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Kyoto University
A Central Extension of a Formal Loop Group

Takashi Hashimoto\(^1\) and Ryuichi Sawae\(^2\)

1. Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Hiroshima, 724, Japan
   e-mail: r3104@math.sci.hiroshima-u.ac.jp
2. Department of Management and Information Science, Shikoku University, Tokushima, 771-11, Japan
   e-mail: sawae@keiei.shikoku-u.ac.jp

0. Introduction

In this article, we prove that there is an elegant relation between the conformal factor and a group 2-cocycle on the formal loop group with values in \( SU(1, N+1) \), and show that the trivial central extension of the Hauser group acts transitively on the space of formal solutions of the Einstein-Maxwell field equations with \( N \) abelian gauge fields. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra \([K]\). This relation was first found by [BM].

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein-Maxwell field equations with \( N \) abelian gauge potentials.

Let \( ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \) be a metric on \( \mathbb{R}^{1+3} \) and \( A = A_\mu dx^\mu \) an abelian gauge potential with values in \( \mathbb{R}^N \). Then the Einstein-Maxwell field equations with \( N \) abelian gauge fields are given by

\[
R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\kappa F^{\mu\kappa} = 0 \quad (\mu, \nu = 0, 1, 2, 3),
\]

where \( R_{\mu\nu} \) is the Ricci curvature and

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
T_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\kappa} tF_{\nu}^\kappa - \frac{1}{4} g_{\mu\nu} F_{\kappa\iota} tF^{\kappa\iota}).
\]

We adopt the coordinates \((x^0, x^1, x^2, x^3) = (x^0, \phi, z, \rho)\) with \( x^0 \) being time and \((\phi, z, \rho)\) the cylindrical coordinates of \( \mathbb{R}^3 \). Stationary axisymmetric space-times amount to the assumption that a metric is of the form

\[
g = \begin{pmatrix}
h_{00} & h_{01} & \lambda & 0 \\
0 & h_{10} & 0 & 0 \\
h_{10} & 0 & h_{11} & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
\]
\[ \det h = -\rho^2, \]

where \( \lambda > 0 \), \( h_{01} = h_{10} \) and \( h = (h_{ij}) \). The field \( \lambda \) is called the conformal factor.

For abelian gauge potentials, we fix the gauge so as to \( A_2 = A_3 = 0 \). Since we assume that the fields are stationary and axisymmetric, the functions \( h_{ij} \)'s, \( \lambda \) and \( A_i \)'s depend only on \( z \) and \( \rho \). Further, we fix the gauge as follows:

\[
 h|_{(z,\rho)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A|_{(z,\rho)=(0,0)} = 0. \quad (0.1)
\]

Introducing the Ernst potentials \( u \in \mathbb{R}, v \in \mathbb{C}^N \) constructed from \( h \) and \( A \) by the standard method (cf. [DO][E]), we obtain

**Proposition 0.1.** The stationary axisymmetric Einstein-Maxwell field equations with \( N \) abelian gauge fields are equivalent to the following equations:

\[
 f(d^*du + \rho^{-1}d\rho \wedge *du) = (du - 2v^*dv) \wedge *du \quad (0.2)
\]
\[
 f(d^*dv + \rho^{-1}d\rho \wedge *dv) = (du - 2v^*dv) \wedge *dv \quad (0.3)
\]

\[
 \frac{\partial_z \lambda}{\lambda} = -\frac{\partial_z f}{2f} + \frac{\rho}{2f^2}(\partial_z f \partial_\rho f)
 - \frac{\rho}{2f^2}(\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)(\partial_z u - \partial_z f - 2v^* \partial_z v)
 + \frac{\rho}{f}(\partial_z v^* \partial_\rho v + \partial_z v^* \partial_\rho v) \quad (0.4)
\]
\[
 \frac{\partial_\rho \lambda}{\lambda} = -\frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2}\{((\partial_\rho f)^2 - (\partial_z f)^2)
 + \frac{\rho}{4f^2}\{((\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2 - (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2
 - \frac{\rho}{f}(\partial_z v^* \partial_\rho v - \partial_\rho v^* \partial_z v), \quad (0.5)
\]

where \( v^* = \bar{v}, |v|^2 = v^*v, f = \text{Re} \ u - |v|^2 \) and \( * \) is the Hodge operator given by \( *dz = d\rho, *d\rho = -dz \).

