots{\to}E_1^\star{\to}a, \Psi^\star, \Psi^\star, \ldots, \Psi^\star \Rightarrow a}\text{ left }\to\\ \vdots \text{ some left contractions}\\ E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, \Psi^\star \Rightarrow a\end{array}$$

Then we get the proof Q as

$$\begin{array}{c}\vdots S\\ \dfrac{E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, B^\star{\to}C^\star, \Pi^\star, E_n^\star{\to}\cdots{\to}E_1^\star{\to}a, \Sigma_0^\star \Rightarrow a}{B^\star{\to}C^\star, \Pi^\star, E_n^\star{\to}\cdots{\to}E_1^\star{\to}a, \Sigma_0^\star \Rightarrow a}\text{ left contraction}\\ \vdots\\ \Gamma^\star \Rightarrow A^\star\end{array}$$

where $p = n$. Lemma 2.12 guarantees that this is the required proof.

(Sub-sub-case 3-2-2) $n > p > 0$. In this case, Lemma 4.1 (2) tells us that P' is of the form

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots P_3\\ \Theta \Rightarrow E_{p+1}^\star\end{array}\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots P_1\\ \Pi \Rightarrow a^1, B^\star\end{array}\quad \begin{array}{c}\vdots P_2\\ C^\star, \Sigma \Rightarrow a^3\end{array}}{B^\star{\to}C^\star, \Pi, \Sigma \Rightarrow a, a}\text{ left }\to}{B^\star{\to}C^\star, \Pi, \Sigma \Rightarrow a}\text{ right contraction}\\ \vdots (*)\\ E_p^\star{\to}\cdots{\to}E_1^\star{\to}a^0, \Lambda \Rightarrow D\end{array}}{E_{p+1}^\star{\to}E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, \Theta, \Lambda \Rightarrow D}\text{ left }\to\\ \vdots (**)\\ \Gamma \Rightarrow A\end{array}$$

where a^0 is a descendant of a partner of a^1 or that of a^3. Then by using the proof S in Sub-sub-case 3-2-1, we get the proof Q as

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots P_3\\ \Theta \Rightarrow E_{p+1}^\star\end{array}\quad \begin{array}{c}\vdots S\\ E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, B^\star{\to}C^\star, \Pi^\star, \Sigma^\star \Rightarrow a\\ \vdots (*)(E_p^\star{\to}\cdots{\to}E_1{\to}a \text{ is untouched})\\ \dfrac{E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, \Lambda^\star \Rightarrow D^\star}{E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, \Lambda^\star \Rightarrow D^\star}\text{ left contraction}\end{array}}{E_{p+1}^\star{\to}E_p^\star{\to}\cdots{\to}E_1^\star{\to}a, \Theta, \Lambda^\star \Rightarrow D^\star}\text{ left }\to\\ \vdots (**)\\ \Gamma^\star \Rightarrow A^\star\end{array}$$

Lemma 2.12 guarantees that this is the required proof.

(Sub-sub-case 3-2-3) $n = 0$ or $p = 0$. In this case, there is the formula a in $\{\Pi, \Sigma\}$. Then we can get the required proof Q as

$$\begin{array}{c} a \Rightarrow a \\ \vdots \text{ some left weakenings} \\ (B' \to C')^\star, \Pi^\star, \Sigma^\star \Rightarrow a \\ \vdots \\ \Gamma^\star \Rightarrow A^\star \end{array}$$

This completes the proof of Lemma 4.7. ∎

Now we can prove the contraction-elimination theorem for $\mathbf{LK}^\star_\to$:

Proof of Theorem 2.8 (1) By Lemma 4.7, Lemma 2.7, transitivity of $\prec$, and induction on the degree of the proof. ∎

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