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Kyoto University
A Central Extension of a Formal Loop Group

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0. Introduction

In this article, we prove that there is an elegant relation between the conformal factor and a group 2-cocycle on the formal loop group with values in $SU(1, N+1)$, and show that the trivial central extension of the Hauser group acts transitively on the space of formal solutions of the Einstein-Maxwell field equations with $N$ abelian gauge fields. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra [K]. This relation was first found by [BM].

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein-Maxwell field equations with $N$ abelian gauge potentials.

Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a metric on $\mathbb{R}^{1+3}$ and $A = A_\mu dx^\mu$ an abelian gauge potential with values in $\mathbb{R}^N$. Then the Einstein-Maxwell field equations with $N$ abelian gauge fields are given by

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\kappa F^{\mu\kappa} = 0 \quad (\mu, \nu = 0, 1, 2, 3),$$

where $R_{\mu\nu}$ is the Ricci curvature and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad T_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\kappa} t F_{\nu}^{\kappa} - \frac{1}{4} g_{\mu\nu} F_{\kappa\iota} t F^{\kappa\iota}).$$

We adopt the coordinates $(x^0, x^1, x^2, x^3) = (x^0, \phi, z, \rho)$ with $x^0$ being time and $(\phi, z, \rho)$ the cylindrical coordinates of $\mathbb{R}^3$. Stationary axisymmetric space-times amount to the assumption that a metric is of the form

$$g = \begin{pmatrix} h_{00} & h_{01} & \lambda & 0 \\ h_{10} & h_{11} & 0 & -\lambda \\ 0 & -\lambda & 0 & 0 \end{pmatrix}.$$
det \( h = -\rho^2 \),

where \( \lambda > 0 \), \( h_{01} = h_{10} \) and \( h = (h_{ij}) \). The field \( \lambda \) is called the conformal factor.

For abelian gauge potentials, we fix the gauge so as to \( A_2 = A_3 = 0 \). Since we assume that the fields are stationary and axisymmetric, the functions \( h_{ij} \)'s, \( \lambda \) and \( A_i \)'s depend only on \( z \) and \( \rho \). Further, we fix the gauge as follows:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad A_{(z,\rho)=(0,0)} = 0. \tag{0.1}
\]

Introducing the Ernst potentials \( u \in \mathbb{R}, v \in \mathbb{C}^N \) constructed from \( h \) and \( A \) by the standard method (cf. [DO][E]), we obtain

**Proposition 0.1.** The stationary axisymmetric Einstein-Maxwell field equations with \( N \) abelian gauge fields are equivalent to the following equations:

\[
\begin{align*}
&f(d \ast du + \rho^{-1}d\rho \wedge *du) = (du - 2v^*dv) \wedge *du & \tag{0.2} \\
&f(d \ast dv + \rho^{-1}d\rho \wedge *dv) = (du - 2v^*dv) \wedge *dv & \tag{0.3}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial_z \lambda}{\lambda} & = -\frac{\partial_z f}{2f} + \frac{\rho}{2f^2} (\partial_z f \partial_\rho f) \\
& \quad - \frac{\rho}{2f^2} (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)(\partial_z u - \partial_z f - 2v^* \partial_z v) \\
& \quad + \frac{\rho}{f} (\partial_z v^* \partial_\rho v + \partial_z v^* \partial_\rho v) \tag{0.4}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial_\rho \lambda}{\lambda} & = -\frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2} \{ (\partial_\rho f)^2 - (\partial_z f)^2 \} \\
& \quad + \frac{\rho}{4f^2} \{ (\partial_z u - \partial_z f - 2v^* \partial_z v)^2 - (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2 \} \\
& \quad - \frac{\rho}{f} (\partial_z v^* \partial_z v - \partial_\rho v^* \partial_\rho v), \tag{0.5}
\end{align*}
\]

where \( v^* = \bar{v}, |v|^2 = v^*v, f = \text{Re} u - |v|^2 \) and \( * \) is the Hodge operator given by \( *dz = d\rho, *d\rho = -dz \).

The first two equations are called the Ernst equations.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the conditions

\[
u|_{(z,\rho)=(0,0)} = 1 \quad \text{and} \quad v|_{(z,\rho)=(0,0)} = 0. \tag{0.6}
\]

It is essential to introduce the function \( \tau = f^{1/2} \lambda \) and we shall consider \( \tau \), in stead of \( \lambda \), throughout this article.
1. Ernst Equation

Let $\theta$ be Cartan involution of $GL(N + 2, \mathbb{C})$ defined by $g \mapsto g^{* -1}$ and $G$ a subgroup of $GL(N + 2, \mathbb{C})$ defined by

$$\{g \in GL(N + 2, \mathbb{C}); g^* J g = J, \det g = 1\},$$

where $J = \begin{pmatrix} 1_N & i \\ i & -1 \end{pmatrix}$ and $1_N$ denotes the $N \times N$ identity matrix. Note that $G$ is isomorphic to $SU(1, N + 1)$. Let $K$ be the subgroup of $G$ such that each element of $K$ is fixed by $\theta$.

