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The configuration space of 6 points in $\mathbb{P}^2$, the moduli space of cubic surfaces and the Weyl group of type $E_6$

Jiro Sekiguchi

1. Introduction

My first plan of the talk is to explain my study on the hypergeometric system $E(3, 6)$ of type $(3, 6)$ ([8]). The system in question admits $\Sigma_6$-action, where $\Sigma_6$ is the symmetric group on 6 letters. This follows from that $E(3, 6)$ lives in the configurations space $\mathbb{P}^2_6$ of 6 points in $\mathbb{P}^2$ which admits $\Sigma_6$-action as permutations of the 6 points. Recently M. Yoshida (Kyushu Univ.) pointed out that the $\Sigma_6$-action on the space $\mathbb{P}^2_6$ is naturally extended to $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type $E_6$ (cf. [3]). Moreover, he told me that B. Hunt studied relations between the $W(E_6)$-action in question and the $W(E_6)$-invariant quintic hypersurface of $\mathbb{P}^5$.

Reading his note [4], I felt that it is an interesting exercise for REDUCE user to show whether his conjecture is true or not. For this reason, I changed the original plan and I restrict my attention to the study on $W(E_6)$-actions on $\mathbb{P}^5$ and on $\mathbb{P}^6_2$, namely, to the birational geometry related with the hypergeometric system $E(3,6)$.

It is better for the readers who are interested in SYMBOLIC COMPUTATION to read section 6 first.

2. The hypergeometric function of type $(3,6)$

Though I don’t treat it in this note, I begin this note with introducing the hypergeometric function of type $(3,6)$:

$$E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A_{m_1,m_2,n_1,n_2} x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2}$$

where

$$A_{m_1,m_2,n_1,n_2} = \frac{(a_2, m_1 + m_2)(a_3, n_1 + n_2)(1 - a_5, m_1 + n_1)(1 - a_6, m_2 + n_2)}{m_1!m_2!n_1!n_2!(a_0, m_1 + m_2 + n_1 + n_2)}.$$
is known that the singularities of the system of differential equations whose solution is $E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2)$ is contained in the union of the 14 hypersurfaces $T_j : p_j = 0 (1 \leq j \leq 14)$ where

\[
p_1 = x_1y_2 - x_2y_1 - x_1 + x_2 + y_1 - y_2, \quad p_2 = y_1 - 1, \quad p_3 = x_1 - 1,
\]
\[
p_4 = y_2 - 1, \quad p_5 = x_2 - 1, \quad p_6 = y_1 - y_2, \quad p_7 = x_1 - x_2, \quad p_8 = x_1 - y_1,
\]
\[
p_9 = x_2 - y_2, \quad p_{10} = x_1y_2 - x_2y_1, \quad p_{11} = x_2, \quad p_{12} = x_1, \quad p_{13} = y_2, \quad p_{14} = y_1.
\]

We define birational transformations $s_j (1 \leq j \leq 5)$ on $C^4$ by

\[
s_1 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{1}{x_1}, \frac{1}{x_2}, \frac{y_1}{x_1}, \frac{y_2}{x_2} \right),
\]
\[
s_2 : (x_1, x_2, y_1, y_2) \rightarrow (y_1, y_2, x_1, x_2),
\]
\[
s_3 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{x_1 - y_1}{1 - y_1}, \frac{x_2 - y_2}{1 - y_2}, \frac{y_1}{y_1 - 1}, \frac{y_2}{y_2 - 1} \right),
\]
\[
s_4 : (x_1, x_2, y_1, y_2) \rightarrow \left( \frac{x_2}{x_1}, \frac{1}{x_1}, \frac{y_2}{y_1} \right),
\]
\[
s_5 : (x_1, x_2, y_1, y_2) \rightarrow (x_2, x_1, y_1, y_2).
\]

Then the group generated by $s_j (1 \leq j \leq 5)$ is identified with $\Sigma_6$ because

\[
s_j^2 = id. (1 \leq j \leq 5), \quad s_js_k = s_ks_j \ (|j - k| > 1),
\]
\[
s_is_js_j = s_js_ks_j \ (|j - k| = 1).
\]

Let $r$ be a birational transformation on $C^4$ defined by

\[
r : (x_1, x_2, y_1, y_2) \rightarrow (1/x_1, 1/x_2, 1/y_1, 1/y_2).
\]

Then the group $\tilde{G}$ generated by $s_1, \ldots, s_5$ and $r$ is isomorphic to the Weyl group $W(E_6)$ of type $E_6$ which will be seen later (cf. [3], [4]).

We define the hypersurface $T_{15} : p_{15} = 0$, where

\[
p_{15} = x_1y_2(x_1 - y_1)(x_2 - 1) - x_2y_1(x_1 - 1)(1 - y_2).
\]

It follows from the definition that $s_1, \ldots, s_5, r$ and therefore all the elements of $\tilde{G}$ are biregular outside the union $T$ of the hypersurfaces $T_j (1 \leq j \leq 15)$.

