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Kyoto University
The configuration space of 6 points in $\mathbb{P}^2$, 
the moduli space of cubic surfaces 
and the Weyl group of type $E_6$

Jiro Sekiguchi

1. Introduction

My first plan of the talk is to explain my study on the hypergeometric system $E(3,6)$ of type (3,6) ([8]). The system in question admits $\Sigma_6$-action, where $\Sigma_6$ is the symmetric group on 6 letters. This follows from that $E(3,6)$ lives in the configuration space $\mathbb{P}^2(6)$ of 6 points in $\mathbb{P}^2$ which admits $\Sigma_6$-action as permutations of the 6 points. Recently M. Yoshida (Kyushu Univ.) pointed out that the $\Sigma_6$-action on the space $\mathbb{P}^2(6)$ is naturally extended to $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type $E_6$ (cf. [3]). Moreover, he told me that B. Hunt studied relations between the $W(E_6)$-action in question and the $W(E_6)$-invariant quintic hypersurface of $\mathbb{P}^5$. 

Reading his note [4], I felt that it is an interesting exercise for REDUCE user to show whether his conjecture is true or not. For this reason, I changed the original plan and I restrict my attention to the study on $W(E_6)$-actions on $\mathbb{P}^5$ and on $\mathbb{P}^6$, namely, to the birational geometry related with the hypergeometric system $E(3,6)$.

It is better for the readers who are interested in SYMBOLIC COMPUTATION to read section 6 first.

2. The hypergeometric function of type (3,6)

Though I don’t treat it in this note, I begin this note with introducing the hypergeometric function of type (3,6):

$$E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A_{m_1, m_2, n_1, n_2} x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2}$$

where

$$A_{m_1, m_2, n_1, n_2} = \frac{(a_2, m_1 + m_2)(a_3, n_1 + n_2)(1 - a_5, m_1 + n_1)(1 - a_6, m_2 + n_2)}{m_1! m_2! n_1! n_2! (a_0, m_1 + m_2 + n_1 + n_2)}$$

By definition, $E(a_0, a_2, a_3, a_5, a_6; x_1, x_2, y_1, y_2)$ has parameters $a_j \ (j = 0, 2, 3, 5, 6)$. This function is one of 4 variables generalizations of Gaussian hypergeometric function. It
is known that the singularities of the system of differential equations whose solution is 
\[ E(a_{0}, a_{2}, a_{5}, a_{6}, x_{1}, x_{2}, y_{1}, y_{2}) \] is contained in the union of the 14 hypersurfaces 
\[ T_{j} : p_{j} = 0 \ (1 \leq j \leq 14), \]
where
\[
\begin{align*}
p_{1} &= x_{1}y_{2} - x_{2}y_{1} - x_{1} + x_{2} + y_{1} - y_{2}, \quad p_{2} = y_{1} - 1, \quad p_{3} = x_{1} - 1, \\
p_{4} &= y_{2} - 1, \quad p_{5} = x_{2} - 1, \quad p_{6} = y_{1} - y_{2}, \quad p_{7} = x_{1} - x_{2}, \quad p_{8} = x_{1} - y_{1}, \\
p_{9} &= x_{2} - y_{2}, \quad p_{10} = x_{1}y_{2} - x_{2}y_{1}, \quad p_{11} = x_{2}, \quad p_{12} = x_{1}, \quad p_{13} = y_{2}, \quad p_{14} = y_{1}.
\end{align*}
\]

We define birational transformations \( s_{j} \) (1 \( \leq j \leq 5 \) ) on \( C^{4} \) by
\[
\begin{align*}
s_{1} &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow \left( \frac{1}{x_{1}}, \frac{1}{x_{2}}, \frac{y_{1}}{x_{1}}, \frac{y_{2}}{x_{2}} \right), \\
s_{2} &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow (y_{1}, y_{2}, x_{1}, x_{2}), \\
s_{3} &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow \left( \frac{x_{1} - y_{1}}{1 - y_{1}}, \frac{x_{2} - y_{2}}{1 - y_{2}}, \frac{y_{1}}{y_{1} - 1}, \frac{y_{2}}{y_{2} - 1} \right), \\
s_{4} &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow \left( \frac{x_{2}}{x_{1}}, \frac{1}{y_{1}}, \frac{y_{2}}{y_{1}} \right), \\
s_{5} &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow (x_{2}, x_{1}, y_{2}, y_{1}).
\end{align*}
\]

Then the group generated by \( s_{j} \) (1 \( \leq j \leq 5 \) ) is identified with \( \Sigma_{6} \) because
\[
\begin{align*}
s_{j}^{3} &= id. \ (1 \leq j \leq 5), \quad s_{j}s_{k} = s_{k}s_{j} \ (|j - k| > 1), \\
s_{j}s_{k}s_{j} &= s_{k}s_{j}s_{k} \ (|j - k| = 1).
\end{align*}
\]

Let \( r \) be a birational transformation on \( C^{4} \) defined by
\[
\begin{align*}
r &: (x_{1}, x_{2}, y_{1}, y_{2}) \rightarrow (1/x_{1}, 1/x_{2}, 1/y_{1}, 1/y_{2}).
\end{align*}
\]

Then the group \( \tilde{G} \) generated by \( s_{1}, \ldots, s_{5} \) and \( r \) is isomorphic to the Weyl group \( W(E_{6}) \) of type \( E_{6} \) which will be seen later (cf. [3], [4]).

