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Towards the Kazhdan-Lusztig multiplicity formula for generalized Kac-Moody algebras

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1. Generalized Kac-Moody Algebras

1.1. Let $A=(a_{ij})_{i,j\in I}$ with $I=\{1,2,\ldots,n\}$ be a real $n\times n$ matrix satisfying the following conditions:

(C1) $a_{ii}=2$, or $a_{ii}\leq 0$ $(i\in I)$;

(C2) $a_{ij}\leq 0$ $(i\neq j)$, and $a_{ij}\in\mathbb{Z}$ if $a_{ii}=2$;

(C3) $a_{ij}=0\Leftrightarrow a_{ji}=0$.

We call such a matrix a GGCM (= generalized, generalized Cartan matrix).

For any GGCM $A=(a_{ij})_{i,j\in I}$, we have a triple $(\mathfrak{h}, \Pi=\{\alpha_i\}_{i\in I}, \Pi^{\vee}=\{\alpha_i^{\vee}\}_{i\in I})$ satisfying the following (see [6, Chap.1]):

(R1) $\mathfrak{h}$ is a finite-dimensional (complex) vector space such that $\dim_{\mathbb{C}}\mathfrak{h}=2n-\operatorname{rank}A$;

(R2) $\Pi=\{\alpha_i\}_{i\in I}\subset\mathfrak{h}^*$ is linearly independent, and $\Pi^{\vee}=\{\alpha_i^{\vee}\}_{i\in I}\subset\mathfrak{h}$ is linearly independent, where $\mathfrak{h}^*:=\operatorname{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$;

(R3) $\langle\alpha_j,\alpha_i^{\vee}\rangle=a_{ij}$ $(i,j\in I)$, where $\langle\cdot,\cdot\rangle$ denotes a duality pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$.

The above triple is called a *realization* of A.

From now on throughout this paper, we assume that the GGCM A is *symmetrizable*, i.e., that there exists a diagonal matrix D such that $\det D\neq 0$ and DA is symmetric.

A *generalized Kac-Moody algebra* (= GKM algebra) associated to a symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$ is a Lie algebra $\mathfrak{g}(A)$ (over the complex number field $\mathbb{C}$)

generated by the above vector space $\mathfrak{h}$ and the elements e_i, f_i $(i \in I)$ satisfying the following relations (see [1], or [6, Chap.11]):

$$\text{(F1)} \begin{cases} [h, h'] = 0 & (h, h' \in \mathfrak{h}), \\ [h, e_i] = \langle \alpha_i, h \rangle e_i, \ \ [h, f_i] = -\langle \alpha_i, h \rangle f_i & (h \in \mathfrak{h}, \ i \in I), \\ [e_i, f_j] = \delta_{ij} \alpha_i^\vee & (i, j \in I), \end{cases}$$

$$\text{(F2)} \ (ad\, e_i)^{1-a_{ij}} e_j = 0, \ \ (ad\, f_i)^{1-a_{ij}} f_j = 0 \quad (a_{ii} = 2, \ j \neq i),$$

$$\text{(F3)} \ [e_i, e_j] = 0, \ \ [f_i, f_j] = 0 \quad (a_{ii}, a_{jj} \leq 0, \ a_{ij} = 0).$$

Then, we have the *root space decomposition* of $\mathfrak{g}(A)$ with respect to the *Cartan subalgebra* $\mathfrak{h}$: $\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Delta_-}^{\oplus} \mathfrak{g}_\alpha$, where Δ_+ $(\subset \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i)$ is the set of *positive roots*, Δ_- $(= -\Delta_+)$ is the set of *negative roots*, and $\mathfrak{g}_\alpha$ is the *root space* attached to a root $\alpha \in \Delta = \Delta_+ \cup \Delta_- \subset \mathfrak{h}^*$. Note that $\text{mult}(\alpha) := \dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} < +\infty$ $(\alpha \in \Delta_+)$.

Put $\mathfrak{n}_+ := \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_\alpha$, $\mathfrak{n}_- := \sum_{\alpha \in \Delta_+}^{\oplus} \mathfrak{g}_{-\alpha}$, and $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$.

1.2. We put $I^{re} := \{i \in I \mid a_{ii} = 2\}$, and $I^{im} := \{i \in I \mid a_{ii} \leq 0\}$. Let $\Pi^{re} := \{\alpha_i \in \Pi \mid i \in I^{re}\}$ be the set of *real simple roots*, and $\Pi^{im} := \{\alpha_i \in \Pi \mid i \in I^{im}\}$ the set of *imaginary simple roots*.