3. The Weyl group $W(E_6)$

Let $E_R$ be a Cartan subalgebra of a compact Lie algebra of type $E_6$, i.e. $E_R \simeq R^6$. Let $t = (t_1, t_2, t_3, t_4, t_5, t_6)$ be a coordinate system of $E_R$ such that the roots of type $E_6$ are:

\[
\pm(t_i \pm t_j), \quad 1 \leq i < j \leq 5
\]
\[
\pm\frac{1}{2}(\delta_1t_1 + \delta_2t_2 + \delta_3t_3 + \delta_4t_4 + \delta_5t_5 + \delta_6t_6)
\]
(where $\delta_j = \pm 1$ and $\prod_j \delta_j = 1$). Note that compared with the notation in [1], our variables $t_i = \epsilon_i$, $i = 1, \ldots, 5$, while our coordinate $t_6$ is denoted $\epsilon_6 - \epsilon_7 - \epsilon_8$ in [1]. We now introduce the following linear forms on $E_R$:

$$h = -\frac{1}{2}(t_1 + \cdots + t_6),$$
$$h_{1j} = -t_{j-1} + h_0, \quad j = 2, \ldots, 6$$
$$h_{jk} = t_{j-1} - t_{k-1}, \quad j, k \neq 1$$
$$h_{jkl} = -t_{j-1} - t_{k-1} - t_{l-1} + h_0, \quad j, k, l \neq 1$$

where

$$h_0 = \frac{1}{2}(t_1 + \cdots + t_5 - t_6).$$

Then the totality of $h, h_{ij}, h_{ijk}$ forms a set of positive roots of type $E_6$. Let $s$ (resp. $s_{ij}, s_{ijk}$) be the reflection on $E_R$ with respect to the hyperplane $h = 0$ (resp. $h_{ij} = 0, h_{ijk} = 0$). Then the Weyl group of type $E_6$ which is denoted by $W(E_6)$ in this note is the group generated by the 36 reflections defined above.

As a system of simple roots, we take

$$\alpha_1 = h_{12}, \quad \alpha_2 = h_{123}, \quad \alpha_3 = h_{23}, \quad \alpha_4 = h_{34}, \quad \alpha_5 = h_{45}, \quad \alpha_6 = h_{56}. $$

Then the Dynkin diagram is:

$$\begin{array}{cccccc}
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\mid & & & & \\
\alpha_2 & & & & \\
\end{array}$$

Let $g_j$ be the reflection on $E_R$ with respect to the root $\alpha_j$ ($j = 1, \ldots, 6$). Then, from the definition,

$$g_1 = s_{12}, \quad g_2 = s_{123}, \quad g_3 = s_{23}, \quad g_4 = s_{34}, \quad g_5 = s_{45}, \quad g_6 = s_{56}.$$

Let $E$ be the complexification of $E_R$ and we extend the action of $W(E_6)$ on $E_R$ to that on $E$ in a natural manner. Moreover let $P^5$ be the projective space associated to $E$. Then the $W(E_6)$-action on $E$ induces a projective linear action of $W(E_6)$ on $P^5$.

4. The configuration space of 6 points in $P^2$

We have already defined a birational action of $W(E_6)$ on $C^4$ in section 2. In this section, we explain that the birational transformations $s_1, \ldots, s_5, r$ naturally arise from the study of the configuration space of 6 points in $P^2$.

For this purpose, we first introduce the linear space $W$ of $3 \times 6$ matrices:
Then $W$ admits a left $GL(3, \mathbb{C})$-action and a right $GL(6, \mathbb{C})$-action in a natural way. For a moment, we identify $(\mathbb{C}^*)^6$ with the maximal torus of $GL(6, \mathbb{C})$ consisting of diagonal matrices and consider the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ on $W$ instead of that of $GL(3, \mathbb{C}) \times GL(6, \mathbb{C})$.

For simplicity, we write $X = (X_1, X_2)$ for the matrix $X \in W$, where both $X_1, X_2$ are $3 \times 3$ matrices. For any $3 \times 3$ matrix $Y = (y_{ij})_{1 \leq i,j \leq 3}$ with the condition $y_{ij} \neq 0$ ($1 \leq i,j \leq 3$), we define a $3 \times 3$ matrix

$$\sigma(Y) = \left( \frac{1}{y_{ij}} \right)_{1 \leq i,j \leq 3}$$

following a suggestion of M. Yoshida. Moreover, we put

$$D(i_1, i_2, i_3) = \det \begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix}$$

for a given matrix $X \in W$.