We define the hypersurface \( T_{15} : p_{15} = 0 \), where
\[
\begin{align*}
p_{15} &= x_{1}y_{2}(1 - y_{1})(1 - x_{2}) - x_{2}y_{1}(1 - x_{1})(1 - y_{2}).
\end{align*}
\]

It follows from the definition that \( s_{1}, \ldots, s_{5}, r \) and therefore all the elements of \( \tilde{G} \) are bi-regular outside the union \( T \) of the hypersurfaces \( T_{j} \) (1 \( \leq j \leq 15 \) ).

3. The Weyl group \( W(E_{6}) \)

Let \( E_{R} \) be a Cartan subalgebra of a compact Lie algebra of type \( E_{6} \), i.e. \( E_{R} \simeq R^{6} \). Let \( t = (t_{1}, t_{2}, t_{3}, t_{4}, t_{5}, t_{6}) \) be a coordinate system of \( E_{R} \) such that the roots of type \( E_{6} \) are:
\[
\begin{align*}
\pm(t_{i} \pm t_{j}), \quad 1 \leq i < j \leq 5 \\
\pm\frac{1}{2}(\delta_{1}t_{1} + \delta_{2}t_{2} + \delta_{3}t_{3} + \delta_{4}t_{4} + \delta_{5}t_{5} + \delta_{6}t_{6})
\end{align*}
\]
(where $\delta_j = \pm 1$ and $\prod_j \delta_j = 1$). Note that compared with the notation in [1], our variables $t_i = \epsilon_i, i = 1, \cdots, 5$, while our coordinate $t_6$ is denoted $\epsilon_6 - \epsilon_7 - \epsilon_8$ in [1]. We now introduce the following linear forms on $E_R$:

$$h = -\frac{1}{2}(t_1 + \cdots + t_6),$$

$$h_{1j} = -t_{j-1} + h_0, \quad j = 2, \cdots, 6$$

$$h_{jk} = t_{j-1} - t_{k-1}, \quad j, k \neq 1$$

$$h_{jkl} = -t_{j-1} - t_{k-1} - t_{l-1} + h_0, \quad j, k, l \neq 1$$

where

$$h_0 = \frac{1}{2}(t_1 + \cdots + t_5 - t_6).$$

Then the totality of $h, h_{ij}, h_{ijk}$ forms a set of positive roots of type $E_6$. Let $s$ (resp. $s_{ij}, s_{ijk}$) be the reflection on $E_R$ with respect to the hyperplane $h = 0$ (resp. $h_{ij} = 0, h_{ijk} = 0$). Then the Weyl group of type $E_6$ which is denoted by $W(E_6)$ in this note is the group generated by the 36 reflections defined above.

As a system of simple roots, we take

$$\alpha_1 = h_{12}, \quad \alpha_2 = h_{123}, \quad \alpha_3 = h_{23}, \quad \alpha_4 = h_{34}, \quad \alpha_5 = h_{45}, \quad \alpha_6 = h_{56}.$$}

Then the Dynkin diagram is:

$$\begin{array}{cccccc}
\alpha_1 & \quad & \alpha_3 & \quad & \alpha_4 & \quad & \alpha_5 & \quad & \alpha_6 \\
\alpha_2
\end{array}$$

Let $g_j$ be the reflection on $E_R$ with respect to the root $\alpha_j (j = 1, \cdots, 6)$. Then, from the definition,

$$g_1 = s_{12}, \quad g_2 = s_{123}, \quad g_3 = s_{23}, \quad g_4 = s_{34}, \quad g_5 = s_{45}, \quad g_6 = s_{56}.$$}

Let $E$ be the complexification of $E_R$ and we extend the action of $W(E_6)$ on $E_R$ to that on $E$ in a natural manner. Moreover let $P^5$ be the projective space associated to $E$. Then the $W(E_6)$-action on $E$ induces a projective linear action of $W(E_6)$ on $P^5$.

4. The configuration space of 6 points in $P^2$

We have already defined a birational action of $W(E_6)$ on $C^4$ in section 2. In this section, we explain that the birational transformations $s_1, \cdots, s_5, r$ naturally arise from the study of the configuration space of 6 points in $P^2$. For this purpose, we first introduce the linear space $W$ of $3 \times 6$ matrices:
$W = \{X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix} ; x_{ij} \in \mathbb{C} (1 \leq i \leq 3, 1 \leq j \leq 6)\}.$

Then $W$ admits a left $GL(3, \mathbb{C})$-action and a right $GL(6, \mathbb{C})$-action in a natural way. For a moment, we identify $({\mathbb{C}^*})^6$ with the maximal torus of $GL(6, \mathbb{C})$ consisting of diagonal matrices and consider the action of $GL(3, \mathbb{C}) \times ({\mathbb{C}^*})^6$ on $W$ instead of that of $GL(3, \mathbb{C}) \times GL(6, \mathbb{C})$.

