<table>
<thead>
<tr>
<th>Title</th>
<th>The configuration space of 6 points in $P^2$, the moduli space of cubic surfaces and the Weyl group of type $E_6$ (Theory and applications in computer algebra)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
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<td>Textversion</td>
<td>Departmental Bulletin Paper</td>
</tr>
</tbody>
</table>

Kyoto University
The configuration space of 6 points in $\mathbb{P}^2$, the moduli space of cubic surfaces and the Weyl group of type $E_6$

1. Introduction

My first plan of the talk is to explain my study on the hypergeometric system $E(3,6)$ of type $(3,6)$ ([8]). The system in question admits $\Sigma_6$-action, where $\Sigma_6$ is the symmetric group on 6 letters. This follows from that $E(3,6)$ lives in the configurations space $\mathbb{P}_2^6$ of 6 points in $\mathbb{P}^2$ which admits $\Sigma_6$-action as permutations of the 6 points. Recently M. Yoshida (Kyushu Univ.) pointed out that the $\Sigma_6$-action on the space $\mathbb{P}_2^6$ is naturally extended to $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type $E_6$ (cf. [3]). Moreover, he told me that B. Hunt studied relations between the $W(E_6)$-action in question and the $W(E_6)$-invariant quintic hypersurface of $\mathbb{P}^5$.

Reading his note [4], I felt that it is an interesting exercise for REDUCE user to show whether his conjecture is true or not. For this reason, I changed the original plan and I restrict my attention to the study on $W(E_6)$-actions on $\mathbb{P}^5$ and on $\mathbb{P}_2^6$, namely, to the birational geometry related with the hypergeometric system $E(3,6)$.

It is better for the readers who are interested in SYMBOLIC COMPUTATION to read section 6 first.

2. The hypergeometric function of type $(3,6)$

Though I don’t treat it in this note, I begin this note with introducing the hypergeometric function of type $(3,6)$:

$$E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A_{m_1,m_2,n_1,n_2} x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2}$$

where

$$A_{m_1,m_2,n_1,n_2} = \frac{(a_2, m_1 + m_2)(a_3, n_1 + n_2)(1 - a_5, m_1 + n_1)(1 - a_6, m_2 + n_2)}{m_1! m_2! n_1! n_2! (a_0, m_1 + m_2 + n_1 + n_2)}.$$ 

By definition, $E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2)$ has parameters $a_j \ (j = 0, 2, 3, 5, 6)$. This function is one of 4 variables generalizations of Gaussian hypergeometric function. It
is known that the singularities of the system of differential equations whose solution is $E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2)$ is contained in the union of the 14 hypersurfaces $T_j : p_j = 0 (1 \leq j \leq 14)$, where

\[
p_1 = x_1 y_2 - x_2 y_1 - x_1 + x_2 + y_1 - y_2, \quad p_2 = y_1 - 1, \quad p_3 = x_1 - 1, \\
p_4 = y_2 - 1, \quad p_5 = x_2 - 1, \quad p_6 = y_1 - y_2, \quad p_7 = x_1 - x_2, \quad p_8 = x_1 - y_1, \\
p_9 = x_2 - y_2, \quad p_{10} = x_1 y_2 - x_2 y_1, \quad p_{11} = x_2, \quad p_{12} = x_1, \quad p_{13} = y_2, \quad p_{14} = y_1.
\]

We define birational transformations $s_j (1 \leq j \leq 5)$ on $C^4$ by

\[
s_1 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{1}{x_1}, \frac{1}{x_2}, \frac{y_1}{x_1}, \frac{y_2}{x_2} \right),
\]

\[
s_2 : (x_1, x_2, y_1, y_2) \rightarrow (y_1, y_2, x_1, x_2),
\]

\[
s_3 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{x_1 - y_1}{1 - y_1}, \frac{x_2 - y_2}{1 - y_2}, \frac{y_1}{1 - y_1}, \frac{y_2}{1 - y_2} \right),
\]

\[
s_4 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{1}{x_1}, \frac{1}{x_2}, \frac{y_2}{y_1}, \frac{1}{y_1} \right),
\]

\[
s_5 : (x_1, x_2, y_1, y_2) \rightarrow (x_2, x_1, y_2, y_1).
\]

Then the group generated by $s_j (1 \leq j \leq 5)$ is identified with $\Sigma_6$ because

\[
s_j^2 = id. \quad (1 \leq j \leq 5), \quad s_j s_k = s_k s_j \quad (|j - k| > 1),
\]

\[
s_j s_k s_j = s_k s_j s_k \quad (|j - k| = 1).
\]

Let $r$ be a birational transformation on $C^4$ defined by

\[
r : (x_1, x_2, y_1, y_2) \rightarrow (1/x_1, 1/x_2, 1/y_1, 1/y_2).
\]

Then the group $\tilde{G}$ generated by $s_1, \ldots, s_5$ and $r$ is isomorphic to the Weyl group $W(E_6)$ of type $E_6$ which will be seen later (cf. [3], [4]).