Using these notation, we define subsets $W', W_0$ of $W$ by

$$W' = \{ X \in W ; D(i_1, i_2, i_3) \neq 0 (1 \leq i_1 < i_2 < i_3 \leq 6) \},$$

$$W_0 = \{ (X_1, X_2) \in W' ; (I_3, \text{Cof}(X_1^{-1}X_2)), (I_3, \sigma(X_1^{-1}X_2)) \in W' \},$$

where $\text{Cof}(Y) = (\det Y)Y^{-1}$ is the cofactor matrix of a given square matrix $Y$.

It is clear that the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ on $W$ naturally induces that on each of $W', W_0$. In the sequel, we mainly consider the quotient space of $W_0$ under the action of $GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$, that is,

$$W_Q = GL(3, \mathbb{C}) \setminus W_0/(\mathbb{C}^*)^6.$$ 

It is clear from the definition that for any element $X \in W_0$, there are $(g, h) \in GL(3, \mathbb{C}) \times (\mathbb{C}^*)^6$ and $(x_1, x_2, y_1, y_2) \in \mathbb{C}^4$ such that

$$gXh = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$ 

In particular $(x_1, x_2, y_1, y_2)$ is uniquely determined for $X \in W_0$. In this sense, $W_Q = GL(3, \mathbb{C}) \setminus W_0/(\mathbb{C}^*)^6$ is identified with an open subset of $\mathbb{C}^4$. Note that $(x_1, x_2, y_1, y_2)$ is the variables $(x_1, x_2, y_1, y_2)$ of section 2. Then $W_Q = \mathbb{C}^4 - T$. 
Changes of column vectors of $X \in W_0$ induce birational transformations on $\mathbb{C}^4$ with coordinate system $(x_1, x_2, y_1, y_2)$. The action $s_j$ ($1 \leq j \leq 5$) introduced in section 2 is nothing but the birational transformation on $\mathbb{C}^4$ corresponding to the change of the $j$-th column vector and $(j + 1)$-column vector of $X \in W_0$. Moreover $W_Q$ admits an involution induced from the action on $W_0$ defined by

$$\tilde{r} : (X_1, X_2) \mapsto (I_3, \sigma(X_1^{-1}X_2))$$

for any $(X_1, X_2) \in W_0$. The involution $r$ defined in section 2 is equal to that induced from $\tilde{r}$.

The following theorem which seems known shows a concrete correspondence between $W(E_6)$ and the group $\tilde{G}$ introduced in section 2.

**Theorem 4.1.** The correspondence

$g_1 \mapsto s_1, \ g_2 \mapsto r, \ g_3 \mapsto s_2, \ g_4 \mapsto s_3, \ g_5 \mapsto s_4, \ g_6 \mapsto s_5$

induces a group isomorphism of $W(E_6)$ to the group $\tilde{G}$.

**Remark.** In [3], it is stated that there is a $W(E_6)$-action on $W_Q$. See also [4].

### 5. $W(E_6)$-equivariant maps

We first define rational functions on $E$ by

\[
\begin{align*}
    x_1(t) &= \frac{h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135}}{h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235}}, \\
    y_1(t) &= \frac{h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125}}{h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235}}, \\
    \lambda(t) &= \frac{h_{34} \cdot h_{345} \cdot h_{26} \cdot h_{256}}{h_{24} \cdot h_{245} \cdot h_{36} \cdot h_{356}}, \\
    \mu(t) &= \frac{h_{456} \cdot h_{235} \cdot h_{134} \cdot h_{126}}{h_{24} \cdot h_{234} \cdot h_{46} \cdot h_{356}}, \\
    \nu(t) &= \frac{h_{25} \cdot h_{235} \cdot h_{46} \cdot h_{356}}{h_{24} \cdot h_{234} \cdot h_{46} \cdot h_{496}}, \\
    \rho(t) &= \frac{h_{25} \cdot h_{235} \cdot h_{46} \cdot h_{496}}{h_{24} \cdot h_{234} \cdot h_{56} \cdot h_{356}}.
\end{align*}
\]

where $h, h_{ij}, h_{ijk}$ denote linear functions on $E$ introduced in section 3. Since all the rational functions above are homogeneous of degree zero, they are regarded as functions on $\mathbb{P}^5$. Therefore defining

\[
F_1(t) = (x_1(t), x_2(t), y_1(t), y_2(t)), \quad F_2(t) = (\lambda(t), \mu(t), \nu(t), \rho(t)),
\]

we obtain two maps $F_1, F_2$ from $\mathbb{P}^5$ to $\mathbb{C}^4$. The roles of $F_1, F_2$ will become clear in Theorem 5.1 which will be given later. To define $F_1, F_2$, I am indebted to [4]. We are going to explain the meaning of $x_j(t), y_j(t)$ following [4].
We begin with defining the cross ratio. Let $\xi_i = [\xi_1 : \xi_2 : \xi_3 : \xi_4] (1 \leq i \leq 5)$ be five points of $\mathbb{P}^2$ and let $l : q_1u_1 + q_2u_2 + q_3u_3 = 0$ be a generic line in $\mathbb{P}^2$. We denote by $[1 : z_i : w_i]$ the intersection of $l$ and the line passing through the points $\xi_1$ and $\xi_i$. Then we put

\begin{equation}
CR(\xi_2, \xi_3, \xi_4, \xi_5; \xi_1) = \frac{(z_2 - z_4)(z_3 - z_5)}{(z_2 - z_5)(z_3 - z_4)}
\end{equation}

which is in fact a cross ratio of $z_2, z_3, z_4, z_5$.