The first two equations are called the Ernst equations.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the conditions

\[
 u|_{(z,\rho)=(0,0)} = 1 \quad \text{and} \quad v|_{(z,\rho)=(0,0)} = 0. \quad (0.6)
\]

It is essential to introduce the function \( \tau = f^{1/2} \lambda \) and we shall consider \( \tau \), in stead of \( \lambda \), throughout this article.
1. Ernst Equation

Let \( \theta \) be Cartan involution of \( GL(N + 2, \mathbb{C}) \) defined by \( g \mapsto g^{*-1} \) and \( G \) a subgroup of \( GL(N + 2, \mathbb{C}) \) defined by

\[
\{ g \in GL(N + 2, \mathbb{C}) ; g^* J g = J, \det g = 1 \},
\]

where \( J = \begin{pmatrix} \mathbb{I}_N & i \\ -i & \mathbb{I}_N \end{pmatrix} \) and \( \mathbb{I}_N \) denotes the \( N \times N \) identity matrix. Note that \( G \) is isomorphic to \( SU(1, N+1) \). Let \( K \) be the subgroup of \( G \) such that each element of \( K \) is fixed by \( \theta \).

We fix subgroups \( A \) and \( N \) of \( G \) as follows:

\[
A = \left\{ \begin{pmatrix} a & 0 \\ 1_N & 1/a \end{pmatrix} ; a > 0 \right\}
\]

\[
N = \left\{ \begin{pmatrix} 1 & v \\ z + i|v|^2/2 & iv^* \end{pmatrix} ; x \in \mathbb{R}, v \in \mathbb{C}^N \right\},
\]

where \(|v|^2 = v^* v\). Then we have \( G = KAN \) (Iwasawa decomposition).

Let \( R \) be a ring of formal power series in \( z \) and \( \rho \) over \( \mathbb{C} \) i.e. \( R = \mathbb{C}[[z, \rho]] \). We extend the complex conjugation \(*\) of \( \mathbb{C} \) to a conjugation of \( R \) by defining \( \bar{z} = z, \bar{\rho} = \rho \). Let \( G_R \) be a subgroup of \( GL(N + 2, R) \) defined by

\[
\{ g \in GL(N + 2, R) ; g^* J g = J, \det g = 1 \}.
\]

Then, corresponding to \( G = KAN \), \( G_R \) decomposes as \( G_R = K_R A_R N_R \), where \( K_R \), \( A_R \) and \( N_R \) denote subgroups of \( G_R \) consisting of matrices with values in \( K \), \( A \) and \( N \) respectively, each of whose components is an element of \( R \).

Now we parametrize an element of \( A_R N_R \) as follows:

\[
P = \begin{pmatrix}
  f^{1/2} & 0 & 0 \\
  \sqrt{2}v & 1_N & 0 \\
  (\psi + i|v|^2)/f^{1/2} & \sqrt{2}iv^*/f^{1/2} & f^{-1/2}
\end{pmatrix}, \tag{1.1}
\]

where \( f \) and \( v \) are the same ones as in (0.2) and (0.3), and \( \psi = \text{Im} u \).

The following fact is well known.

**Proposition 1.1.** Under the parametrization of (1.1), we put \( M = P^* P \). Then the Ernst equations (0.2) and (0.3) are equivalent to the following equation:

\[
d(\rho \ast dM M^{-1}) = 0. \tag{1.2}
\]
Moreover the function \( \tau \) is a solution of (0.4) and (0.5) if and only if it is a solution of the following equations:

\[
\tau^{-1} \partial_{z} \tau = \frac{\rho}{4} \text{tr}(\partial_{z}MM^{-1} \partial_{\rho}MM^{-1}) \tag{1.3}
\]

\[
\tau^{-1} \partial_{\rho} \tau = \frac{\rho}{8} \text{tr}((\partial_{\rho}MM^{-1})^2 - (\partial_{z}MM^{-1})^2). \tag{1.4}
\]

The integrability of \( \tau \) follows easily from (1.3) and (1.4). Equation (1.2) is also called the Ernst equation. We shall consider the solutions satisfying

\[ P|_{(z, \rho)=(0,0)} = 1, \]

which corresponds to the gauge fixing condition (0.6).