We define the hypersurface $T_{15} : p_{15} = 0$, where

\[
p_{15} = x_1 y_2 (1 - y_1) (1 - x_2) - x_2 y_1 (1 - x_1) (1 - y_2).
\]

It follows from the definition that $s_1, \ldots, s_5, r$ and therefore all the elements of $\tilde{G}$ are biregular outside the union $T$ of the hypersurfaces $T_j (1 \leq j \leq 15)$.

3. The Weyl group $W(E_6)$

Let $E_R$ be a Cartan subalgebra of a compact Lie algebra of type $E_6$, i.e. $E_R \simeq R^6$. Let $t = (t_1, t_2, t_3, t_4, t_5, t_6)$ be a coordinate system of $E_R$ such that the roots of type $E_6$ are:

\[
\pm (t_i \pm t_j), \quad 1 \leq i < j \leq 5
\]

\[
\pm \frac{1}{2} (\delta_1 t_1 + \delta_2 t_2 + \delta_3 t_3 + \delta_4 t_4 + \delta_5 t_5 + \delta_6 t_6)
\]
(where $\delta_j = \pm 1$ and $\prod_j \delta_j = 1$). Note that compared with the notation in [1], our variables $t_i = \epsilon_i$, $i = 1, \ldots, 5$, while our coordinate $t_6$ is denoted $\epsilon_6 - \epsilon_7 - \epsilon_8$ in [1]. We now introduce the following linear forms on $E_R$:

$$h = -\frac{1}{2}(t_1 + \cdots + t_6),$$

$$h_{ij} = -t_{j-1} + h_0, \quad j = 2, \ldots, 6$$

$$h_{jk} = t_{j-1} - t_{k-1}, \quad j, k \neq 1$$

$$h_{jkl} = -t_{j-1} - t_{k-1} - t_{l-1} + h_0, \quad j, k, l \neq 1$$

where

$$h_0 = \frac{1}{2}(t_1 + \cdots + t_5 - t_6).$$

Then the totality of $h, h_{ij}, h_{ijk}$ forms a set of positive roots of type $E_6$. Let $s$ (resp. $s_{ij}, s_{ijk}$) be the reflection on $E_R$ with respect to the hyperplane $h = 0$ (resp. $h_{ij} = 0, h_{ijk} = 0$). Then the Weyl group of type $E_6$ which is denoted by $W(E_6)$ in this note is the group generated by the 36 reflections defined above.

As a system of simple roots, we take

$$\alpha_1 = h_{12}, \quad \alpha_2 = h_{123}, \quad \alpha_3 = h_{23}, \quad \alpha_4 = h_{34}, \quad \alpha_5 = h_{45}, \quad \alpha_6 = h_{56}.$$

Then the Dynkin diagram is:

$$\begin{align*}
\alpha_1 & \quad \alpha_3 & \quad \alpha_4 & \quad \alpha_5 & \quad \alpha_6 \\
\quad & \quad \mid \\
\quad & \quad \alpha_2
\end{align*}$$

Let $g_j$ be the reflection on $E_R$ with respect to the root $\alpha_j$ ($j = 1, \ldots, 6$). Then, from the definition,

$$g_1 = s_{12}, \quad g_2 = s_{123}, \quad g_3 = s_{23}, \quad g_4 = s_{34}, \quad g_5 = s_{45}, \quad g_6 = s_{56}.$$

Let $E$ be the complexification of $E_R$ and we extend the action of $W(E_6)$ on $E_R$ to that on $E$ in a natural manner. Moreover let $P^5$ be the projective space associated to $E$. Then the $W(E_6)$-action on $E$ induces a projective linear action of $W(E_6)$ on $P^5$.

### 4. The configuration space of 6 points in $P^2$

We have already defined a birational action of $W(E_6)$ on $C^4$ in section 2. In this section, we explain that the birational transformations $s_1, \ldots, s_5, r$ naturally arise from the study of the configuration space of 6 points in $P^2$.

For this purpose, we first introduce the linear space $W$ of $3 \times 6$ matrices:
$W = \{ X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix} ; x_{ij} \in \mathbb{C} \ (1 \leq i \leq 3, 1 \leq j \leq 6) \}.$

Then $W$ admits a left $GL(3, \mathbb{C})$-action and a right $GL(6, \mathbb{C})$-action in a natural way. For a moment, we identify $(\mathbb{C}^*)^6$ with the maximal torus of $GL(6, \mathbb{C})$ consisting of diagonal matrices and consider the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ on $W$ instead of that of $GL(3, \mathbb{C}) \times GL(6, \mathbb{C})$.

For simplicity, we write $X = (X_1, X_2)$ for the matrix $X \in W$, where both $X_1, X_2$ are $3 \times 3$ matrices. For any $3 \times 3$ matrix $Y = (y_{ij})_{1 \leq i, j \leq 3}$ with the condition $y_{ij} \neq 0 \ (1 \leq i, j \leq 3)$, we define a $3 \times 3$ matrix

$$\sigma(Y) = \begin{pmatrix} 1 \\ y_{ij} \end{pmatrix}_{1 \leq i, j \leq 3}.$$

following a suggestion of M. Yoshida. Moreover, we put

$$D(i_1, i_2, i_3) = \det \begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix}$$

for a given matrix $X \in W$.