Now we consider a matrix of the form

\[
X = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1
0 & 1 & 0 & 1 & x_1 & x_2
0 & 0 & 1 & 1 & y_1 & y_2
\end{pmatrix}.
\]

From the matrix $X$, we define six points $\xi_i (i = 1, \cdots, 6)$ in $\mathbb{P}^2$ in a usual manner, that is,

$\xi_1 = [1 : 0 : 0]$, $\xi_2 = [0 : 1 : 0]$, $\xi_3 = [0 : 0 : 1]$,

$\xi_4 = [1 : 1 : 0]$, $\xi_5 = [1 : x_1 : y_1]$, $\xi_6 = [1 : x_2 : y_2]$.

Then we can compute $CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}; \xi_{i_1})$ explicitly for various $i_1, i_2, i_3, i_4, i_5$.

On the other hand, we put

\begin{equation}
CR'(i_2, i_3, i_4, i_5; i_1) = \frac{h_{i_2i_4}h_{i_1i_2i_4}h_{i_3i_5}h_{i_1i_3i_5}}{h_{i_3i_4}h_{i_1i_3i_4}h_{i_2i_5}h_{i_1i_2i_5}}.
\end{equation}

By definition, $CR'(i_2, i_3, i_4, i_5; i_1)$ is a function on $\mathbb{P}^5$. Then from the equation

\begin{equation}
CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}; \xi_{i_1}) = CR'(i_2, i_3, i_4, i_5; i_1),
\end{equation}

we obtain various equalities. In particular, by computing the cases

$$(i_1, i_2, i_3, i_4, i_5) = (3, 2, 1, 4, 5), (3, 2, 1, 4, 6), (2, 1, 3, 4, 5), (2, 1, 3, 4, 6),$$

we have the definition of $x_1(t), x_2(t), y_1(t), y_2(t)$ at the beginning of this section.

Let $F_3$ be the birational transformation on $\mathbb{C}^4$ defined by $F_3(x_1, x_2, y_1, y_2) = (\lambda, \mu, \nu, \rho)$, where

\[
\lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)},
\]

\[
\mu = \frac{((y_1 - 1)(x_2 - y_2) - (y_2 - 1)(x_1 - y_1))x_2y_2}{x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 + x_1y_2 + x_2y_1y_2 - x_2y_1},
\]

\[
\nu = \frac{(x_1y_2 - x_2y_1)(x_2 - 1)(y_2 - 1)}{(x_1 - x_2)(x_2 - y_2)(y_1 - y_2)},
\]

\[
\rho = \frac{(x_1 - x_2)(x_2 - y_2)(y_1 - 1)}{((x_1 - 1)(x_2 - y_2) - (x_1 - y_1)(x_2 - 1))(y_2 - 1)x_2}.
\]
It is easy to show that $F_3$ is birational, because its inverse is given by

$$F_3^{-1}(\lambda, \mu, \nu, \rho) = \frac{(\lambda \rho - 1)(\lambda \mu \nu \rho - 1)(\lambda \mu - 1)(\lambda \mu \nu \rho - 1)^2(\rho - 1)(\rho - 1)(\rho - 1)(\rho - 1)(\rho - 1)}{(\lambda \mu - 1)(\lambda \mu \nu - 1)(\lambda \mu \nu \rho - 1)}.$$ 

By the map $F_3$, the action of $W(E_6)$ on the $(x_1, x_2, y_1, y_2)$-space implies that on the $(\lambda, \mu, \nu, \rho)$-space. In fact, we define the following six birational transformations on the $(\lambda, \mu, \nu, \rho)$-space (cf. [7]):

$\tilde{g}_1 : \begin{cases} 
\lambda \rightarrow \lambda \mu \nu \rho^2 (1 - \lambda)/(\lambda \mu \nu \rho^2 - 1) \\
\mu \rightarrow (\lambda \mu \rho - 1)/(\mu (\lambda \mu \rho - 1)(\lambda \mu \nu - 1)) \\
\nu \rightarrow (\mu \nu \rho - 1)/(\nu (\lambda \nu \rho - 1)(\lambda \mu \nu - 1)) \\
\rho \rightarrow (\lambda \rho - 1)(\lambda \mu \nu \rho - 1)/(\rho (\lambda \mu \rho - 1)(\lambda \nu \rho - 1)) 
\end{cases}$