It is also known that the equation (1.2) can be rewritten as the integrability condition of a 1-form with values in \( g \) each of whose component is an element of \( C(z, \rho) \otimes C[[t]] \), where \( C(z, \rho) \) is the quotient field of \( R = C[[z, \rho]] \) and \( t \) an indeterminate called "spectral parameter". Namely, let \( \mathcal{A} \) and \( \mathcal{I} \) be 1-forms defined by

\[ \mathcal{A} = \frac{1}{2} (dPP^{-1} - (dPP)^*) \quad \mathcal{I} = \frac{1}{2} (dPP^{-1} + (dPP)^*) \]

for any \( P \in A_{R}N_{R} \), and put

\[ \Omega_{P} = \mathcal{A} + \left( \frac{1 - t^{2}}{1 + t^{2}} - \frac{2t}{1 + t^{2}} \right) \mathcal{I}, \]

where \( \ast \) is the Hodge operator given by \( \ast dz = d\rho \), \( \ast d\rho = -dz \). We extend the canonical exterior derivative \( d \) on \( C(z, \rho) \) to that on \( C(z, \rho) \otimes C[[t]] \) by defining

\[ dt = \frac{t}{(1 + t^{2})\rho} ((1 - t^{2})d\rho + 2tdz). \tag{1.5} \]

Note then that \( d^2 t = 0 \). Now we have

**Proposition 1.2.** \( \Omega_{P} \) satisfies the integrability condition, i.e.,

\[ d\Omega_{P} - \Omega_{P} \wedge \Omega_{P} = 0 \tag{1.6} \]

if and only if \( P \) is a solution of (1.2).

It follows from Proposition 1.2 that if \( P \) is a solution of the Ernst equation, then there exists a potential \( p = \sum_{n \geq 0} p_{n}t^{n} \) such that each entry of \( p_{n} \) is an element of \( C(z, \rho) \) and

\[ dp = \Omega_{P} \cdot p \quad \text{and} \quad p_{0} = P. \tag{1.7} \]
2. Hauser Group

We introduce formal loop algebras and formal loop groups, following [T].

Put $F_0 = R = \mathbb{C}[z, \rho]$ and $F_n = \rho^n | R$ for a nonzero integer $n$. We introduce a topology in $R$ by declaring that $\{F_n\}_{n \geq 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m, n \geq 0$.

Then we define a formal loop algebra $\mathcal{F} \mathfrak{g} l$ by

$$\mathcal{F} \mathfrak{g} l = \left\{ X = \sum_{n \in \mathbb{Z}} X_n t^n ; \; X_n \in \mathfrak{g} l(N+2, F_n) \right\} . \quad (2.1)$$

Let $^*$ be an anti-involution of $\mathcal{F} \mathfrak{g} l$ defined by

$$X^* = \sum_{n \in \mathbb{Z}} X_n^*(-1/t)^n$$

for $X = \sum_{n \in \mathbb{Z}} X_n t^n$. This is well-defined by the definition of our filtration $\{F_n\}_{n \in \mathbb{Z}}$.

We define a formal loop group $\mathcal{F} \mathcal{G} 0$, following [T], by

$$\mathcal{F} \mathcal{G} 0 = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathcal{F} \mathfrak{g} l; \; g^* J g = J, \; \det g = 1, \; g_0|_{(z, \rho)=(0,0)} = 1 \right\} \quad (2.2)$$

and its subgroups by

$$\mathcal{F} \mathcal{K} = \left\{ k = \sum_{n \in \mathbb{Z}} k_n t^n \in \mathcal{F} \mathcal{G} 0; \; \theta^{(\infty)} k = k \right\} \quad (2.3)$$

$$\mathcal{F} \mathcal{P} = \left\{ p = \sum_{n \in \mathbb{Z}} p_n t^n \in \mathcal{F} \mathcal{G} 0; \; p_0 \in A_R N_R, \; p_n = 0 \text{ if } n < 0 \right\} . \quad (2.4)$$