For simplicity, we write $X = (X_1, X_2)$ for the matrix $X \in W$, where both $X_1, X_2$ are $3 \times 3$ matrices. For any $3 \times 3$ matrix $Y = (y_{ij})_{1 \leq i,j \leq 3}$ with the condition $y_{ij} \neq 0$ $(1 \leq i, j \leq 3)$, we define a $3 \times 3$ matrix

$$\sigma(Y) = \begin{pmatrix} 1 \\ y_{ij} \end{pmatrix}_{1 \leq i,j \leq 3}.$$

following a suggestion of M. Yoshida. Moreover, we put

$$D(i_1, i_2, i_3) = \det(\begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix})$$

for a given matrix $X \in W$.

Using these notation, we define subsets $W', W_0$ of $W$ by

$$W' = \{X \in W ; D(i_1, i_2, i_3) \neq 0 (1 \leq i_1 < i_2 < i_3 \leq 6)\},$$

$$W_0 = \{(X_1, X_2) \in W' ; (I_3, \text{Cof}(X_1^{-1}X_2)), (I_3, \sigma(X_1^{-1}X_2)) \in W'\},$$

where $\text{Cof}(Y) = (\det Y)^t Y^{-1}$ is the cofactor matrix of a given square matrix $Y$.

It is clear that the action of $GL(3, \mathbb{C}) \times ({\mathbb{C}^*})^6$ on $W$ naturally induces that on each of $W', W_0$. In the sequel, we mainly consider the quotient space of $W_0$ under the action of $GL(3, \mathbb{C}) \times ({\mathbb{C}^*})^6$, that is,

$$W_Q = GL(3, \mathbb{C}) \backslash W_0/({\mathbb{C}^*})^6.$$

It is clear from the definition that for any element $X \in W_0$, there are $(g, h) \in GL(3, \mathbb{C}) \times ({\mathbb{C}^*})^6$ and $(x_1, x_2, y_1, y_2) \in \mathbb{C}^4$ such that

$$gXh = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

In particular $(x_1, x_2, y_1, y_2)$ is uniquely determined for $X \in W_0$. In this sense, $W_Q = GL(3, \mathbb{C}) \backslash W_0/({\mathbb{C}^*})^6$ is identified with an open subset of $\mathbb{C}^4$. Note that $(x_1, x_2, y_1, y_2)$ is the variables $(x_1, x_2, y_1, y_2)$ of section 2. Then $W_Q = \mathbb{C}^4 - T$. 
Changes of column vectors of $X \in W_0$ induce birational transformations on $\mathbb{C}^4$ with coordinate system $(x_1, x_2, y_1, y_2)$. The action $s_j (1 \leq j \leq 5)$ introduced in section 2 is nothing but the birational transformation on $\mathbb{C}^4$ corresponding to the change of the $j$-th column vector and $(j + 1)$-column vector of $X \in W_0$. Moreover $W_q$ admits an involution induced from the action on $W_0$ defined by

$$\tilde{r} : (X_1, X_2) \rightarrow (I_3, \sigma(X_1^{-1}X_2))$$

for any $(X_1, X_2) \in W_0$. The involution $r$ defined in section 2 is equal to that induced from $\tilde{r}$.

The following theorem which seems known shows a concrete correspondence between $W(E_6)$ and the group $\tilde{G}$ introduced in section 2.

**Theorem 4.1.** The correspondence

$$g_1 \rightarrow s_1, \quad g_2 \rightarrow r, \quad g_3 \rightarrow s_2, \quad g_4 \rightarrow s_3, \quad g_5 \rightarrow s_4, \quad g_6 \rightarrow s_5$$

induces a group isomorphism of $W(E_6)$ to the group $\tilde{G}$.