$\tilde{g}_2 : (\lambda, \mu, \nu, \rho) \rightarrow (\lambda, 1/\mu, \nu, \mu \rho)$

$\tilde{g}_3 : (\lambda, \mu, \nu, \rho) \rightarrow (1/\lambda, \mu, \nu, \lambda \rho)$

$\tilde{g}_4 : (\lambda, \mu, \nu, \rho) \rightarrow (\nu \rho - 1)(\lambda \mu \nu \rho^2 - 1)/(\rho (\nu - 1)(\lambda \mu \nu - 1))$

$\tilde{g}_5 : (\lambda, \mu, \nu, \rho) \rightarrow (\lambda, \mu, 1/\nu, \nu \rho)$

$\tilde{g}_6 : \begin{cases} 
\lambda \rightarrow \lambda \nu \rho - 1)(\lambda \mu \nu \rho - 1)/(\lambda (\nu \rho - 1)(\mu \nu - 1)) \\
\mu \rightarrow (\mu \nu \rho - 1)/(\nu (\lambda \nu \rho - 1)(\lambda \nu \rho - 1)) \\
\nu \rightarrow \lambda \mu \nu \rho^2 (1 - \nu)/(\lambda \mu \nu \rho^2 - 1) \\
\rho \rightarrow (\nu - 1)(\lambda \mu \nu \rho^2 - 1)/(\rho (\nu - 1)(\lambda \mu \nu - 1)) 
\end{cases}$

Let $G_1$ be the group generated by $\tilde{g}_j$ ($j = 1, \cdots, 6$). Then the correspondence

$$g_j \rightarrow \tilde{g}_j \quad j = 1, \cdots, 6$$

is an isomorphism between $W(E_6)$ and $G_1$.

Needless to say, $F_1$ (resp. $F_2$) is regarded as a map from $P^5$ to the $(x_1, x_2, y_1, y_2)$-space (resp. the $(\lambda, \mu, \nu, \rho)$-space.) Moreover, $F_3$ is regarded as a map from the $(x_1, x_2, y_1, y_2)$-space to the $(\lambda, \mu, \nu, \rho)$-space.

**Theorem 5.1.** The three maps $F_j$ ($j = 1, 2, 3$) are $W(E_6)$-equivariant and

$$F_3 \circ F_1(g(t)) = F_2(g(t)) \quad (\forall t \in P^5, \forall g \in W(E_6)).$$

The $W(E_6)$-equivariances of $F_1$, $F_2$ are stated in [4] implicitly.

We now mention the meaning of the $(\lambda, \mu, \nu, \rho)$-space. In [2], A. Cayley defined a 4-dimensional family of cubic surfaces. Modifying his family, we introduce a family of cubic surfaces of $P^3$ with homogeneous coordinate $(X : Y : Z : W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [7]):

$$\rho W[\lambda X^2 + \mu Y^2 + \nu Z^2 + (\rho - 1)^2(\lambda \mu \nu \rho - 1)^2 W^2 + (\mu \nu + 1)YZ + (\lambda \nu + 1)ZX + (\lambda \mu + 1)XY$$

$$+ (\mu \nu + 1)Z + (\lambda \nu + 1)X + (\lambda \mu + 1)Y]$$

$$= 0$$
The family of cubic surfaces above admits a \( W(E_6) \)-action as given in [7]. In particular, the \( W(E_6) \)-action in [7] preserves the parameter space. For this reason, we obtain a \( W(E_6) \)-action on the \((\lambda, \mu, \nu, \rho)\)-space which actually coincides with the \( W(E_6) \)-action on the \((\lambda, \mu, \nu, \rho)\)-space explained before Theorem 5.1.

6. A Conjecture of B. Hunt

It is known (cf.[1]) that there is a unique \( W(E_6) \)-invariant homogeneous polynomial of \( t = (t_1, \cdots, t_6) \) of degree 5 up to a constant factor. For example, we take \( Q_5(t) \) below as such a polynomial (cf. [4]):

\[
Q_5(t) = -\frac{5}{108} t_6^5 + \frac{5}{18} \sigma_1 t_6^3 + \frac{5}{4} (\sigma_1^2 - 4 \sigma_2) t_6 + 30 \sqrt{\sigma_5},
\]

where \( \sigma_i = \sigma_i(t_1^2, \cdots, t_5^2) \) is the \( i \)-th elementary symmetric polynomial in \( t_1^2, \cdots, t_5^2 \) and \( \sqrt{\sigma_5} = t_1 \cdots t_5 \).

Let \( I_5 \) be the hypersurface in \( \mathbb{P}^5 \) defined by \( Q_5(t) = 0 \). Since \( Q_5(t) \) is \( W(E_6) \)-invariant, so is \( I_5 \). Moreover, since \( \dim I_5 = 4 \), the restrictions \( F_1|I_5, F_2|I_5 \) are generically finite maps from \( I_5 \) to \( C^4 \). In [4], B. Hunt stated conjectures on these maps which turn out to be one conjecture below.