For $\alpha_i, \alpha_j \in \Pi^{im}$, we say that α_i is *perpendicular* to α_j if $a_{ij} = 0$. (Remark that an imaginary simple root $\alpha_i \in \Pi^{im}$ is perpendicular to itself if $a_{ii} = 0$.) For $\lambda \in \mathfrak{h}^*$ and $\alpha_i \in \Pi^{im}$, we say that α_i is *perpendicular* to λ if $\langle \lambda, \alpha_i^\vee \rangle = 0$.

Now, fix an element $\Lambda \in P_+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i \in I)$, and $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ if $a_{ii} = 2\}$. Then, we define a subset $\mathcal{A}(\Lambda)$ of $\mathfrak{h}^*$ to be the set of all sums of (not necessarily distinct,) pairwise perpendicular, imaginary simple roots perpendicular to Λ. Note that $\mathcal{A} := \mathcal{A}(0)$ contains the set $\{0\} \cup \Pi^{im} \cup \{m\alpha_j \mid m \in \mathbb{Z}_{\geq 2}, \alpha_j \in \Pi^{im}$ with $a_{jj} = 0\}$ by definition. For an element $\beta = \sum_{i \in I^{im}} k_i \alpha_i$ $(k_i \in \mathbb{Z}_{\geq 0})$, we put $\text{ht}(\beta) = \sum_{i \in I^{im}} k_i$.

1.3. For $i \in I^{re}$, let r_i be the *simple reflection* of $\mathfrak{h}^*$ given by: $r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ $(\lambda \in \mathfrak{h}^*)$. The *Weyl group* W of $\mathfrak{g}(A)$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I^{re})$. For an element $w \in W$, $\ell(w)$ denotes the *length* of w.

Let $\Delta^{re} := W \cdot \Pi^{re}$ be the set of *real roots*, $\Delta^{im} := \Delta \setminus \Delta^{re}$ the set of *imaginary roots*. For a real root $\alpha = w(\alpha_i)$ $(w \in W, \alpha_i \in \Pi^{re})$, we define the reflection r_α of

$\mathfrak{h}^*$ with respect to α by: $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ $(\lambda \in \mathfrak{h}^*)$, where $\alpha^\vee := w(\alpha_i^\vee) \in \mathfrak{h}$ is the *dual real root* of α. Note that $r_\alpha = w r_i w^{-1} \in W$.

1.4. For $\lambda \in \mathfrak{h}^*$, we denote by $V(\lambda)$ the *Verma module* $U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{b})} \mathbb{C}(\lambda)$ with highest weight λ over the GKM algebra $\mathfrak{g}(A)$. Here, $\mathbb{C}(\lambda)$ is the one-dimensional $\mathfrak{h}$-module with weight λ, on which $\mathfrak{n}_+$ acts trivially. As is well-known, the Verma module $V(\lambda)$ is the universal highest weight $\mathfrak{g}(A)$-module with highest weight λ, and has a unique maximal proper $\mathfrak{g}(A)$-submodule $V'(\lambda)$. Then, we define $L(\lambda)$ to be the quotient $\mathfrak{g}(A)$-module of $V(\lambda)$ by $V'(\lambda)$, so that $L(\lambda)$ is the irreducible highest weight $\mathfrak{g}(A)$-module with highest weight λ.

2. Bruhat Ordering and Kazhdan-Lusztig Polynomials

2.1. Here, we extend the notion of the Bruhat ordering on the Weyl group W to that on the direct product set $W \times \mathcal{A}$ of W and $\mathcal{A} = \mathcal{A}(0)$ as follows.

Definition 2.1 (*Bruhat ordering*). Let $w_1, w_2 \in W$. We write $w_1 \leftarrow w_2$ if there exists some $\gamma \in \Delta^{re} \cap \Delta_+$ such that $w_1 = r_\gamma w_2$ and $\ell(w_1) = \ell(w_2) + 1$. Moreover, for $w, w' \in W$, we write $w \geqslant w'$ if $w = w'$ or if there exist $w_1, \dots, w_r \in W$ such that

$$w \leftarrow w_1 \leftarrow \cdots \leftarrow w_r \leftarrow w'.$$

Definition 2.2 (cf. [11, Definition 2.2]). Let $\beta_1, \beta_2 \in \mathcal{A}$. We write $\beta_1 \leftarrow \beta_2$ if there exists some $\alpha_j \in \Pi^{im}$ such that $\beta_1 = \beta_2 + \alpha_j$. Moreover, for $\beta = \sum_{k \in I^{im}} m_k \alpha_k$, $\beta' = \sum_{k \in I^{im}} m'_k \alpha_k \in \mathcal{A}$, we write $\beta \geqslant \beta'$ if $m_k \geq m'_k$ for all $k \in I^{im}$.