Using these notation, we define subsets $W'$, $W_0$ of $W$ by

$$W' = \{ X \in W ; D(i_1, i_2, i_3) \neq 0 \ (1 \leq i_1 < i_2 < i_3 \leq 6) \},$$

$$W_0 = \{(X_1, X_2) \in W' ; (I_3, Cof(X_1^{-1}X_2)), (I_3, \sigma(X_1^{-1}X_2)) \in W' \},$$

where $Cof(Y) = (\det Y)^tY^{-1}$ is the cofactor matrix of a given square matrix $Y$.

It is clear that the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ on $W$ naturally induces that on each of $W'$, $W_0$. In the sequel, we mainly consider the quotient space of $W_0$ under the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$, that is,

$$W_Q = GL(3, \mathbb{C}) \backslash W_0/(\mathbb{C}^*)^6.$$

It is clear from the definition that for any element $X \in W_0$, there are $(g, h) \in GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ and $(x_1, x_2, y_1, y_2) \in \mathbb{C}^4$ such that

$$gXh = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

In particular $(x_1, x_2, y_1, y_2)$ is uniquely determined for $X \in W_0$. In this sense, $W_Q = GL(3, \mathbb{C}) \backslash W_0/(\mathbb{C}^*)^6$ is identified with an open subset of $\mathbb{C}^4$. Note that $(x_1, x_2, y_1, y_2)$ is the variables $(x_1, x_2, y_1, y_2)$ of section 2. Then $W_Q = \mathbb{C}^4 - T$. 
Changes of column vectors of $X \in W_0$ induce birational transformations on $\mathbb{C}^4$ with coordinate system $(x_1, x_2, y_1, y_2)$. The action $s_j (1 \leq j \leq 5)$ introduced in section 2 is nothing but the birational transformation on $\mathbb{C}^4$ corresponding to the change of the $j$-th column vector and $(j+1)$-column vector of $X \in W_0$. Moreover $W_Q$ admits an involution induced from the action on $W_0$ defined by

$$\tilde{r} : (X_1, X_2) \rightarrow (I_3, \sigma(X_1^{-1}X_2))$$

for any $(X_1, X_2) \in W_0$. The involution $r$ defined in section 2 is equal to that induced from $\tilde{r}$.

The following theorem which seems known shows a concrete correspondence between $W(E_6)$ and the group $\tilde{G}$ introduced in section 2.

**Theorem 4.1.** The correspondence

$$g_1 \rightarrow s_1, \quad g_2 \rightarrow r, \quad g_3 \rightarrow s_2, \quad g_4 \rightarrow s_3, \quad g_5 \rightarrow s_4, \quad g_6 \rightarrow s_5$$

induces a group isomorphism of $W(E_6)$ to the group $\tilde{G}$.

**Remark.** In [3], it is stated that there is a $W(E_6)$-action on $W_Q$. See also [4].

5. $W(E_6)$-equivariant maps

We first define rational functions on $E$ by

$$x_1(t) = \frac{h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135}}{h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235}}, \quad x_2(t) = \frac{h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136}}{h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}},$$

$$y_1(t) = \frac{h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125}}{h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235}}, \quad y_2(t) = \frac{h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126}}{h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236}},$$

$$\lambda(t) = \frac{h_{34} \cdot h_{345} \cdot h_{26} \cdot h_{256}}{h_{24} \cdot h_{245} \cdot h_{16} \cdot h_{136}} \cdot \frac{h_{13} \cdot h_{136} \cdot h_{24} \cdot h_{246}}{h_{12} \cdot h_{126} \cdot h_{34} \cdot h_{346}},$$

$$\mu(t) = \frac{h_{456} \cdot h_{235} \cdot h_{134} \cdot h_{126}}{h \cdot h_{15} \cdot h_{24} \cdot h_{36}}, \quad \nu(t) = \frac{h_{25} \cdot h_{235} \cdot h_{46} \cdot h_{346}}{h_{24} \cdot h_{234} \cdot h_{56} \cdot h_{356}} \cdot \frac{h_{16} \cdot h_{136} \cdot h_{24} \cdot h_{234}}{h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}},$$

$$\rho(t) = \frac{h_{24} \cdot h_{235} \cdot h_{36} \cdot h_{356}}{h_{23} \cdot h_{235} \cdot h_{46} \cdot h_{496}} \cdot \frac{h_{15} \cdot h_{156} \cdot h_{24} \cdot h_{246}}{h_{14} \cdot h_{146} \cdot h_{23} \cdot h_{236}},$$

where $h, h_{ij}, h_{ijk}$ denote linear functions on $E$ introduced in section 3. Since all the rational functions above are homogeneous of degree zero, they are regarded as functions on $\mathbb{P}^5$. Therefore defining