**Conjecture 6.1.** ([4]) Both \( F_1|I_5, F_2|I_5 \) are generically bijective.

How to attack Conjecture 6.1 with the help of REDUCE? In virtue of Theorem 5.1, it suffices to show Conjecture 6.1 for one of \( F_1|I_5, F_2|I_5 \). Noting the definition of \( F_1(t) \), we find that Conjecture 6.1 is rewritten as follows:

**Problem 6.2.** Let \( x_1, x_2, y_1, y_2 \) be constants. At least assume that \((x_1, x_2, y_1, y_2)\) is outside the set \( T \). Using \( x_1, x_2, y_1, y_2 \), we define four polynomials of \( t \) by

\[
\begin{align*}
  f_1 &= h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135} - x_1 \cdot h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235}, \\
  f_2 &= h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136} - x_2 \cdot h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236}, \\
  g_1 &= h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125} - y_1 \cdot h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235}, \\
  g_2 &= h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126} - y_2 \cdot h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236},
\end{align*}
\]

where \( h, h_{ij}, h_{ijk} \) are linear functions of \( t \) defined in section 3. Then how many solutions are there for the simultaneous equations of \( t \) defined by

\[
(4) \quad f_1 = f_2 = g_1 = g_2 = Q_5 = 0
\]

under the condition \( F_1(t) \not\in T \)?

Needles to say, there is a gap between Conjecture 6.1 and Problem 6.2, that is, Conjecture 6.1 claims that for generic \( x_1, x_2, y_1, y_2 \), equation (4) has a unique projective solution. Since I don't know whether Conjecture 6.1 is true or not, I reformulate it as a problem.
I tried to solve Problem 6.2 directly by using REDUCE3.4 on TOSHIBA J3100 once and at last abandoned to do because of out of capacity.

From now on, I am going to explain results related with Problem 6.2 and the moduli of cubic surfaces. We consider the hypersurface $H$ in $\mathbb{P}^5$ defined by $\lambda(t) - 1 = 0$, that is,

$$ (5) \quad P(t) = h_{345} \cdot h_{26} \cdot h_{256} \cdot h_{13} \cdot h_{136} \cdot h_{246} - h_{245} \cdot h_{36} \cdot h_{356} \cdot h_{12} \cdot h_{126} \cdot h_{346} = 0. $$

Then it is easy to show that the polynomial $P(t)$ of equation (4) is decomposed into two factors (up to a constant):

$$ P(t) = h_{23} \cdot P_5(t), $$

where $P_5(t)$ is homogeneous of degree 5. Moreover $P_5$ is so taken that

$$ P_5(t_1, t_2, t_3, t_4, t_5, t_6) = \text{const.} \cdot Q_5(t_1, t_2, t_3, t_4, t_6, -3t_5). $$

From this remarkable relation, we easily imply the following (cf. [4], [6]).

**Proposition 6.3.**

(i) There are 45 hypersurfaces in $\mathbb{P}^5$ as the $W(E_6)$-orbit of $H$. Moreover, the isotropy subgroup of $H$ in $W(E_6)$ is isomorphic to the Weyl group of type $F_4$.

(ii) The intersection $H \cap I_5$ is decomposed into two irreducible components. One is defined by $t_5 = t_6 = 0$ therefore is isomorphic to $\mathbb{P}^3$. The other is defined by an equation of degree 24.

(iii) If $t \in H$, then $F_2(t) = (1, 1, 1, 1)$, that is, $\lambda(t) = \mu(t) = \nu(t) = \rho(t) = 1$. The corresponding cubic surface has Eckard points.

It follows from Proposition 6.3 (i) that there is a natural 1-1 correspondence between the $W(E_6)$-orbit of $H$ and the 45 exceptional divisors of Naruki’s cross ratio variety [6].

We mention Proposition 6.3 (ii) in detail. We first introduce symmetric polynomials of $t_1, t_2, t_3, t_4$ by

$$ s_2 = t_1^2 + t_2^2 + t_3^2 + t_4^2, $$
$$ s_4 = t_1^2(t_2^2 + t_3^2 + t_4^2) + t_2^2(t_3^2 + t_4^2) + t_3^2t_4^2, $$
$$ s'_4 = t_1t_2t_3t_4. $$