Definition 2.3 (cf. [11, Definition 2.3]). For $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{A}$, we write $(w_1, \beta_1) \leftarrow (w_2, \beta_2)$

$$\text{if} \quad w_1 \leftarrow w_2 \text{ and } \beta_1 = \beta_2, \quad \text{or} \quad \text{if} \quad w_1 = w_2 \text{ and } \beta_1 \leftarrow \beta_2.$$

Moreover, for $(w, \beta), (w', \beta') \in W \times \mathcal{A}$, we write $(w, \beta) \geqslant (w', \beta')$ if $w \geqslant w'$ and $\beta \geqslant \beta'$.

2.2. Here, we review the definitions of the Kazhdan-Lusztig polynomials and the inverse Kazhdan-Lusztig polynomials, and then give their certain extensions. We first note that the Weyl group W of the GKM algebra $\mathfrak{g}(A)$ is a Coxeter group with canonical generator system $\{r_i \mid i \in I^{re}\}$. The *Hecke algebra* $\mathcal{H}(W)$ of W is the associative algebra over the Laurent polynomial ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (in the indeterminate $q^{\frac{1}{2}}$) which has a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-basis $\{T_w\}_{w \in W}$ with the following relations:

(H1) $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ $(w, w' \in W)$;

(H2) $(T_{r_i} + 1)(T_{r_i} - q) = 0$ $(i \in I^{re})$.

Let $\imath$ be the involutive automorphism of the ring $\mathcal{H}(W)$ defined by: $\imath(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$, $\imath(T_w) = (T_{w^{-1}})^{-1}$ $(w \in W)$. Then, we know the following proposition due to Kazhdan and Lusztig [9].

Proposition 2.4 ([9]). *For each $w \in W$, there exists a unique element $C_w \in \mathcal{H}(W)$ having the following properties:*

(1) $\imath(C_w) = C_w$;

(2) $C_w = (-1)^{\ell(w)} q^{\frac{\ell(w)}{2}} \cdot \sum_{y \leqslant w} (-1)^{\ell(y)} q^{-\ell(y)}\, \imath(P_{y,w}(q))\, T_y$,

where $P_{w,w} = 1$, and $P_{y,w}(q)$ is a polynomial with integer coefficients in the indeterminate q of degree $\leq (1/2) \cdot (\ell(w) - \ell(y) - 1)$ for $y \leqslant w$.
Moreover, the elements C_w $(w \in W)$ form a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-basis of $\mathcal{H}(W)$.

The above polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$ $(y \leqslant w)$ are called the *Kazhdan-Lusztig polynomials.* We set $P_{y,w}(q) := 0$ unless $y \leqslant w$.

Now, for $\beta, \beta' \in \mathcal{A} = \mathcal{A}(0)$, we define a polynomial $P_{\beta,\beta'}(q)$ in q by

$$P_{\beta,\beta'}(q) := \begin{cases} 1 & \text{if } \beta' \geqslant \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $(w, \beta), (w', \beta') \in W \times \mathcal{A}$, we put

$$P_{(w,\beta),(w',\beta')}(q) := P_{w,w'}(q) \cdot P_{\beta,\beta'}(q),$$

and call this polynomial the *extended Kazhdan-Lusztig polynomial.*

It is also known (see [10], and also [8, §5]) that there exist the *inverse Kazhdan-Lusztig polynomials* $Q_{w,y}(q) \in \mathbb{Z}[q]$ $(w \leqslant y \in W)$ for the Coxeter group W such that

$$\sum_{w \leqslant y \leqslant w'} (-1)^{\ell(y)-\ell(w)} Q_{w,y}(q) P_{y,w'}(q) = \delta_{w,w'} \qquad (w \leqslant w').$$

We set $Q_{w,y}(q) := 0$ unless $w \leqslant y$.

For $(w,\beta),(w',\beta')\in W\times\mathcal{A}$, we put

$$Q_{(w,\beta),(w',\beta')}(q) := Q_{w,w'}(q)\cdot Q_{\beta,\beta'}(q),$$

where $Q_{\beta,\beta'}(q) := P_{\beta,\beta'}(q)$, and call this polynomial the *extended inverse Kazhdan-Lusztig polynomial.*

Then, it is easy to see the following.

For $(w,\beta),(w',\beta')\in W\times\mathcal{A}$, we have

$$\sum_{(y,\gamma)\in W\times\mathcal{A}} (-1)^{(\ell(y)+\mathrm{ht}(\gamma))-(\ell(w)+\mathrm{ht}(\beta))} Q_{(w,\beta),(y,\gamma)}(q) P_{(y,\gamma),(w',\beta')}(q) = \delta_{(w,\beta),(w',\beta')}.$$

3. Irreducible Subquotients and Embeddings of Verma Modules

A $\mathfrak{g}(A)$-module V is said to be $\mathfrak{h}$-*diagonalizable* if V admits a *weight space decomposition*: $V=\sum^{\oplus}_{\tau\in\mathfrak{h}^*}V_\tau$, where V_τ is the *weight space* of weight $\tau\in\mathfrak{h}^*$. We denote by $P(V)$ the set of all weights of V. We call an $\mathfrak{h}$-diagonalizable module $V=\sum^{\oplus}_{\tau\in P(V)}V_\tau$ a *weight module* if $\dim_{\mathbb{C}} V_\tau<+\infty$ for all $\tau\in\mathfrak{h}^*$.