Since $\mathcal{F} \mathcal{G} 0$ is canonically embedded in $\mathcal{F} \mathfrak{g} l$, we can define an involution $\theta^{(\infty)}$ of $\mathcal{F} \mathcal{G} 0$ by

$$\theta^{(\infty)}(g) = (g^*)^{-1} \quad \text{for } g \in \mathcal{F} \mathcal{G} 0,$$

which we call Cartan involution of $\mathcal{F} \mathcal{G} L$.

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathcal{F} \mathcal{G}$ as

$$g = kp \quad (k \in \mathcal{F} \mathcal{K}, \; p \in \mathcal{F} \mathcal{P}) . \quad (2.5)$$

Let $s$ be another indeterminate. Define an infinite dimensional group $G^{(\infty)}$, which we call Hauser group, by

$$G^{(\infty)} = \left\{ g = \sum_{n \geq 0} g_n s^n \in GL(N+2, \mathbb{C}[[s]]); \; g^* J g = J, \; \det g = 1, g_0 = 1 \right\} ,$$

but

$$\theta^{(\infty)} = \theta^{(\infty)}(g^*)^{-1}$$

for $g \in \mathcal{F} \mathcal{G} 0$, which we call Cartan involution of $\mathcal{F} \mathcal{G} L$.

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$$G^{(\infty)} = \left\{ g = \sum_{n \geq 0} g_n s^n \in GL(N+2, \mathbb{C}[[s]]); \; g^* J g = J, \; \det g = 1, g_0 = 1 \right\} ,$$
where $\mathbb{C}[[s]]$ is a ring of formal power series in $s$ over $\mathbb{C}$ and $g^* = \sum g_n s^n$.

Let $j$ be a homomorphism of $GL(N+2, \mathbb{C}[[s]])$ into $\mathcal{F}GL$ given by

$$j : g = \sum_{n \geq 0} g_n s^n \mapsto j(g) = \sum_{n \geq 0} g_n \left( \rho \left( \frac{1}{t} - t \right) + 2z \right)^n.$$ 

Then it is easy to see that $j$ is injective and that the image of $G^{(\infty)}$ by $j$ is in $\mathcal{F}G_0$. We denote by $\mathcal{F}H$ the image of $G^{(\infty)}$ by $j$. The following equations characterize the elements of $\mathcal{F}H$ in $\mathcal{F}G$.

**Lemma 2.1.** An element $g \in \mathcal{F}G$ belongs to $\mathcal{F}H$ if and only if $g$ satisfies the following equations:

$$\partial_t g = -\rho \left( \partial_z + \frac{1}{t} \partial_{\rho} \right) g \quad (2.6)$$
$$\partial_t g = -\frac{\rho}{2} \left( 1 + \frac{1}{t^2} \right) \partial_z g. \quad (2.7)$$

This characterization will play an important role in the proof of our main theorem.

**Definition.** Let $\mathcal{F}P$ be as in (2.4). We define $\mathcal{S}P$ to be a subset of $\mathcal{F}P$ consisting of elements $p = \sum_{n \geq 0} p_n t^n$ which satisfy the following conditions:

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z,\rho)=(0,0)} = 1. \quad (2.8)$$

We call $\mathcal{S}P$ the space of potentials.

It follows from (2.8) that $p_0$ is a solution of the Ernst equation (1.2) for $p = \sum_{n \geq 0} p_n t^n \in \mathcal{S}P$.

**Theorem 2.2.** Let $p \in \mathcal{F}P$. Then $p \in \mathcal{S}P$ if and only if $p^* p \in \mathcal{F}H$.

Let $p \in \mathcal{S}P$ and $g \in G^{(\infty)}$. By (2.5) there exist $k \in \mathcal{F}K$ and $p_g \in \mathcal{F}P$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g. \quad (2.9)$$

Then, it follows immediately from Theorem 2.2 that $p_g$ is in $\mathcal{S}P$. Thus we can define an action of the Hauser group $G^{(\infty)}$ on $\mathcal{S}P$ to the right by

$$\mathcal{S}P \times G^{(\infty)} \longrightarrow \mathcal{S}P \quad (p, g) \longmapsto p_g, \quad (2.10)$$

where $p_g$ is given by (2.9).