Using $s_2$, $s_4$, $s'_4$, we define the polynomial $h$ of degree 24 by

$$ h = c_{10}t_5^{20} + c_9t_5^{18} + c_8t_5^{16} + c_7t_5^{14} + c_6t_5^{12} + c_5t_5^{10} + c_4t_5^8 + c_3t_5^6 + c_2t_5^4 + c_1t_5^2 + c_0, $$

where

$$ c_{10} = 1728s_2^2, $$
$$ c_9 = 432s_2(-21s_2^2 + 20s_4), $$
$$ c_8 = 27(4800s_4^2 + 761s_2^4 - 1736s_2^2s_4 + 400s_4^2), $$
$$ c_7 = 8s_2(-46656s_4^2 - 3217s_2^4 + 12852s_2^2s_4 - 10368s_4^2), $$
$$ c_6 = -12s_4(-5832s_2^2 + 979s_2^4 + 6912s_2^2s_4 - 2048s_4^2), $$
$$ c_5 = 27s_2(-49856s_4^2 + 400s_4^3), $$
$$ c_4 = 8s_2(-384s_4^3 - 239s_2^2s_4^2 + 776s_2^2s_4 - 448s_4^2), $$
$$ c_3 = 27s_2(-7088s_4^2 + 2048s_4^3), $$
$$ c_2 = 8s_2(-108s_4^2 + 576s_2^2s_4 - 2048s_4^2), $$
$$ c_1 = -12s_4(-3072s_2^2 + 2048s_4^2), $$
$$ c_0 = 1728s_4^2, $$
\[ c_0 = 2(-190080s_4^2s_2^2 - 336960s_4^2s_4 + 9251s_2^6 - 55955s_4s_2s_4) \\
+ 91368s_4^2s_2^4 - 28080s_4^4, \]
\[ c_5 = 2s_2(825360s_4^2s_2^2 - 1582848s_4^2s_4 - 3256s_2^6 + 27143s_2^6s_4 \\
- 72496s_4^2s_2^2 + 61776s_4^4), \]
\[ c_4 = -59833728s_4^4 - 1370994s_4^2s_2^4 + 5809680s_4^2s_2^2s_4 - 4732128s_4^2s_4^2 - 193s_2^8 \\
+ 3054s_2^6s_4 - 12981s_2^4s_4^2 + 10120s_2^2s_4^3 + 21168s_4^4, \]
\[ c_3 = 2s_2(-2191104s_4^4s_2^2 - 199476s_4^2s_2^6 - 1263024s_4^2s_2^4s_4 + 1990080s_4^2s_2^3s_4^2 \\
+ 496s_2^8 - 7327s_2^6s_4 + 40443s_2^4s_4^2 - 98824s_2^2s_4^3 + 90160s_4^4), \]
\[ c_2 = -907200s_4^4s_2^2 + 2491776s_4^2s_2^6 - 54714s_2^4s_4^2 + 554274s_2^2s_4^3 + 138764s_2^4s_4^4 \\
- 1854576s_2^2s_4^2 + 2051616s_2^2s_4^4 - 256s_2^6 + 4640s_2^4s_4^2 - 33505s_2^2s_4^3 + 1232s_4^3), \]
\[ c_1 = 6s_4^2s_2(-4968s_4^2s_2^2 + 14688s_4^2s_4 - 26s_2^6 + 285s_4^2s_4 - 1032s_2^2s_4^2 + 1232s_4^3), \]
\[ c_0 = 27s_4^4(192s_4^4s_2^4 - 8s_2^2s_4 + 16s_2^2). \]

Moreover,
\[ N = -2\{(5s_2^5 - 1602s_2^4s_4^2 - 34s_2^3s_4^2 + 4134s_2^2s_4^3 + 10037s_2s_4s_2^4 - 3005s_2^6s_4^2 \\
+ 56s_2^2s_4^4 - 12820s_2s_4t_5^2 + 828s_2t_5^6 - 15764s_2^4t_5^2 + 1980s_4t_5^6 - 360t_5^{10})t_5^2 \\
-(s_2^2 + 164s_2^2t_5^2 - 4s_4 + 7368t_5^4)s_2^2s_4^2t_5^2 \\
- (s_2^2 - 16s_2^2t_5^2 + 4s_4 + 7368t_5^4)s_2^2s_4^2t_5^2 \}, \]
\[ D = -\{3(31s_2^3 + 650s_2^2t_5^2 - 92s_2s_4 + 2320s_2t_5^4 - 1752s_4t_5^2 + 5648t_5^6)s_2^2t_5^2 \\
+ 2(2464s_2^4 - 2055s_2^2t_5^2 + 187t_5^4)s_2^2t_5^2 - 4(1687s_2^6 - 415s_4t_5^4 + 12s_2^4t_5^4) \\
- (1465s_4 - 1044t_5^4)s_2^2t_5^2 + 15(269s_4 - 61t_5^4)s_2^2t_5^2 - 16s_4^4 + 144s_2^4t_5^4 \\
- 599s_4^4 - 5488s_4^3t_5^4 + 2072s_4^2t_5^4 - 120s_4t_5^{12}) \}. \]

Then from the equations
\[ P_5 = Q_5 = 0, \]
we obtain
\[ t_6 = N/D, \quad h = 0. \]

The equation \( h = 0 \) is the one stated in Proposition 6.3 (ii).