$$F_1(t) = (x_1(t), x_2(t), y_1(t), y_2(t)), \quad F_2(t) = (\lambda(t), \mu(t), \nu(t), \rho(t)),$$

we obtain two maps $F_1, F_2$ from $\mathbb{P}^5$ to $\mathbb{C}^4$. The roles of $F_1, F_2$ will become clear in Theorem 5.1 which will be given later. To define $F_1, F_2$, I am indebted to [4]. We are going to explain the meaning of $x_j(t), y_j(t)$ following [4].
We begin with defining the cross ratio. Let $\xi_i = [\xi_{1i} : \xi_{2i} : \xi_{3i} : \xi_{4i}] (1 \leq i \leq 5)$ be five points of $\mathbb{P}^2$ and let $l : q_1 u_1 + q_2 u_2 + q_3 u_3 = 0$ be a generic line in $\mathbb{P}^2$. We denote by $[1 : z_i : w_i]$ the intersection of $l$ and the line passing through the points $\xi_1$ and $\xi_i$. Then we put

$$ CR(\xi_2, \xi_3, \xi_4, \xi_5; \xi_1) = \frac{(z_2 - z_4)(z_3 - z_5)}{(z_2 - z_5)(z_3 - z_4)} $$

which is in fact a cross ratio of $z_2, z_3, z_4, z_5$.

Now we consider a matrix of the form

$$ X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & x_1 & x_2 & 0 & 0 & 1 & 1 & y_1 & y_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}. $$

From the matrix $X$, we define six points $\xi_i (i = 1, \cdots, 6)$ in $\mathbb{P}^2$ in a usual manner, that is,

$$ \xi_1 = [1 : 0 : 0], \quad \xi_2 = [0 : 1 : 0], \quad \xi_3 = [0 : 0 : 1], $$

$$ \xi_4 = [1 : 1 : 1], \quad \xi_5 = [1 : x_1 : y_1], \quad \xi_6 = [1 : x_2 : y_2]. $$

Then we can compute $CR(\xi_2, \xi_3, \xi_4, \xi_5; \xi_1)$ explicitly for various $i_1, i_2, i_3, i_4, i_5$.

On the other hand, we put

$$ CR'(i_2, i_3, i_4, i_5; i_1) = \frac{h_{i_2 i_4} h_{i_1 i_2 i_4} h_{i_3 i_5} h_{i_1 i_3 i_5}}{h_{i_3 i_4} h_{i_1 i_3 i_4} h_{i_2 i_5} h_{i_1 i_2 i_5}}. $$

By definition, $CR'(i_2, i_3, i_4, i_5; i_1)$ is a function on $\mathbb{P}^5$. Then from the equation

$$ CR(\xi_2, \xi_3, \xi_4, \xi_5; \xi_1) = CR'(i_2, i_3, i_4, i_5; i_1), $$

we obtain various equalities. In particular, by computing the cases

$$(i_1, i_2, i_3, i_4, i_5) = (3, 2, 1, 4, 5), (3, 2, 1, 4, 6), (2, 1, 3, 4, 5), (2, 1, 3, 4, 6),$$

we have the definition of $x_1(t), x_2(t), y_1(t), y_2(t)$ at the beginning of this section.

Let $F_3$ be the birational transformation on $\mathbb{C}^4$ defined by $F_3(x_1, x_2, y_1, y_2) = (\lambda, \mu, \nu, \rho)$, where

$$ \lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)}, $$

$$ \mu = \frac{(y_1 - 1)(x_2 - y_2) - (y_2 - 1)(x_1 - y_1)}{x_1 x_2 y_1 - x_1 x_2 y_2 - x_1 y_1 y_2 + x_1 y_2 + x_2 y_1 y_2 - x_2 y_1}, $$

$$ \nu = \frac{(x_1 y_2 - x_2 y_1)(x_2 - 1)(y_2 - 1)}{(x_1 - x_2)(x_2 - y_2)(y_1 - y_2)}, $$

$$ \rho = \frac{(x_1 - x_2)(x_2 - y_2)(y_1 - 1)}{((x_1 - 1)(x_2 - y_2) - (x_1 - y_1)(x_2 - 1))(y_2 - 1)x_2}. $$
It is easy to show that $F_3$ is birational, because its inverse is given by

$$F_3^{-1}(\lambda, \mu, \nu, \rho) = \left(\frac{(\lambda \rho - 1)(\lambda \mu \nu \rho - 1)}{(\lambda \mu \rho - 1)(\lambda \nu \rho - 1)}\right) \left(\frac{(\lambda \rho - 1)(\lambda \mu \nu - 1)(\rho - 1)}{(\lambda \mu \rho - 1)(\lambda \nu \rho - 1)}\right) \left(\frac{\mu - 1}{\mu \rho - 1}\right) \left(\frac{\nu - 1}{\nu \rho - 1}\right) \left(\frac{\rho - 1}{\rho \mu - 1}\right).$$