**Remark.** In [3], it is stated that there is a $W(E_6)$-action on $W_q$. See also [4].

**5. $W(E_6)$-equivariant maps**

We first define rational functions on $E$ by

$$
\begin{align*}
\lambda(t) &= \frac{h_{34} \cdot h_{345} \cdot h_{26} \cdot h_{256}}{h_{24} \cdot h_{245} \cdot h_{46} \cdot h_{456}} \cdot \frac{h_{15} \cdot h_{156} \cdot h_{24} \cdot h_{246}}{h_{14} \cdot h_{146} \cdot h_{23} \cdot h_{236}}, \\
\mu(t) &= \frac{h_{24} \cdot h_{245} \cdot h_{36} \cdot h_{356}}{h_{23} \cdot h_{235} \cdot h_{46} \cdot h_{456}} \cdot \frac{h_{16} \cdot h_{165} \cdot h_{25} \cdot h_{256}}{h_{15} \cdot h_{156} \cdot h_{24} \cdot h_{246}}, \\
\nu(t) &= \frac{h_{25} \cdot h_{235} \cdot h_{46} \cdot h_{456}}{h_{23} \cdot h_{235} \cdot h_{46} \cdot h_{456}} \cdot \frac{h_{14} \cdot h_{146} \cdot h_{23} \cdot h_{236}}{h_{13} \cdot h_{136} \cdot h_{24} \cdot h_{246}}, \\
\rho(t) &= \frac{h_{24} \cdot h_{245} \cdot h_{36} \cdot h_{356}}{h_{23} \cdot h_{235} \cdot h_{46} \cdot h_{456}} \cdot \frac{h_{15} \cdot h_{156} \cdot h_{24} \cdot h_{246}}{h_{14} \cdot h_{146} \cdot h_{23} \cdot h_{236}}.
\end{align*}
$$

where $h, h_{ij}, h_{ijk}$ denote linear functions on $E$ introduced in section 3. Since all the rational functions above are homogeneous of degree zero, they are regarded as functions on $\mathbb{P}^5$. Therefore defining

$$F_1(t) = (x_1(t), x_2(t), y_1(t), y_2(t)), \quad F_2(t) = (\lambda(t), \mu(t), \nu(t), \rho(t)),$$

we obtain two maps $F_1, F_2$ from $\mathbb{P}^5$ to $\mathbb{C}^4$. The roles of $F_1, F_2$ will become clear in Theorem 5.1 which will be given later. To define $F_1, F_2$, I am indebted to [4]. We are going to explain the meaning of $x_j(t), y_j(t)$ following [4].
We begin with defining the cross ratio. Let $\xi_i = [\xi_{1i} : \xi_{2i} : \xi_{3i}]$ (1 \leq i \leq 5) be five points of $\mathbb{P}^2$ and let $l : g_1 u_1 + g_2 u_2 + g_3 u_3 = 0$ be a generic line in $\mathbb{P}^2$. We denote by $[1 : z_i : w_i]$ the intersection of $l$ and the line passing through the points $\xi_1$ and $\xi_i$. Then we put

\begin{align*}
(1) \quad CR(\xi_2, \xi_3, \xi_4, \xi_5 ; \xi_1) &= \frac{(z_2 - z_4)(z_3 - z_5)}{(z_2 - z_5)(z_3 - z_4)}
\end{align*}

which is in fact a cross ratio of $z_2, z_3, z_4, z_5$.

Now we consider a matrix of the form

\[
X = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & x_1 & x_2 \\
0 & 1 & 0 & 1 & x_1 & x_2 \\
0 & 0 & 1 & 1 & y_1 & y_2
\end{pmatrix}.
\]

From the matrix $X$, we define six points $\xi_i (i = 1, \cdots, 6)$ in $\mathbb{P}^2$ in a usual manner, that is,

\[
\xi_1 = [1 : 0 : 0], \quad \xi_2 = [0 : 1 : 0], \quad \xi_3 = [0 : 0 : 1],
\]

\[
\xi_4 = [1 : 1 : 1], \quad \xi_5 = [1 : x_1 : y_1], \quad \xi_6 = [1 : x_2 : y_2].
\]

Then we can compute $CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5} ; \xi_{i_1})$ explicitly for various $i_1, i_2, i_3, i_4, i_5$.

On the other hand, we put

\begin{align*}
(2) \quad CR'(i_2, i_3, i_4, i_5 ; i_1) &= \frac{h_{i_2i_4} h_{i_1i_2i_4} h_{i_3i_5} h_{i_1i_3i_5}}{h_{i_3i_4} h_{i_1i_3i_4} h_{i_2i_5} h_{i_1i_2i_5}}.
\end{align*}

By definition, $CR'(i_2, i_3, i_4, i_5 ; i_1)$ is a function on $\mathbb{P}^5$. Then from the equation

\begin{align*}
(3) \quad CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5} ; \xi_{i_1}) &= CR'(i_2, i_3, i_4, i_5 ; i_1),
\end{align*}

we obtain various equalities. In particular, by computing the cases

\[(i_1, i_2, i_3, i_4, i_5) = (3, 2, 1, 4, 5), (3, 2, 1, 4, 6), (2, 1, 3, 4, 5), (2, 1, 3, 4, 6),\]

we have the definition of $x_1(t), x_2(t), y_1(t), y_2(t)$ at the beginning of this section.