If we consider the equation \( \lambda - 1 = 0 \) in the \((x_1, x_2, y_1, y_2)\)-space, we obtain a hypersurface \( H_0 \) defined by
\[ x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1) - y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1) = 0. \]
Now we formulate a problem simplified from Problem 6.2, noting Proposition 6.3 (ii). Namely, we consider Problem 6.2 in the case $t_5 = t_6 = 0$ and $t_1 = 1$. (The condition $t_1 = 1$ is not essential. From the homogeneity, we may assume $t_j = 1$ for some $j$.)

**Problem 6.2'.** Define four polynomials of $t_2, t_3, t_4$ by

\[ f_{10} = (t_2 + t_3 - t_4 + 1)^2(t_2 + t_4)(t_3 - 1) - x_1(t_2 + t_3)(t_2 - t_3 + t_4 + 1)^2(t_4 - 1), \]
\[ f_{20} = (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_3 - 1)t_2 + x_2(t_2 + t_3)(t_2 - t_3 + t_4 + 1)(t_2 - t_3 - t_4 + 1), \]
\[ g_{10} = (t_2 + t_3 - t_4 + 1)^2(t_2 - t_3)(t_4 + 1) - y_1(t_2 - t_3 + t_4 + 1)^2(t_2 - t_4)(t_3 + 1), \]
\[ g_{20} = (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_2 - t_3) - y_2(t_2 - t_3 + t_4 + 1)(t_2 - t_3 - t_4 + 1)(t_3 + 1)t_2, \]

where $x_1, x_2, y_1, y_2$ are constants with the condition (6) and $(x_1, x_2, y_1, y_2) \not\in T$. (In particular, we assume that $x_1$ is a rational function of $x_2, y_1, y_2$.) Then how many solutions are there for the equations (7) of $t_2, t_3, t_4$ below

(7) \[ f_{10} = f_{20} = g_{10} = g_{20} = 0 \]

under the condition $t \not\in T$?

It is possible to give an answer to Problem 6.2'. In fact, erasing $t_3, t_4$ from (7), we obtain an equation for $t_2$ defined by

(8) \[ \sum_{j=0}^{9} b_j t_2^j = 0, \]

where

\[ b_0 = (x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)^2(x_2 y_2 - 2 y_2 + 1)y_2^4, \]
\[ b_3 = 4(x_2^2 y_1 y_2 - x_2^2 y_2 + x_2 y_1^2 + x_2 y_1 y_2^2 - 4 x_2 y_1 y_2 + x_2 y_1 + x_2 y_2^2 - y_1^2 y_2 + y_1 y_2)(x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2, \]
\[ b_4 = 6(x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2) \times (x_2 y_2 - 2 y_2 + 1) x_2 y_1 y_2^2, \]
\[ b_5 = -6(x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2) \times (x_2 y_2 - 2 y_2 + 1) x_2 y_1 y_2, \]
\[ b_6 = -4(x_2^2 y_1 y_2 - x_2^2 y_2 + x_2 y_1^2 + x_2 y_1 y_2^2 - 4 x_2 y_1 y_2 + x_2 y_1 + x_2 y_2^2 - y_1^2 y_2 + y_1 y_2)(x_2 y_2 - 2 x_2 + 1)y_2^3 \times (x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2, \]
\[ b_7 = -3(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2)(x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2, \]
\[ b_8 = 3(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2)^2(x_2 y_2 - 2 x_2 + 1)y_2^4, \]
\[ b_9 = (x_2^2 y_1 y_2 - x_2^2 y_2 + x_2 y_1^2 + x_2 y_1 y_2^2 - 4 x_2 y_1 y_2 + x_2 y_1 + x_2 y_2^2 - y_1^2 y_2 + y_1 y_2)^2(x_2 y_2 - 2 x_2 + 1)y_2^4. \]
Moreover, if $t_2$ is a solution of (8), $t_3, t_4$ are uniquely determined by (7).
I checked that equation (8) for $t_2$ is irreducible of degree 9 and that for generic $x_2, y_1, y_2$, (8) has no multiple factor. As a consequence, we obtain the following.

**Theorem 6.4.** The restriction of $F_1$ to the subspace $t_5 = t_6 = 0$ is generically 9 to 1.
I am not sure whether Theorem 6.4 induces the invalidity of Conjecture 6.1 or not.

**Acknowledgements**

Last I mention that the note [4] and the communications with Prof. B. Hunt are valuable when I formulate the maps $F_j (j = 1, 2, 3)$ and solve Problem 6.2'. Moreover, I am indebted to Prof. K. Okubo because without his help, I could not use REDUCE 3.4 (not REDUCE 3.2 !) which is a powerful tool in obtaining the results of this note.

**References**