By the map $F_3$, the action of $W(E_6)$ on the $(x_1, x_2, y_1, y_2)$-space implies that on the $(\lambda, \mu, \nu, \rho)$-space. In fact, we define the following six birational transformations on the $(\lambda, \mu, \nu, \rho)$-space (cf. [7]):

$$\tilde{g}_1: \begin{cases} 
\lambda \rightarrow \frac{\lambda \mu \nu \rho^2(1 - \lambda)}{(\lambda \mu \nu \rho^2 - 1)} \\
\mu \rightarrow \frac{(\lambda \mu \rho - 1)(\lambda \mu \nu - 1)/((\nu \rho - 1)(\lambda \mu \nu - 1))}
\nu \rightarrow \frac{(\lambda \nu \rho - 1)(\lambda \mu \nu - 1)/((\mu \rho - 1)(\lambda \mu \nu - 1))}
\rho \rightarrow \frac{(\lambda \rho - 1)(\lambda \mu \nu \rho - 1)/((\rho \lambda - 1)(\lambda \mu \nu \rho - 1))}
\end{cases}$$

$$\tilde{g}_2: (\lambda, \mu, \nu, \rho) \rightarrow (\lambda, 1/\mu, \nu, \mu \rho)$$

$$\tilde{g}_3: (\lambda, \mu, \nu, \rho) \rightarrow (1/\lambda, \mu, \nu, \lambda \rho)$$

$$\tilde{g}_4: (\lambda, \mu, \nu, \rho) \rightarrow (\lambda \rho, \mu \rho, \nu \rho, 1/\rho)$$

$$\tilde{g}_5: (\lambda, \mu, \nu, \rho) \rightarrow (\lambda, \mu, 1/\nu, \nu \rho)$$

$$\tilde{g}_6: \begin{cases} 
\lambda \rightarrow \frac{(\lambda \nu \rho - 1)(\lambda \mu \nu \rho - 1)/((\mu \nu - 1)(\lambda \mu \nu - 1))}
\mu \rightarrow \frac{(\mu \nu \rho^2(1 - \nu)}{(\mu \nu \rho^2 - 1)}
\nu \rightarrow \frac{(\nu \rho - 1)(\lambda \mu \nu \rho - 1)/((\rho \nu - 1)(\lambda \mu \nu - 1))}
\rho \rightarrow \frac{(\nu \rho - 1)(\lambda \mu \nu \rho - 1)/((\rho \nu - 1)(\lambda \mu \nu - 1))}
\end{cases}$$

Let $G_1$ be the group generated by $\tilde{g}_j$ $(j = 1, \cdots, 6)$. Then the correspondence

$$g_j \rightarrow \tilde{g}_j \quad j = 1, \cdots, 6$$

is an isomorphism between $W(E_6)$ and $G_1$.

Needless to say, $F_1$ (resp. $F_2$) is regarded as a map from $\mathbb{P}^5$ to the $(x_1, x_2, y_1, y_2)$-space (resp. the $(\lambda, \mu, \nu, \rho)$-space.) Moreover, $F_3$ is regarded as a map from the $(x_1, x_2, y_1, y_2)$-space to the $(\lambda, \mu, \nu, \rho)$-space.

**Theorem 5.1.** The three maps $F_j$ $(j = 1, 2, 3)$ are $W(E_6)$-equivariant and

$$F_3 \circ F_1(g(t)) = F_2(g(t)) \quad (\forall t \in \mathbb{P}^5, \forall g \in W(E_6)).$$

The $W(E_6)$-equivariances of $F_1, F_2$ are stated in [4] implicitly.

We now mention the meaning of the $(\lambda, \mu, \nu, \rho)$-space. In [2], A. Cayley defined a 4-dimensional family of cubic surfaces. Modifying his family, we introduce a family of cubic surfaces of $\mathbb{P}^3$ with homogeneous coordinate $(X : Y : Z : W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [7]):

$$\rho W[\lambda X^2 + \mu Y^2 + \nu Z^2 + (\rho - 1)^2(\lambda \mu \nu \rho - 1)^2 W^2 + (\mu \nu + 1)YZ + (\lambda \nu + 1)ZX + (\lambda \mu + 1)XY$$

$$+ (\lambda \nu + 1)ZX + (\lambda \mu + 1)XY]$$

$$\rightarrow \tilde{g}_j \quad \tilde{g}_j \rightarrow g_j \quad j = 1, \cdots, 6$$
The family of cubic surfaces above admits a $W(E_6)$-action as given in [7]. In particular, the $W(E_6)$-action in [7] preserves the parameter space. For this reason, we obtain a $W(E_6)$-action on the $(\lambda, \mu, \nu, \rho)$-space which actually coincides with the $W(E_6)$-action on the $(\lambda, \mu, \nu, \rho)$-space explained before Theorem 5.1.