Let $F_3$ be the birational transformation on $\mathbb{C}^4$ defined by $F_3(x_1, x_2, y_1, y_2) = (\lambda, \mu, \nu, \rho)$, where

\[
\lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)},
\]

\[
\mu = \frac{(y_1 - 1)(x_2 - y_2) - (y_2 - 1)(x_1 - y_1))x_2y_2}{x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 + x_1y_2 + x_2y_1y_2 - x_2y_1},
\]

\[
\nu = \frac{(x_1y_2 - x_2y_1)(x_2 - 1)(y_2 - 1)}{(x_1 - x_2)(x_2 - y_2)(y_1 - y_2)},
\]

\[
\rho = \frac{(x_1 - 1)(x_2 - y_2) - (x_1 - y_1)(x_2 - 1))(y_2 - 1)x_2}.
\]
It is easy to show that $F_3$ is birational, because its inverse is given by

$$F_3^{-1}(\lambda, \mu, \nu, \rho) = \left( \frac{(\lambda \rho - 1)(\lambda \mu \nu - 1)}{(\lambda \mu - 1)(\lambda \nu - 1)}, \lambda \mu \nu \rho - 1, \lambda \mu \nu \rho - 1, \lambda \mu \nu \rho - 1 \right).$$

By the map $F_3$, the action of $W(E_6)$ on the $(x_1, x_2, y_1, y_2)$-space implies that on the $(\lambda, \mu, \nu, \rho)$-space. In fact, we define the following six birational transformations on the $(\lambda, \mu, \nu, \rho)$-space (cf. [7]):

\begin{align*}
\tilde{g}_1 : & \begin{cases} 
\lambda & \mapsto \frac{\lambda \nu \rho^2 (1 - \lambda)}{(\lambda \mu \nu \rho^2 - 1)} \\
\mu & \mapsto \frac{(\lambda \mu \rho - 1)(\lambda \mu \nu \rho - 1)}{(\mu (\lambda \nu - 1) (\lambda \mu \rho - 1))} \\
\nu & \mapsto \frac{(\lambda \nu \rho - 1)(\lambda \mu \nu - 1)}{(\nu (\lambda \rho - 1) (\lambda \mu \nu - 1))} \\
\rho & \mapsto \frac{(\lambda \rho - 1)(\lambda \mu \nu \rho - 1)}{(\rho (\lambda - 1)(\lambda \mu \nu \rho - 1))}
\end{cases} \\
\tilde{g}_2 : & (\lambda, \mu, \nu, \rho) \mapsto (\lambda, 1/\mu, \nu, \mu \rho) \\
\tilde{g}_3 : & (\lambda, \mu, \nu, \rho) \mapsto (1/\lambda, \mu, \nu, \lambda \rho) \\
\tilde{g}_4 : & (\lambda, \mu, \nu, \rho) \mapsto (\lambda \rho, \mu \rho, \nu \rho, 1/\rho) \\
\tilde{g}_5 : & (\lambda, \mu, \nu, \rho) \mapsto (\lambda, \mu, 1/\nu, \nu \rho) \\
\tilde{g}_6 : & \begin{cases} 
\lambda & \mapsto \frac{(\lambda \nu \rho - 1)(\lambda \mu \nu - 1)}{(\lambda (\nu \rho - 1) (\mu \nu \rho - 1))} \\
\mu & \mapsto \frac{(\mu \nu \rho - 1)(\mu \nu \rho - 1)}{(\mu (\nu \rho - 1) (\mu \nu \rho - 1))} \\
\nu & \mapsto \frac{\lambda \mu \nu \rho^2 (1 - \nu)}{(\lambda \mu \nu \rho^2 - 1)} \\
\rho & \mapsto \frac{(\nu \rho - 1)(\lambda \mu \nu \rho - 1)}{(\rho (\nu - 1)(\lambda \mu \nu \rho - 1))}
\end{cases}
\end{align*}

Let $G_1$ be the group generated by $\tilde{g}_j$ ($j = 1, \cdots, 6$). Then the correspondence

$$g_j \mapsto \tilde{g}_j \quad j = 1, \cdots, 6$$

is an isomorphism between $W(E_6)$ and $G_1$.

Needless to say, $F_1$ (resp. $F_2$) is regarded as a map from $\mathbb{P}^5$ to the $(x_1, x_2, y_1, y_2)$-space (resp. the $(\lambda, \mu, \nu, \rho)$-space.) Moreover, $F_3$ is regarded as a map from the $(x_1, x_2, y_1, y_2)$-space to the $(\lambda, \mu, \nu, \rho)$-space.

**Theorem 5.1.** The three maps $F_j$ ($j = 1, 2, 3$) are $W(E_6)$-equivariant and

$$F_3 \circ F_1(g(t)) = F_2(g(t)) \quad (\forall t \in \mathbb{P}^5, \forall g \in W(E_6)).$$

The $W(E_6)$-equivariances of $F_1$, $F_2$ are stated in [4] implicitly.

We now mention the meaning of the $(\lambda, \mu, \nu, \rho)$-space. In [2], A. Cayley defined a 4-dimensional family of cubic surfaces. Modifying his family, we introduce a family of cubic surfaces of $\mathbb{P}^3$ with homogeneous coordinate $(X : Y : Z : W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [7]):

$$\rho W[\lambda X^2 + \mu Y^2 + \nu Z^2 + (\rho - 1)^2(\lambda \mu \nu - 1)^2 W^2 + (\mu \nu + 1)YZ + (\lambda \nu + 1)ZX + (\lambda \mu + 1)XY$$
\[-(\rho - 1)(\lambda \mu \nu \rho - 1)W\{(\lambda + 1)X + (\mu + 1)Y + (\nu + 1)Z\} + XYZ = 0.\]

The family of cubic surfaces above admits a $W(E_6)$-action as given in [7]. In particular, the $W(E_6)$-action in [7] preserves the parameter space. For this reason, we obtain a $W(E_6)$-action on the $(\lambda, \mu, \nu, \rho)$-space which actually coincides with the $W(E_6)$-action on the $(\lambda, \mu, \nu, \rho)$-space explained before Theorem 5.1.