Now, for $\lambda\in\mathfrak{h}^*$, following [13, §2], we define the category $\mathcal{C}(\lambda)$ to be the full-subcategory of the category of all $\mathfrak{g}(A)$-modules whose objects are weight modules V such that $P(V)\subset\lambda-\sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. For $\lambda,\mu\in\mathfrak{h}^*$, we write $\mu\leq\lambda$ if $\lambda-\mu\in\sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$.

Here, we recall from [3, Definition 3.5] the definition of the multiplicity $[V:L(\mu)]$ of $L(\mu)$ in V for a module $V\in\mathcal{C}(\lambda)$ (in [3], the multiplicity $[V:L(\mu)]$ is defined for V in a wider category $\mathcal{O}$).

Proposition 3.1 ([3, Proposition 3.2]). *Let $\lambda,\mu\in\mathfrak{h}^*$, and $V\in\mathcal{C}(\lambda)$. Then, there exists a finite increasing filtration*

$$0=V_0\subset V_1\subset\cdots\subset V_t=V$$

of $\mathfrak{g}(A)$-submodules of V such that for each j $(1\leq j\leq t)$ the quotient module V_j/V_{j-1} either is isomorphic to some $L(\mu_j)$ $(\mu_j\in\mathfrak{h}^*)$, or has no weights τ with $\tau\geq\mu$.

We call the above filtration a *local composition series* of V at μ. We know that the cardinality of the set $\{1 \leq j \leq t \mid V_j/V_{j-1} \cong L(\mu)\}$ is independent of the choice of the local composition series of V at μ. So, we call it the *multiplicity* of $L(\mu)$ in V, and denote it by $[V : L(\mu)]$.

Now, we choose and fix an element $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ $(i \in I)$. From now on, we shall use the notation

$$(w, \beta) \circ \Lambda := w(\Lambda + \rho - \beta) - \rho$$

for $(w, \beta) \in W \times \mathcal{A}$ and $\Lambda \in P_+$.

We recall the following two theorems, which are essentially proved in [11].

Theorem 3.2 (cf. [11, Proposition 2.11]). *Fix $\Lambda \in P_+$. Let $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{A}(\Lambda)$. Then, we have*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}(A)}(V((w_1, \beta_1) \circ \Lambda), V((w_2, \beta_2) \circ \Lambda)) \leq 1.$$

Note that any nonzero $\mathfrak{g}(A)$-module homomorphism between two Verma modules is injective. So, we may write

$$V((w_1, \beta_1) \circ \Lambda) \subset V((w_2, \beta_2) \circ \Lambda)$$

when the equality holds in the above theorem.

Theorem 3.3 (cf. [11, Proposition 2.12]). *Let $\Lambda \in P_+$, $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{A}(\Lambda)$. Then,*

$$\begin{aligned} & V((w_1, \beta_1) \circ \Lambda) \subset V((w_2, \beta_2) \circ \Lambda) \\ \Longleftrightarrow \quad & (w_1, \beta_1) \geqslant (w_2, \beta_2) \\ \Longleftrightarrow \quad & [V((w_2, \beta_2) \circ \Lambda) : L((w_1, \beta_1) \circ \Lambda)] \neq 0. \end{aligned}$$

4. Translation Functors

Here, for $\lambda, \mu \in \mathfrak{h}^*$, we define the translation functor T_μ^λ from the category $\mathcal{C}(\mu)$ to the category $\mathcal{C}(\lambda)$, which is a generalization to GKM algebras of the one defined in [12] by W. Neidhardt. This functor enables us to show that the multiplicity

$[V((w,\beta)\circ\Lambda) : L((w',\beta')\circ\Lambda)]$ $((w,\beta),(w',\beta')\in W\times\mathcal{A}(\Lambda))$ does not depend on the choice of $\Lambda\in P_+$.