6. A Conjecture of B. Hunt

It is known (cf.[1]) that there is a unique $W(E_6)$-invariant homogeneous polynomial of $t = (t_1, \cdots, t_6)$ of degree 5 up to a constant factor. For example, we take $Q_5(t)$ below as such a polynomial (cf. [4]):

$$Q_5(t) = -\frac{5}{108}t_6^5 + \frac{5}{18}\sigma_1 t_6^3 + \frac{5}{4}(\sigma_1^2 - 4\sigma_2)t_6 + 30\sqrt{\sigma_5},$$

where $\sigma_i = \sigma_i(t_1^2, \cdots, t_6^2)$ is the $i$-th elementary symmetric polynomial in $t_1^2, \cdots, t_6^2$ and $\sqrt{\sigma_5} = t_1 \cdots t_5$.

Let $I_5$ be the hypersurface in $\mathbb{P}^5$ defined by $Q_5(t) = 0$. Since $Q_5(t)$ is $W(E_6)$-invariant, so is $I_5$. Moreover, since $\dim I_5 = 4$, the restrictions $F_1|I_5, F_2|I_5$ are generically finite maps from $I_5$ to $\mathbb{C}^4$. In [4], B. Hunt stated conjectures on these maps which turn out to be one conjecture below.

**Conjecture 6.1.** ([4]) Both $F_1|I_5, F_2|I_5$ are generically bijective.

How to attack Conjecture 6.1 with the help of REDUCE? In virtue of Theorem 5.1, it suffices to show Conjecture 6.1 for one of $F_1|I_5, F_2|I_5$. Noting the definition of $F_1(t)$, we find that Conjecture 6.1 is rewritten as follows:

**Problem 6.2.** Let $x_1, x_2, y_1, y_2$ be constants. At least assume that $(x_1, x_2, y_1, y_2)$ is outside the set $T$. Using $x_1, x_2, y_1, y_2$, we define four polynomials of $t$ by

$$f_1 = h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135} - x_1 \cdot h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235},$$
$$f_2 = h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136} - x_2 \cdot h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236},$$
$$g_1 = h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125} - y_1 \cdot h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235},$$
$$g_2 = h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126} - y_2 \cdot h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236},$$

where $h, h_{ij}, h_{ijk}$ are linear functions of $t$ defined in section 3. Then how many solutions are there for the simultaneous equations of $t$ defined by

$$f_1 = f_2 = g_1 = g_2 = Q_5 = 0$$

under the condition $F_1(t) \not\in T$?

Needles to say, there is a gap between Conjecture 6.1 and Problem 6.2, that is, Conjecture 6.1 claims that for generic $x_1, x_2, y_1, y_2$, equation (4) has a unique projective solution. Since I don’t know whether Conjecture 6.1 is true or not, I reformulate it as a problem.
I tried to solve Problem 6.2 directly by using REDUCE3.4 on TOSHIBA J3100 once and at last abandoned to do because of out of capacity.

From now on, I am going to explain results related with Problem 6.2 and the moduli of cubic surfaces. We consider the hypersurface $H$ in $\mathbf{P}^5$ defined by $\lambda(t) - 1 = 0$, that is,

$$P(t) = h_{345} \cdot h_{26} \cdot h_{256} \cdot h_{13} \cdot h_{136} \cdot h_{12} \cdot h_{126} \cdot h_{346} = 0.$$  

Then it is easy to show that the polynomial $P(t)$ of equation (4) is decomposed into two factors (up to a constant):

$$P(t) = h_{23} \cdot P_5(t),$$

where $P_5(t)$ is homogeneous of degree 5. Moreover $P_5$ is so taken that

$$P_5(t_1, t_2, t_3, t_4, t_5, t_6) = \text{const.} Q_5(t_1, t_2, t_3, t_4, t_6, -3t_5).$$

From this remarkable relation, we easily imply the following (cf. [4], [6]).

**Proposition 6.3.**

(i) There are 45 hypersurfaces in $\mathbf{P}^5$ as the $W(E_6)$-orbit of $H$. Moreover, the isotropy subgroup of $H$ in $W(E_6)$ is isomorphic to the Weyl group of type $F_4$.

(ii) The intersection $H \cap I_5$ is decomposed into two irreducible components. One is defined by $t_5 = t_6 = 0$ therefore is isomorphic to $\mathbf{P}^3$. The other is defined by an equation of degree 24.

(iii) If $t \in H$, then $F_2(t) = (1, 1, 1, 1)$, that is, $\lambda(t) = \mu(t) = \nu(t) = \rho(t) = 1$. The corresponding cubic surface has Eckard points.

It follows from Proposition 6.3 (i) that there is a natural 1-1 correspondence between the $W(E_6)$-orbit of $H$ and the 45 exceptional divisors of Naruki’s cross ratio variety [6].