6. A Conjecture of B. Hunt

It is known (cf.[1]) that there is a unique $W(E_6)$-invariant homogeneous polynomial of $t = (t_1, \cdots, t_6)$ of degree 5 up to a constant factor. For example, we take $Q_5(t)$ below as such a polynomial (cf. [4]):

\[Q_5(t) = \frac{5}{108} t_6^5 + \frac{5}{18} \sigma_1 t_6^3 + \frac{5}{4} (\sigma_1^2 - 4 \sigma_2) t_6 + 30 \sqrt{\sigma_5},\]

where $\sigma_i = \sigma_i(t_1^2, \cdots, t_5^2)$ is the i-th elementary symmetric polynomial in $t_1^2, \cdots, t_5^2$ and $\sqrt{\sigma_5} = t_1 \cdots t_5$.

Let $I_5$ be the hypersurface in $\mathbb{P}^5$ defined by $Q_5(t) = 0$. Since $Q_5(t)$ is $W(E_6)$-invariant, so is $I_5$. Moreover, since dim $I_5 = 4$, the restrictions $F_1|I_5, F_2|I_5$ are generically finite maps from $I_5$ to $\mathbb{C}^4$. In [4], B. Hunt stated conjectures on these maps which turn out to be one conjecture below.

**Conjecture 6.1.**([4]) Both $F_1|I_5, F_2|I_5$ are generically bijective.

How to attack Conjecture 6.1 with the help of REDUCE? In virtue of Theorem 5.1, it suffices to show Conjecture 6.1 for one of $F_1|I_5, F_2|I_5$. Noting the definition of $F_1(t)$, we find that Conjecture 6.1 is rewritten as follows:

**Problem 6.2.** Let $x_1, x_2, y_1, y_2$ be constants. At least assume that $(x_1, x_2, y_1, y_2)$ is outside the set $T$. Using $x_1, x_2, y_1, y_2$, we define four polynomials of $t$ by

\[f_1 = h_{24} \cdot h_{234} \cdot h_{15} \cdot h_{135} - x_1 \cdot h_{14} \cdot h_{134} \cdot h_{25} \cdot h_{235},\]
\[f_2 = h_{24} \cdot h_{234} \cdot h_{16} \cdot h_{136} - x_2 \cdot h_{14} \cdot h_{134} \cdot h_{26} \cdot h_{236},\]
\[g_1 = h_{34} \cdot h_{234} \cdot h_{15} \cdot h_{125} - y_1 \cdot h_{14} \cdot h_{124} \cdot h_{35} \cdot h_{235},\]
\[g_2 = h_{34} \cdot h_{234} \cdot h_{16} \cdot h_{126} - y_2 \cdot h_{14} \cdot h_{124} \cdot h_{36} \cdot h_{236},\]

where $h, h_{ij}, h_{ijk}$ are linear functions of $t$ defined in section 3. Then how many solutions are there for the simultaneous equations of $t$ defined by

\[f_1 = f_2 = g_1 = g_2 = Q_5 = 0\]

under the condition $F_1(t) \notin T$ ?

Needles to say, there is a gap between Conjecture 6.1 and Problem 6.2, that is, Conjecture 6.1 claims that for generic $x_1, x_2, y_1, y_2$, equation (4) has a unique projective solution. Since I don’t know whether Conjecture 6.1 is true or not, I reformulate it as a problem.
I tried to solve Problem 6.2 directly by using REDUCE3.4 on TOSHIBA J3100 once and at last abandoned to do because of out of capacity.

From now on, I am going to explain results related with Problem 6.2 and the moduli of cubic surfaces. We consider the hypersurface $H$ in $\mathbf{P}^5$ defined by $\lambda(t) - 1 = 0$, that is,

\begin{equation}
(5) \quad P(t) = h_{345} \cdot h_{26} \cdot h_{256} \cdot h_{13} \cdot h_{136} \cdot h_{246} - h_{245} \cdot h_{36} \cdot h_{356} \cdot h_{12} \cdot h_{126} \cdot h_{346} = 0.
\end{equation}

Then it is easy to show that the polynomial $P(t)$ of equation (4) is decomposed into two factors (up to a constant):

\[ P(t) = h_{23} \cdot P_5(t), \]

where $P_5(t)$ is homogeneous of degree 5. Moreover $P_5$ is so taken that

\[ P_5(t_1, t_2, t_3, t_4, t_5, t_6) = \text{const}.Q_5(t_1, t_2, t_3, t_4, t_6, -3t_5). \]

From this remarkable relation, we easily imply the following (cf. [4], [6]).