Since we assume that the GGCM A is symmetrizable, there exists a nondegenerate, symmetric, invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$. Recall that the restriction of this bilinear form $(\cdot|\cdot)$ to $\mathfrak{h}$ is also nondegenerate, so that it induces on $\mathfrak{h}^*$ a nondegenerate, W-invariant bilinear form, which we again denote by $(\cdot|\cdot)$. Then, we can define the so-called (generalized) *Casimir operator* Ω on the modules V in the category $\mathcal{C}(\lambda)$ $(\lambda\in\mathfrak{h}^*)$, or more generally in the category $\mathcal{O}$ (see [6, Chaps.2 and 9]). Further, under the action of Ω, a module $V\in\mathcal{C}(\lambda)$ decomposes into the direct sum

$$V=\sum\nolimits_{k\in\mathbf{C}}^{\oplus}V^{(k)}$$

of generalized eigenspaces $V^{(k)}$ for the eigenvalue $k\in\mathbf{C}$ of Ω. Note that on a highest weight $\mathfrak{g}(A)$-module V with highest weight $\lambda\in\mathfrak{h}^*$, Ω acts as the scalar operator $(|\lambda+\rho|^2-|\rho|^2)I_V$, where $|\mu|^2$ denotes $(\mu|\mu)$ for $\mu\in\mathfrak{h}^*$.

Definition 4.1. For $\lambda,\mu\in\mathfrak{h}^*$, define the functor T_μ^λ from the category $\mathcal{C}(\mu)$ to the category $\mathcal{C}(\lambda)$ by

$$T_\mu^\lambda(V):=(V\otimes_{\mathbf{C}}L(\lambda-\mu))^{(|\lambda+\rho|^2-|\rho|^2)}\qquad(V\in\mathcal{C}(\mu)),$$

which we call the *translation functor* from μ to λ.

Remark. By [4, Proposition 4.6], we see that the functor T_μ^λ is an exact functor.

Now, following the general line of [12], we can prove a series of propositions below.

Proposition 4.2. *Let $\Lambda\in P_+,(w,\beta)\in W\times\mathcal{A}$. Then, we have*

$$T_0^\Lambda(V((w,\beta)\circ 0))\cong\begin{cases}V((w,\beta)\circ\Lambda) & \text{if}\quad \beta\in\mathcal{A}(\Lambda),\\ 0 & \text{if}\quad \beta\notin\mathcal{A}(\Lambda).\end{cases}$$

Proposition 4.3. *Let $\Lambda\in P_+,(w,\beta)\in W\times\mathcal{A}$. Then, we have*

$$T_0^\Lambda(L((w,\beta)\circ 0))\cong L((w,\beta)\circ\Lambda)\quad\text{or}\quad 0.$$

Remark. $T_0^\Lambda(L((w,\beta)\circ 0))=0$ unless $\beta\in\mathcal{A}(\Lambda)$.

PROPOSITION 4.4. *Let $\Lambda \in P_+, \mu \in \mathfrak{h}^*, (w,\beta) \in W \times \mathcal{A}(\Lambda)$. If*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_t = V((w,\beta) \circ 0)$$

is a local composition series of $V((w,\beta) \circ 0)$ at $\mu - \Lambda$, then

$$0 = T_0^\Lambda(V_0) \subset T_0^\Lambda(V_1) \subset \cdots \subset T_0^\Lambda(V_t) \cong V((w,\beta) \circ \Lambda)$$

is a local composition series of $V((w,\beta) \circ \Lambda)$ at μ.

Using Propositions 4.3 and 4.4, we can show the following, which is one of our main results.

THEOREM 4.5. *Let $\Lambda \in P_+$. Then, for any $(w,\beta), (w',\beta') \in W \times \mathcal{A}(\Lambda)$, we have*

$$[V((w,\beta) \circ \Lambda) : L((w',\beta') \circ \Lambda)] = [V((w,\beta) \circ 0) : L((w',\beta') \circ 0)].$$

As a corollary of the proof of Theorem 4.5, we obtain

COROLLARY 4.6. *Let $\Lambda \in P_+, (w,\beta) \in W \times \mathcal{A}(\Lambda)$. Then, we have*

$$T_0^\Lambda(L((w,\beta) \circ 0)) \cong L((w,\beta) \circ \Lambda).$$

5. GENERALIZATION OF THE KAZHDAN-LUSZTIG CONJECTURE

5.1. Here, let $A_J = (a_{ij})_{i,j \in J}$ be a symmetrizable GCM (= generalized Cartan matrix) indexed by a finite set J, and let $\mathfrak{g}_J := \mathfrak{g}(A_J)$ be a Kac-Moody algebra over $\mathbb{C}$ associated to A_J with the Cartan subalgebra $\mathfrak{h}_J$, simple roots $\Pi_J = \{\alpha_i\}_{i \in J}$ ($\subset \mathfrak{h}_J^*$), simple coroots $\Pi_J^\vee = \{\alpha_i^\vee\}_{i \in J}$ ($\subset \mathfrak{h}_J$), and the Weyl group W_J ($\subset GL(\mathfrak{h}_J^*)$). In addition, let $P_{w,w'}(q)$ ($w, w' \in W_J$) be the Kazhdan-Lusztig polynomials for the Coxeter group W_J (see §2.2).