From the fact that an element $g = \sum_{n \geq 0} g_n s^n \in G^{(\infty)}$ such that $g^* = g$ and such that $g_0$ is positive definite decomposes as $g = h^* h$ for some $h \in G^{(\infty)}$, we have
Corollary 2.3. The action of $G^{(\infty)}$ on $\mathcal{S}\mathcal{P}$ given by (2.10) is transitive.

Remark. As we mentioned in [S], our group $G^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (1.2) through the action (2.10).

3. 2-Cocycle on $\mathcal{F}G_0$

The formal loop algebra $\mathcal{F}gl$ becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map 
\[
\exp : \mathcal{F}gl \to \mathcal{F}GL
\]
given by
\[
\exp X = e^X = \sum_{n \geq 0} \frac{X^n}{n!}
\]
is called the formal exponential map. Note that for any $g \in \mathcal{F}G_0$ we can find a unique element $X$ in $\mathcal{F}gl$ such that $g = e^X$, since the logarithm given by
\[
\log (1 + A) = \sum_{n > 1} \frac{(-1)^{n-1}}{n} A^n
\]
is well-defined and satisfies
\[
e^{\log(1+A)} = 1 + A
\]
for $A = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathcal{F}gl$ with $a_0 \in gl(N + 2, m)$, where $m$ is the maximal ideal of $R$.

For $X, Y$ in $\mathcal{F}gl$, let $c_n(X, Y)$ $(n = 1, 2, \cdots)$ be the elements in $\mathcal{F}gl$ which are determined by
\[
\exp vX \exp vY = \exp \sum_{n \geq 0} c_n(X, Y)v^n,
\]
where $v$ is an indeterminate. Furthermore $c_n$'s are uniquely determined by the following recursion formulas (see [V]):
\[
c_1(X, Y) = X + Y
\]
\[(n + 1)c_{n+1}(X, Y) = \frac{1}{2}[X - Y, c_n(X, Y)] + \sum_{p \geq 1, 2p \leq n} \sum_{k_1, \ldots, k_{2p} > 0} \left[ c_{k_1}(X, Y), \ldots, [c_{k_{2p}}(X, Y), X + Y] \ldots \right] (n \geq 1),
\]
where $K_{2p}$'s are determined by
\[
\frac{x}{1 - e^{-x}} - \frac{1}{2} x = 1 + \sum_{p \geq 1} K_{2p} x^{2p}.
\]
We set $C(X, Y) = \sum_{n \geq 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathcal{F}gl$ for $X, Y$ such that $X_0, Y_0 \in gl(N + 2, m)$. 
Lemma 3.1. For $n \geq 2$, there exists a $\mathfrak{gl}$-valued function $L_n(\cdot, \cdot)$ which satisfies
\begin{equation}
 c_n(X, Y) = [X, L_n(X, Y)] + [Y, L_n(-Y, -X)].
\end{equation}
for $X, Y \in \mathfrak{gl}$.

Note that $L_n$'s are not uniquely determined, however, we fix $L_n$'s so that there holds
\begin{equation}
 L(X, v Y) = \left( \frac{e^{-adX} - 1 + adX}{adX(1 - e^{-adX})} \right) v Y + O(v^2),
\end{equation}
where we put $L(X, Y) = \sum_{n \geq 2} L_n(X, Y)$ for $X, Y \in \mathfrak{gl}$ such that $X_0, Y_0 \in g(N+2, m)$. Thus, we obtain
\begin{equation}
 C(X, Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)].
\end{equation}

For a series $f = \sum_{n \in \mathbb{Z}} f_n t^n \in R[[t, t^{-1}]]$, we write
\begin{equation}
 \text{Res}_t f = f_{-1} \in R.
\end{equation}

Let $R_0 = R[[z, \rho]] \subset R$, the formal power series in $z$ and $\rho$ over $R$. We define a $R_0$-valued 2-cocycle $\omega$ on $\mathfrak{gl}$ by
\begin{equation}
 \omega(X, Y) = \text{Res}_t \text{Re} \text{tr} X \partial_t Y
\end{equation}
for $X, Y \in \mathfrak{gl}$. Note that
\begin{equation}
 \omega(X^*, Y^*) = -\omega(X, Y)
\end{equation}
for $X, Y \in \mathfrak{gl}$.