**Proposition 6.3.**

(i) There are 45 hypersurfaces in $\mathbf{P}^5$ as the $W(E_6)$-orbit of $H$. Moreover, the isotropy subgroup of $H$ in $W(E_6)$ is isomorphic to the Weyl group of type $F_4$.

(ii) The intersection $H \cap I_5$ is decomposed into two irreducible components. One is defined by $t_5 = t_6 = 0$ therefore is isomorphic to $\mathbf{P}^3$. The other is defined by an equation of degree 24.

(iii) If $t \in H$, then $F_2(t) = (1,1,1,1)$, that is, $\lambda(t) = \mu(t) = \nu(t) = \rho(t) = 1$. The corresponding cubic surface has Eckard points.

It follows from Proposition 6.3 (i) that there is a natural 1-1 correspondence between the $W(E_6)$-orbit of $H$ and the 45 exceptional divisors of Naruki's cross ratio variety [6].

We mention Proposition 6.3 (ii) in detail. We first introduce symmetric polynomials of $t_1, t_2, t_3, t_4$ by

\[ s_2 = t_1^2 + t_2^2 + t_3^2 + t_4^2, \]
\[ s_4 = t_1^2(t_2^2 + t_3^2 + t_4^2) + t_2^2(t_3^2 + t_4^2) + t_3^2t_4^2, \]
\[ s'_4 = t_1t_2t_3t_4. \]

Using $s_2, s_4, s'_4$, we define the polynomial $h$ of degree 24 by

\[ h = c_{10}t_5^{20} + c_9t_5^{18} + c_8t_5^{16} + c_7t_5^{14} + c_6t_5^{12} + c_5t_5^{10} + c_4t_5^8 + c_3t_5^6 + c_2t_5^4 + c_1t_5^2 + c_0, \]

where

\[ c_{10} = 1728s_2^2, \]
\[ c_9 = 432s_2(-21s_2^2 + 20s_4), \]
\[ c_8 = 27(4800s_4^2 + 761s_2^4 - 1736s_2^2s_4 + 400s_4^2), \]
\[ c_7 = 8s_2(-46656s_4^2 - 3217s_2^4 + 12852s_2^2s_4 - 10368s_4^2), \]
\[c_6 = 2(-190080s_4^2s_2^2 - 336960s_4^2s_4 + 9251s_2^6 - 55955s_4^2s_2^4),\]
\[c_6 = 2s_2(825360s_4^2s_2^2 - 1582848s_4^2s_4 - 336960s_4^2s_2^4 + 27143s_2^6),\]
\[c_4 = -59833728s_4^4 + 1370994s_4^2s_2^4 - 4732128s_4^2s_4^2 - 193s_2^8 + 3054s_2^6s_4 - 12981s_2^4s_4^2 + 10120s_2^2s_4^3 + 21168s_4^3),\]
\[c_3 = 2s_2(-2191104s_4^4s_2^2 + 199476s_4^2s_2^4 - 1263024s_4^2s_2^2s_4 + 1990080s_4^2s_4^2),\]
\[c_2 = -907200s_4^4s_2^2 + 2491776s_4^4s_4^2 - 54714s_4^2s_2^6 + 554274s_4^2s_2^4s_4 - 1854576s_4^2s_2^2s_4^2 + 2051616s_4^4s_4^3 - 256s_{10} + 4640s_2^8s_4 - 33505s_2^6s_4^2 + 120460s_2^4s_4^3 - 215600s_2^2s_4^4 + 153664s_4^{12}),\]
\[c_1 = 6s_4^2s_2(-4968s_4^2s_2^2 + 14688s_4^2s_4^2 - 26s_2^6 + 285s_2^4s_4^2 - 1032s_4^2s_4^2 + 1232s_3^2),\]
\[c_0 = 27s_4^4(192s_4^2 + s_2^4 - 8s_2^2s_4 + 16s_4^2).\]

Moreover,
\[N = -2\{(5s_2^6 - 1602s_4^2s_2^4 - 34s_4^4 - 4134s_4^2s_4^2 + 10037s_4^4s_2^4s_4^2 - 3005s_2^6s_4^2 \}
\[+56s_2s_4^2 - 12820s_2s_4^2t_5^2 + 828s_2^4s_4^2 + 15764s_4^2s_4^2 + 1980s_4^4t_5^2 - 360t_5^{10})t_5^2 - (s_2^2 + 164s_2^4t_5^2 - 4s_4^2 + 7368t_5^2)\}
\[t_4^2s_4^4\},\]
\[D = -\{3(31s_2^2 + 650s_2^4t_5^2 - 92s_2s_4^4 + 2320s_2t_5^4 - 1752s_4^2t_5^2 + 5648s_4^4t_5^2 + 12s_4^2t_5^4 - (1465s_4^2 - 1044t_5^2)s_4^2t_5^2 + 15(269s_4^2 - 61t_5^4))s_2^4t_5^2 + 144s_2^4t_5^2 + 599s_2^4t_5^2 - 5488s_4^2t_5^2 + 2072s_2^4t_5^2 - 120s_4t_5^{12}\}.\]

Then from the equations
\[P_5 = Q_5 = 0,\]
we obtain
\[t_6 = N/D, \quad h = 0.\]

The equation \(h = 0\) is the one stated in Proposition 6.3 (ii).