We mention Proposition 6.3 (ii) in detail. We first introduce symmetric polynomials of $t_1, t_2, t_3, t_4$ by

$$s_2 = t_1^2 + t_2^2 + t_3^2 + t_4^2,$$

$$s_4 = t_1^2(t_2^2 + t_3^2 + t_4^2) + 2t_2^2(t_3^2 + t_4^2) + 2t_3^2t_4^2,$$

$$s'_4 = t_1t_2t_3t_4.$$

Using $s_2$, $s_4$, $s'_4$, we define the polynomial $h$ of degree 24 by

$$h = c_{10}t_5^{20} + c_9t_5^{18} + c_8t_5^{16} + c_7t_5^{14} + c_6t_5^{12} + c_5t_5^{10} + c_4t_5^8 + c_3t_5^6 + c_2t_5^4 + c_1t_5^2 + c_0,$$

where

$$c_{10} = 1728s_2^2,$$

$$c_9 = 432s_2(-21s_2^2 + 20s_4),$$

$$c_8 = 27(4800s_4^2 + 761s_2^4 - 1736s_2^2s_4 + 400s_4^2),$$

$$c_7 = 8s_2(-46656s_4^2 - 3217s_2^4 + 12852s_2^2s_4 - 10368s_4^2),$$

$$c_6 = 540s_2^2(537s_2^4 - 518s_2^2s_4 + 140s_4^2),$$

$$c_5 = 54s_2(830s_2^2 - 789s_4^2),$$

$$c_4 = 54s_2(-51s_2^2 + 50s_4),$$

$$c_3 = 2s_2^2(17s_2^2 - 16s_4),$$

$$c_2 = 2s_2(-9s_2^2 + 8s_4),$$

$$c_1 = s_2(-s_4),$$

$$c_0 = s_4^2.$$
\[ c_0 = 2(-190080s_4^2s_2^2 - 336960s_4^2s_4 + 9251s_2^6 - 55955s_2^4s_4 \\
+ 91368s_2^2s_4^2 - 28080s_4^4), \]
\[ c_5 = 2s_2(825360s_4^2s_2^2 - 1582848s_4^2s_4 - 3256s_2^6 + 7143s_2^4s_4 \\
- 72496s_2^2s_4^2 + 61776s_4^4), \]
\[ c_4 = -59833728s_4^4 - 1370994s_4^2s_2^4 + 5809680s_4^2s_2^2s_4 - 4732128s_4^2s_4^2 - 193s_2^8 \\
+ 3054s_2^6s_4 - 12981s_2^4s_4^2 + 10120s_2^2s_4^3 + 21168s_4^6, \]
\[ c_3 = 2s_2(-2191104s_4^2s_2^2 - 119476s_4^2s_2^4 - 1263024s_4^2s_4s_2^4 + 1990080s_4^2s_4^2 \\
+ 496s_2^6 - 7327s_2^4s_4 + 40443s_2^2s_4^2 - 98824s_2^2s_4^2 + 90160s_4^4), \]
\[ c_2 = -907200s_4^2s_2^2 + 2491776s_4^2s_4s_4 - 54714s_4^2s_2^4 + 554274s_4^2s_2^2s_4 \\
- 1854576s_4^2s_2^2s_4^2 + 2051616s_4^2s_4^2 - 256s_2^8 + 4640s_2^6s_4 - 33505s_2^4s_4^2 \\
+ 120460s_2^4s_4^2 - 215600s_2^2s_4^3 + 153664s_4^5, \]
\[ c_1 = 6s_4^2s_2(-4968s_4^2s_2^2 + 14688s_4^2s_4 - 26s_2^6 + 285s_4^2s_4 - 1032s_2^2s_4^2 + 1232s_4^3), \]
\[ c_0 = 27s_4^4(192s_4^2 + s_2^4 - 8s_2^4s_4 + 16s_4^2). \]

Moreover,
\[ N = -2\{(5s_2^6 - 1602s_2^4s_4^2 - 34s_2^3s_4^4 + 4134s_2^2s_4^3 + 10037s_2^2s_4^2 - 3005s_2^2s_4^6 \\
+ 56s_2s_4^2 - 12820s_2s_4t_5^2 + 828s_2t_5^8 - 15764s_4^2t_5^2 + 1980s_4t_5^6 - 360t_5^{10})t_5^2 \\
- (s_2^2 + 164s_2t_5^2 - 4s_4 + 7368t_5^4)s_4^2t_5^2 - (s_2^2 + 164s_2t_5^2 - 4s_4 + 7368t_5^4)s_4^2t_5^2 \\
+ 2\{31s_2^3 + 650s_2^2s_4^2 - 92s_2s_4^2 + 2320s_2t_5^2 - 1752s_4^2 + 5648t_5^6)s_4^2t_5^2 \\
+ 2(2464s_4^2 - 2055s_4t_5^2 + 1878t_5^6)s_4^2t_5^2 - 4(1687s_2^2 - 415s_4t_5^2 + 12s_4^2)s_2^6 \\
- (1465s_4 - 1044t_5^2)s_4^2t_5^2 + 15(269s_4 - 61t_5^2)s_4^2t_5^2 - 16s_4^4 + 144s_2^4t_5^2 \\
- 599s_4^2t_5^2 - 5488s_4^2t_5^2 + 2072s_4^2t_5^2 - 120s_4t_5^{12} \} \]

Then from the equations
\[ P_5 = Q_5 = 0, \]
we obtain
\[ t_6 = N/D, \quad h = 0. \]

The equation \( h = 0 \) is the one stated in Proposition 6.3 (ii).