Now we introduce a group 2-cocycle on $\mathcal{F} \mathcal{G}_0$, following [BM]. Note that, from (3.3), any element $g \in \mathcal{F} \mathcal{G}_0$ can be uniquely written as $g = e^X$ for $X \in \mathfrak{gl}$ with $X_0 \in \mathfrak{g}(N+2, m)$.

Definition. Let $\Xi$ be a $R_0$-valued function on $\mathcal{F} \mathcal{G}_0 \times \mathcal{F} \mathcal{G}_0$ defined by
\begin{equation}
 \Xi(e^X, e^Y) = \omega(X, L(X, Y)) + \omega(Y, L(-Y, -X)).
\end{equation}
Then $\Xi$ defines a 2-cocycle on $\mathcal{F} \mathcal{G}_0$, i.e. satisfies the cocycle condition:
\begin{equation}
 \Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z)
\end{equation}
for $X, Y, Z \in \mathfrak{gl}$. 
4. Central Extension

For any \( p \in \mathcal{SP} \), we can find an element \( g \in \mathcal{FH} \) which sends the identity element \( 1 \in \mathcal{SP} \) to \( p \) by Corollary 2.2. Then we have \( p = kg \) for some \( k \in \mathcal{FK} \).

**Proposition 4.1.** For \( p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP} \), let \( g \in \mathcal{FH} \) and \( k \in \mathcal{FK} \) be such that \( p = kg \). Let \( \tau \) be a solution of (1.3) and (1.4) corresponding to \( P = p_0 \). Then we have the following relations:

\[
\tau^{-1} \partial_z \tau = \partial_z \Xi(kg, g^{-1}) \\
\tau^{-1} \partial_p \tau = \partial_p \Xi(kg, g^{-1}).
\]

(4.1) (4.2)

Now we define a central extension of \( \mathcal{FG}_0 \) in terms of the cocycle \( \Xi \).

**Definition.** Let \( (\mathcal{FG}_0)^{\sim} \) be the set given by

\[
(\mathcal{FG}_0)^{\sim} = \{(g, e^\mu) ; g \in \mathcal{FG}_0, \mu \in R_0 \}.
\]

Define a product of any two elements of \( (\mathcal{FG}_0)^{\sim} \) by

\[
(g_1, e^{\mu_1}) \cdot (g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1 + \mu_2 + \Xi(g_1, g_2)})
\]

(4.3)

for \((g_1, e^{\mu_1}), (g_2, e^{\mu_2}) \in (\mathcal{FG}_0)^{\sim} \). Since \( \Xi \) satisfies the cocycle condition (3.7), \( (\mathcal{FG}_0)^{\sim} \) forms a group with group multiplication given by (4.3). Namely, \( (\mathcal{FG}_0)^{\sim} \) is a central extension of \( \mathcal{FG}_0 \).

Let \( \tilde{\theta}^{(\infty)} \) be an involution of \( (\mathcal{FG}_0)^{\sim} \) given by

\[
\tilde{\theta}^{(\infty)}(g, e^\mu) = (\theta^{(\infty)}(g), e^{-\mu})
\]

If we denote by \( (\mathcal{FK})^{\sim} \) the subgroup of \( (\mathcal{FG}_0)^{\sim} \) consisting of elements which are fixed by \( \tilde{\theta}^{(\infty)} \), then we have

\[
(\mathcal{FK})^{\sim} = \{(k, 1) \in (\mathcal{FG}_0)^{\sim} ; k \in \mathcal{FK} \}.
\]