For $\lambda \in \mathfrak{h}_J^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_J, \mathbb{C})$, we denote by $V_J(\lambda)$ the Verma module with highest weight λ over $\mathfrak{g}_J$, and by $L_J(\lambda)$ its unique irreducible quotient. For $\lambda, \mu \in \mathfrak{h}_J^*$, we denote by $[V_J(\lambda) : L_J(\mu)]$ the multiplicity of $L_J(\mu)$ in $V_J(\lambda)$ (see §3).

First, we recall the following celebrated result due to Kashiwara [7, 8], or Casian [2].

Theorem 5.1 ([2], [7, 8]). *Let $\mathfrak{g}_J = \mathfrak{g}(A_J)$ be a Kac-Moody algebra associated to a symmetrizable* GCM *A_J. Assume that Λ is an element of $\mathfrak{h}_J^*$ such that $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in J$. Then, for any $w, w' \in W_J$, we have*

$$[V_J(w(\Lambda+\rho_J)-\rho_J) : L_J(w'(\Lambda+\rho_J)-\rho_J)] = P_{w,w'}(1).$$

Here, ρ_J is a fixed element of $\mathfrak{h}_J^*$ such that $\langle \rho_J, \alpha_i^\vee \rangle = 1$ for all $i \in J$.

5.2. We now return to the setting of §1–§4. Note that we assume that the GGCM A is symmetrizable. In [11], we have essentially proved the following theorem.

Theorem 5.2 (cf. [11, Proposition 2.9]). *Let $\Lambda \in P_+, (w,\beta) \in W \times \mathcal{A}(\Lambda)$. Then, any irreducible subquotient of $V((w,\beta)\circ\Lambda)$ is isomorphic to $L((w',\beta')\circ\Lambda)$ for some $(w',\beta') \in W \times \mathcal{A}(\Lambda)$ with $(w',\beta') \geqslant (w,\beta)$. Moreover, the converse statement also holds.*

Therefore, the multiplicities $[V((w,\beta)\circ\Lambda) : L((w',\beta')\circ\Lambda)]$ $((w,\beta),(w',\beta') \in W \times \mathcal{A}(\Lambda))$ are of great interest. Here, we shall derive some partial information about the above multiplicities from Theorem 5.1, which is for the case of Kac-Moody algebras.

Remark that the submatrix $A_{I^{re}} := (a_{ij})_{i,j\in I^{re}}$ of a symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$ is a symmetrizable GCM. Let $\mathfrak{g}_{I^{re}}$ be the Lie subalgebra of $\mathfrak{g}(A)$ generated by $\mathfrak{h}_{I^{re}} \cup \{e_i, f_i \mid i \in I^{re}\}$, where $\mathfrak{h}_{I^{re}}$ is a certain good subspace of $\mathfrak{h}$, such that the triple $(\mathfrak{h}_{I^{re}}, \{\alpha_i|_{\mathfrak{h}_{I^{re}}}\}_{i\in I^{re}}, \{\alpha_i^\vee\}_{i\in I^{re}})$ is a realization of the GCM $A_{I^{re}}$. (Here, $\alpha_i|_{\mathfrak{h}_{I^{re}}}$ denotes the restriction of α_i to $\mathfrak{h}_{I^{re}}$.) Then, $\mathfrak{g}_{I^{re}}$ is canonically isomorphic to a Kac-Moody algebra $\mathfrak{g}(A_{I^{re}})$ over $\mathbb{C}$ associated to the GCM $A_{I^{re}}$ with the Cartan subalgebra $\mathfrak{h}_{I^{re}}$. In fact, we have

$$\mathfrak{g}_{I^{re}} = \mathfrak{h}_{I^{re}} \bigoplus \sum_{\alpha\in\Delta_{I^{re}}}^{\oplus} \mathfrak{g}_\alpha,$$

where $\Delta_{I^{re}} := \Delta \cap (\sum_{i\in I^{re}} \mathbb{Z}_{\geq 0}\alpha_i)$, or rather its restriction to $\mathfrak{h}_{I^{re}}$, can be regarded as the root system of $(\mathfrak{g}_{I^{re}}, \mathfrak{h}_{I^{re}})$. From now on, we canonically identify the subalgebra $\mathfrak{g}_{I^{re}}$ of $\mathfrak{g}(A)$ with $\mathfrak{g}(A_{I^{re}})$.

Then, we have the following by exactly the same argument as the one for [15, Theorem 2.3], abusing the notation of §5.1 under the above identification.