If we consider the equation \( \lambda - 1 = 0 \) in the \((x_1, x_2, y_1, y_2)\)-space, we obtain a hypersurface \( H_0 \) defined by
\[ x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1) - y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1) = 0. \]
Now we formulate a problem simplified from Problem 6.2, noting Proposition 6.3 (ii). Namely, we consider Problem 6.2 in the case $t_5 = t_6 = 0$ and $t_1 = 1$. (The condition $t_1 = 1$ is not essential. From the homogeneity, we may assume $t_j = 1$ for some $j$.)

**Problem 6.2'.** Define four polynomials of $t_2, t_3, t_4$ by

\[ f_{10} = (t_2 + t_3 - t_4 + 1)^2(t_2 + t_4)(t_3 - 1) - x_1(t_2 + t_3)(t_2 - t_3 + t_4 + 1)^2(t_4 - 1), \]
\[ f_{20} = (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_3 - 1)t_2 + x_2(t_2 + t_3)(t_2 - t_3 + t_4 + 1)(t_2 - t_3 - t_4 + 1), \]
\[ g_{10} = (t_2 + t_3 - t_4 + 1)^2(t_2 - t_3)(t_4 + 1) - y_1(t_2 - t_3 + t_4 + 1)^2(t_2 - t_4)(t_3 + 1), \]
\[ g_{20} = (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_2 - t_3) - y_2(t_2 - t_3 + t_4 + 1)(t_2 - t_3 - t_4 + 1)(t_3 + 1)t_2, \]

where $x_1, x_2, y_1, y_2$ are constants with the condition (6) and $(x_1, x_2, y_1, y_2) \notin T$. (In particular, we assume that $x_1$ is a rational function of $x_2, y_1, y_2$.) Then how many solutions are there for the equations (7) of $t_2, t_3, t_4$ below

\[ f_{10} = f_{20} = g_{10} = g_{20} = 0 \]

under the condition $t \notin T$?

It is possible to give an answer to Problem 6.2'. In fact, erasing $t_3, t_4$ from (7), we obtain an equation for $t_2$ defined by

\[ \sum_{j=0}^{9} b_j t_2^j = 0, \]

where

\[ b_0 = (x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)^2(x_2y_2 - 2y_2 + 1)y_2^4, \]
\[ b_8 = 3(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)^2(x_2y_2 - 2x_2 + 1)y_2^4, \]
\[ b_6 = -4(x_2^2y_1y_2 - x_2y_2^2 + x_2y_2 + x_2y_1)^2(x_2y_1 + x_2y_2^2 - y_1y_2 + y_2)(x_2y_2 - 2y_2 + 1)x_2y_1y_2^2, \]
\[ b_5 = -6(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2) \times (x_2y_2 - 2y_2 + 1)x_2y_1y_2^2, \]
\[ b_4 = 6(x_2y_1y_2 - x_2y_2 - 2y_1y_2 + y_1 + y_2^2)(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2) \times (x_2y_2 - 2x_2 + 1)x_2y_1y_2^2, \]
\[ b_3 = 4(x_2^2y_1y_2 - x_2y_2^2 + x_2y_1^2 + x_2y_1y_2^2 - 4x_2y_1y_2 + x_2y_1 + x_2y_2^2 - y_1y_2 + y_1y_2) \times (x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2)(x_2y_2 - 2y_2 + 1)x_2y_1y_2, \]
\[ b_1 = -3(x_2y_1 + x_2y_2^2 - 2x_2y_2 - y_1y_2 + y_2^2)(x_2y_2 - 2y_2 + 1)x_2y_1y_2^2, \]
Moreover, if \( t_2 \) is a solution of (8), \( t_3, t_4 \) are uniquely determined by (7).

I checked that equation (8) for \( t_2 \) is irreducible of degree 9 and that for generic \( x_2, y_1, y_2 \), (8) has no multiple factor. As a consequence, we obtain the following.

**Theorem 6.4.** The restriction of \( F_1 \) to the subspace \( t_5 = t_6 = 0 \) is generically 9 to 1.

I am not sure whether Theorem 6.4 induces the invalidity of Conjecture 6.1 or not.

**Acknowledgements**

Last I mention that the note [4] and the communications with Prof. B. Hunt are valuable when I formulate the maps \( F_j (j = 1, 2, 3) \) and solve Problem 6.2'. Moreover, I am indebted to Prof. K. Okubo because without his help, I could not use REDUCE 3.4 (not REDUCE 3.2!) which is a powerful tool in obtaining the results of this note.

**References**


[8] Sekiguchi, J. “A birational action of \( S_6 \) on \( \mathbf{C}^4 \) and the hypergeometric system \( E(3, 6) \)” in preparation.

\[
b_0 = -(x_2 y_1 + x_2 y_2^2 - 2x_2 y_2 - y_1 y_2 + y_2)^2(x_2 y_2 - 2x_2 + 1)x_2^2 y_1^2, \\
b_7 = b_2 = 0.
\]