If we consider the equation \(\lambda - 1 = 0\) in the \((x_1, x_2, y_1, y_2)\)-space, we obtain a hypersurface \(H_0\) defined by
\[x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1) - y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1) = 0.\]
Now we formulate a problem simplified from Problem 6.2, noting Proposition 6.3 (ii). Namely, we consider Problem 6.2 in the case \( t_5 = t_6 = 0 \) and \( t_1 = 1 \). (The condition \( t_1 = 1 \) is not essential. From the homogeneity, we may assume \( t_j = 1 \) for some \( j \).)

**Problem 6.2'.** Define four polynomials of \( t_2, t_3, t_4 \) by

\[
\begin{align*}
    f_{10} &= (t_2 + t_3 - t_4 + 1)^2(t_2 + t_4)(t_3 - 1) - x_1(t_2 + t_3)(t_2 - t_3 + t_4 + 1)^2(t_4 - 1), \\
    f_{20} &= (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_2 - t_3 + 1), \\
    g_{10} &= (t_2 + t_3 - t_4 + 1)^2(t_2 - t_3)(t_4 + 1) - y_1(t_2 - t_3 + t_4 + 1)^2(t_2 - t_4)(t_3 + 1), \\
    g_{20} &= (t_2 + t_3 + t_4 + 1)(t_2 + t_3 - t_4 + 1)(t_2 - t_3).
\end{align*}
\]

where \( x_1, x_2, y_1, y_2 \) are constants with the condition (6) and \((x_1, x_2, y_1, y_2) \not\in T\). (In particular, we assume that \( x_1 \) is a rational function of \( x_2, y_1, y_2 \).) Then how many solutions are there for the equations (7) of \( t_2, t_3, t_4 \) below

\[
(7) \quad f_{10} = f_{20} = g_{10} = g_{20} = 0
\]

under the condition \( t \not\in T \)?

It is possible to give an answer to Problem 6.2'. In fact, erasing \( t_3, t_4 \) from (7), we obtain an equation for \( t_2 \) defined by

\[
(8) \quad \sum_{j=0}^{9} b_{j} t_2^j = 0,
\]

where

\[
\begin{align*}
    b_0 &= (x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)^2(x_2 y_2 - 2 y_2 + 1)y_2^4, \\
    b_1 &= 3(x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)^2(x_2 y_2 - 2 x_2 + 1)y_2^4, \\
    b_6 &= -4(x_2 y_1 y_2 - x_2 y_2 + x_2 y_1^2 + x_2 y_1 y_2^2 - 4 x_2 y_1 y_2 + x_2 y_2 - y_2^2 y_2 + y_1 y_2) \\
    &\quad \times (x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_2 - 2 x_2 + 1)y_2^3, \\
    b_5 &= -6(x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2) \\
    &\quad \times (x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2, \\
    b_4 &= 6(x_2 y_1 y_2 - x_2 y_2 - 2 y_1 y_2 + y_1 + y_2^2)(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2) \\
    &\quad \times (x_2 y_2 - 2 x_2 + 1)x_2 y_1 y_2, \\
    b_3 &= 4(x_2 y_1 y_2 - x_2 y_2 + x_2 y_1^2 + x_2 y_1 y_2^2 - 4 x_2 y_1 y_2 + x_2 y_2 - y_2^2 y_2 + y_1 y_2) \\
    &\quad \times (x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2)(x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2, \\
    b_1 &= -3(x_2 y_1 + x_2 y_2^2 - 2 x_2 y_2 - y_1 y_2 + y_2)(x_2 y_2 - 2 y_2 + 1)x_2 y_1 y_2^2.
\end{align*}
\]
Moreover, if $t_2$ is a solution of (8), $t_3, t_4$ are uniquely determined by (7).

I checked that equation (8) for $t_2$ is irreducible of degree 9 and that for generic $x_2, y_1, y_2$, (8) has no multiple factor. As a consequence, we obtain the following.

**Theorem 6.4.** The restriction of $F_1$ to the subspace $t_5 = t_6 = 0$ is generically 9 to 1.

I am not sure whether Theorem 6.4 induces the invalidity of Conjecture 6.1 or not.

**Acknowledgements**

Last I mention that the note [4] and the communications with Prof. B. Hunt are valuable when I formulate the maps $F_j (j = 1, 2, 3)$ and solve Problem 6.2'. Moreover, I am indebted to Prof. K. Okubo because without his help, I could not use REDUCE 3.4 (not REDUCE 3.2!) which is a powerful tool in obtaining the results of this note.

**References**