PROPOSITION 5.3. *Let $\lambda, \mu \in \mathfrak{h}^*$. Assume that $\lambda - \mu \in \sum_{i \in I^{re}} \mathbb{Z}\alpha_i$. Then, we have*

$$[V(\lambda) : L(\mu)] = [V_{I^{re}}(\lambda|_{\mathfrak{h}_{I^{re}}}) : L_{I^{re}}(\mu|_{\mathfrak{h}_{I^{re}}})].$$

Here, for $\tau \in \mathfrak{h}^*$, $V_{I^{re}}(\tau|_{\mathfrak{h}_{I^{re}}})$ is the Verma module over the Kac-Moody algebra $\mathfrak{g}_{I^{re}}$ ($\cong \mathfrak{g}(A_{I^{re}})$), whose highest weight $\tau|_{\mathfrak{h}_{I^{re}}} \in \mathfrak{h}^*_{I^{re}}$ is the restriction of τ to $\mathfrak{h}_{I^{re}}$, and $L_{I^{re}}(\tau|_{\mathfrak{h}_{I^{re}}})$ is its unique irreducible quotient.

As a direct consequence of Theorem 5.1 and Proposition 5.3, using Theorem 3.3, we obtain the following theorem.

THEOREM 5.4. *Let $\Lambda \in P_+$, $(w, \beta), (w', \beta') \in W \times \mathcal{A}(\Lambda)$. Then, we have*

$$[V((w, \beta) \circ \Lambda) : L((w', \beta') \circ \Lambda)] \geq P_{(w,\beta),(w',\beta')}(1),$$

where $P_{(w,\beta),(w',\beta')}(q)$ is the extended Kazhdan-Lusztig polynomial (introduced in §2.2). Moreover, the equality holds if $\beta = \beta'$, or if $w = w' = 1$.

Now, recall that the Weyl group W of the GKM algebra $\mathfrak{g}(A)$ is by definition the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections r_i ($i \in I^{re}$). However, W by itself seems to be too small for the description of the representation theory of $\mathfrak{g}(A)$. Actually, from Theorems 3.3 and 5.2, we have an impression that the direct product $W \times \mathcal{A}$ of W and $\mathcal{A}$ behaves as if it were the true "Weyl group" of the GKM algebra $\mathfrak{g}(A)$.

On the other hand, in the case where $a_{ii} \neq 0$ ($i \in I$), the set $\mathcal{A} = \mathcal{A}(0)$ consists of all sums of distinct, pairwise perpendicular, imaginary simple roots. So, in this case, $\mathcal{A} = \mathcal{A}(0)$ can be embedded into the Coxeter group $(\mathbb{Z}/2\mathbb{Z})^m$ with m the cardinality of the set I^{im}, via the identification of an imaginary simple root $\alpha_j \in \Pi^{im}$ with a generator $\bar{1} \in \mathbb{Z}/2\mathbb{Z}$. Hence, $W \times \mathcal{A}$ can be embedded into the direct product $W \times (\mathbb{Z}/2\mathbb{Z})^m$ of Coxeter groups, together with the Bruhat ordering (see Definitions 2.1–2.3).

Under this embedding, the Kazhdan-Lusztig polynomial associated to the elements $(w, \beta), (w', \beta') \in W \times \mathcal{A}$ should be just the extended Kazhdan-Lusztig polynomial $P_{(w,\beta),(w',\beta')}(q)$ defined in §2.2. (Here, note that the Kazhdan-Lusztig polynomial $P_{\bar{0},\bar{1}}(q)$ ($\bar{0}, \bar{1} \in \mathbb{Z}/2\mathbb{Z}$) for the Coxeter group $\mathbb{Z}/2\mathbb{Z}$ with generator $\bar{1}$ is identically equal to 1.) Therefore, it seems natural to us to suggest the following conjecture.

Conjecture. *Assume that the GGCM $A = (a_{ij})_{i,j\in I}$ satisfies the condition that $a_{ii} \neq 0$ $(i \in I)$. Let $\Lambda \in P_+$, $(w,\beta), (w',\beta') \in W\times\mathcal{A}(\Lambda)$. Then, we have*

$$[V((w,\beta)\circ\Lambda) : L((w',\beta')\circ\Lambda)] = P_{(w,\beta),(w',\beta')}(1).$$

5.3. Since this paper was prepared, we have succeeded in proving that the above conjecture is true. We now sketch briefly the idea of the proof. From now on, we assume that the GGCM $A = (a_{ij})_{i,j\in I}$ is symmetrizable, and satisfies the condition that $a_{ii} \neq 0$ $(i \in I)$. Note that in this case the set $\mathcal{A}(\Lambda)$ consists of all sums of distinct, pairwise perpendicular, imaginary simple roots perpendicular to $\Lambda \in P_+$.