Let \( (\mathcal{FP})^{\sim} \) be a subgroup of \( (\mathcal{FG}_0)^{\sim} \) given by

\[
(\mathcal{FP})^{\sim} = \{(p, e^\mu) \in (\mathcal{FG}_0)^{\sim} ; p \in \mathcal{FP}, \mu \in R_0 \}.
\]

It follows immediately from the decomposition (2.5) of \( \mathcal{FG} \) that \( (\mathcal{FG}_0)^{\sim} \) has a unique decomposition:

\[
(\mathcal{FG}_0)^{\sim} = (\mathcal{FK})^{\sim} \cdot (\mathcal{FP})^{\sim}.
\]

(4.4)

Furthermore, we put

\[
(\mathcal{FH})^{\sim} = \{(g, e^\gamma) \in (\mathcal{FG}_0)^{\sim} ; g \in \mathcal{FH}, \gamma \in R \}.
\]
It follows from Lemma 3.2, [HS2] that $\mathcal{F}\mathcal{H}$ can be regarded as a subgroup of $(\mathcal{F}\mathcal{H})^\sim$ by

$$\mathcal{F}\mathcal{H} \longrightarrow (\mathcal{F}\mathcal{H})^\sim, \quad g \longmapsto (g, 1).$$

Let $(SP)^\sim$ be the subset of $(\mathcal{F}\mathcal{P})^\sim$ given by

$$(SP)^\sim = \left\{ (p, e^\tau) \in (\mathcal{F}\mathcal{P})^\sim; \quad p = \sum_{n \geq 0} p_n t^n \in SP, \quad \tau = e^{-\mu} \right\} \tau = e^{-\mu} \text{ satisfies (1.3) and (1.4) with } P = p_0 \right\}.$$  (4.5)

We call $(SP)^\sim$ the space of potentials with conformal factor.

**Proposition 4.2.** For $p \in SP$, let $k \in F_K$ and $g \in F_H$ be as above, i.e. $p = kg$. Then we have

$$\Xi(p^*, p) = 2\Xi(kg, g^{-1}).$$  (4.6)

Therefore, any element of $(SP)^\sim$ can be written as $(p, e^{-\frac{1}{2}\Xi(p^*, p) + \gamma})$ for $p \in SP, \gamma \in \mathbb{R}$.

Define an action of $(\mathcal{F}\mathcal{H})^\sim$ on the space of potentials with conformal factor $(SP)^\sim$ to the right through the decomposition (4.4):

$$(SP)^\sim \times (\mathcal{F}\mathcal{H})^\sim \longrightarrow (SP)^\sim, \quad ((p, e^\mu), (g, e^\gamma)) \longmapsto (p_g, e^\alpha).$$  (4.7)

Namely, we can find a unique element $(k, 1) \in (\mathcal{F}\mathcal{K})^\sim$ and $(p_g, e^\alpha) \in (\mathcal{F}\mathcal{P})^\sim$ such that

$$(p, e^\mu)(g, e^\gamma) = (k, 1)^{-1}(p_g, e^\alpha),$$

where $k$ and $p_g$ are the elements given in (2.9). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^\mu)(g, e^\gamma))^{-1} \cdot (p, e^\mu)(g, e^\gamma) = (g^* p^* p_g, e^{2(\mu + \gamma + \Xi(p^*, p))})$$

and

$$\tilde{\theta}^{(\infty)}(p_g, e^\alpha)^{-1} \cdot (p_g, e^\alpha) = (p_g^* p_g, e^{2\alpha + \Xi(p_g^*, p_g)}),$$

we obtain

$$\alpha = \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p_g^*, p_g))$$

$$= \gamma' - \frac{1}{2}\Xi(p_g^*, p_g)$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus $(p_g, e^\alpha)$ belongs to $(SP)^\sim$, i.e. the action (4.7) of $(\mathcal{F}\mathcal{H})^\sim$ is well-defined.

Now we state our main theorem:

**Theorem 4.3.** The group $(\mathcal{F}\mathcal{H})^\sim$ acts transitively on the space of potentials with conformal factor $(SP)^\sim$ by (4.7).
REFERENCES