Then, we can prove the following generalization to GKM algebras of Jantzen's character sum formula corresponding to a quotient of two Verma modules (cf. [5] and [14]).

Theorem 5.5. *Let $\mathfrak{g}(A)$ be a GKM algebra associated to a symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$ satisfying the condition that $a_{ii} \neq 0$ $(i \in I)$. Fix $\Lambda \in P_+$. Let $\alpha = w(\alpha_j) \in \Delta_+$, where $w \in W$ and $\alpha_j \in \Pi^{im}$ with $\langle \Lambda, \alpha_j^\vee\rangle = 0$. We set $\lambda := w(\Lambda+\rho)-\rho = (w,0)\circ\Lambda$, $\mu := \lambda - \alpha = w(\Lambda+\rho-\alpha_j)-\rho = (w,\alpha_j)\circ\Lambda$, and $N(\lambda) := V(\lambda)/V(\mu)$ (see Theorem 3.3). Then, $N(\lambda)$ has a $\mathfrak{g}(A)$-module filtration*

$$N(\lambda) = N(\lambda)_0 \supset N(\lambda)_1 \supset N(\lambda)_2 \supset \cdots$$

such that:

(1) $N(\lambda)/N(\lambda)_1 \cong L(\lambda)$ *as a $\mathfrak{g}(A)$-module;*

(2) $\sum_{i\geq 1} \mathrm{ch}\, N(\lambda)_i$

$$= \sum_{\beta\in\Delta_+} \sum_{\substack{j\geq 1 \\ 2(\lambda+\rho|\beta)=j(\beta|\beta)}} \mathrm{ch}\, V(\lambda - j\beta) - \sum_{\gamma\in\Delta_+} \sum_{\substack{m\geq 1 \\ 2(\lambda-\alpha+\rho|\gamma)=m(\gamma|\gamma)}} \mathrm{ch}\, V(\lambda-\alpha-m\gamma)$$

$$- \mathrm{ch}\, V(\lambda-\alpha).$$

Here, ch *denotes the formal character.*

By double induction on $\ell(w') - \ell(w)$ and $\mathrm{ht}(\beta') - \mathrm{ht}(\beta)$, using Theorem 5.4 as the starting point of the induction and Theorem 5.5 for the induction step, we can prove that the above conjecture holds under the condition on the GGCM $A = (a_{ij})_{i,j\in I}$ that $a_{ii} \neq 0$ $(i \in I)$.

As a consequence, we obtain the following theorem.

THEOREM 5.6. *Let $\mathfrak{g}(A)$ be a GKM algebra. Assume that the symmetrizable GGCM $A = (a_{ij})_{i,j\in I}$ satisfies the condition that $a_{ii} \neq 0$ $(i \in I)$. Let $\Lambda \in P_+$. Then, for $(w,\beta) \in W\times\mathcal{A}(\Lambda)$, we have*

$$\mathrm{ch}\, V((w,\beta)\circ\Lambda) = \sum_{(w',\beta')\in W\times\mathcal{A}(\Lambda)} P_{(w,\beta),(w',\beta')}(1)\, \mathrm{ch}\, L((w',\beta')\circ\Lambda).$$

Equivalently, for $(w,\beta) \in W\times\mathcal{A}(\Lambda)$, we have

$$\begin{aligned}&\mathrm{ch}\, L((w,\beta)\circ\Lambda)\\ =& \sum_{(w',\beta')\in W\times\mathcal{A}(\Lambda)} (-1)^{(\ell(w')+\mathrm{ht}(\beta'))-(\ell(w)+\mathrm{ht}(\beta))} Q_{(w,\beta),(w',\beta')}(1)\, \mathrm{ch}\, V((w',\beta')\circ\Lambda),\end{aligned}$$

where $Q_{(w,\beta),(w',\beta')}(q)$ $((w',\beta') \in W\times\mathcal{A}(\Lambda))$ are the extended Kazhdan-Lusztig polynomials.

Remark. It is well-known that $\mathrm{ch}\, V(\lambda) = e(\lambda)\cdot \prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{-\mathrm{mult}(\alpha)}$ $(\lambda \in \mathfrak{h}^*)$, where $e(\tau)$ is a formal exponential for $\tau \in \mathfrak{h}^*$ (see [6, Chap.10]). Moreover, we know that $Q_{1,w'}(1) = 1$ $(w' \in W)$. Therefore, in view of the Weyl-Kac-Borcherds character formula for $L(\Lambda)$ $(\Lambda \in P_+)$ (see [1], or [6, Chap.11]), the condition on the GGCM $A = (a_{ij})_{i,j\in I}$ that $a_{ii} \neq 0$ $(i \in I)$ of Theorem 5.6 seems to be necessary.

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