We fix subgroups $A$ and $N$ of $G$ as follows:

$$A = \left\{ \begin{pmatrix} a \\ 1_N \\ 1/a \end{pmatrix}; a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 \\ v \\ z + i|v|^2/2 \\ iv^* \\ 1 \end{pmatrix}; z \in \mathbb{R}, v \in \mathbb{C}^N \right\},$$

where $|v|^2 = v^* v$. Then we have $G = KAN$ (Iwasawa decomposition).

Let $R$ be a ring of formal power series in $z$ and $\rho$ over $\mathbb{C}$ i.e. $R = \mathbb{C}[[z, \rho]]$. We extend the complex conjugation $^*$ of $\mathbb{C}$ to a conjugation of $R$ by defining $\bar{z} = z, \bar{\rho} = \rho$. Let $G_R$ be a subgroup of $GL(N + 2, R)$ defined by

$$\{g \in GL(N + 2, R); g^* J g = J, \det g = 1\}.$$

Then, corresponding to $G = KAN$, $G_R$ decomposes as $G_R = K_R A_R N_R$, where $K_R, A_R$ and $N_R$ denote subgroups of $G_R$ consisting of matrices with values in $K, A$ and $N$ respectively, each of whose components is an element of $R$.

Now we parametrize an element of $A_R N_R$ as follows:

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1_N & 0 \\ (\psi + i|v|^2)/f^{1/2} & \sqrt{2}iv^*/f^{1/2} & f^{-1/2} \end{pmatrix},$$

(1.1)

where $f$ and $v$ are the same ones as in (0.2) and (0.3), and $\psi = \text{Im} u$.

The following fact is well known.

**Proposition 1.1.** Under the parametrization of (1.1), we put $M = P^* P$. Then the Ernst equations (0.2) and (0.3) are equivalent to the following equation:

$$d(\rho \ast dM M^{-1}) = 0.$$  

(1.2)
Moreover the function $\tau$ is a solution of (0.4) and (0.5) if and only if it is a solution of the following equations:

\[
\tau^{-1} \partial_z \tau = \frac{\rho}{4} \text{tr}((\partial_z MM^{-1})^2 - (\partial_z MM^{-1})^2) \tag{1.3}
\]

\[
\tau^{-1} \partial_\rho \tau = \frac{\rho}{8} \text{tr}((\partial_\rho MM^{-1})^2 - (\partial_z MM^{-1})^2) \tag{1.4}
\]

The integrability of $\tau$ follows easily from (1.3) and (1.4). Equation (1.2) is also called the Ernst equation. We shall consider the solutions satisfying

\[P|_{(z, \rho)=(0,0)} = 1,\]

which corresponds to the gauge fixing condition (0.6).

It is also known that the equation (1.2) can be rewritten as the integrability condition of a 1-form with values in $\mathfrak{g}$ each of whose component is an element of $C(z, \rho) \otimes C[[t]]$, where $C(z, \rho)$ is the quotient field of $R = C[[z, \rho]]$ and $t$ an indeterminate called "spectral parameter". Namely, let $A$ and $I$ be 1-forms defined by

\[A = \frac{1}{2}(dPP^{-1} - (dPP)^*) \quad I = \frac{1}{2}(dPP^{-1} + (dPP)^*) \]

for any $P \in A_R N_R$, and put

\[\Omega_P = A + \left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} *\right) I,
\]

where $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$. We extend the canonical exterior derivative $d$ on $C(z, \rho)$ to that on $C(z, \rho) \otimes C[[t]]$ by defining

\[dt = \frac{t}{(1+t^2)\rho} ((1-t^2)d\rho + 2tdz). \tag{1.5}\]

Note then that $d^2 t = 0$. Now we have

**Proposition 1.2.** $\Omega_P$ satisfies the integrability condition, i.e.,

\[d\Omega_P - \Omega_P \wedge \Omega_P = 0 \tag{1.6}\]

if and only if $P$ is a solution of (1.2).

It follows from Proposition 1.2 that if $P$ is a solution of the Ernst equation, then there exists a potential $p = \sum_{n \geq 0} p_n t^n$ such that each entry of $p_n$ is an element of $C(z, \rho)$ and

\[dp = \Omega_P \cdot p \quad \text{and} \quad p_0 = P. \tag{1.7}\]
2. Hauser Group

We introduce formal loop algebras and formal loop groups, following [T].

Put $F_0 = R = C[[z, \rho]]$ and $F_n = \rho^{|n|}R$ for a nonzero integer $n$. We introduce a topology in $R$ by declaring that $\{F_n\}_{n \geq 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m, n \geq 0$.

Then we define a formal loop algebra $\mathcal{F}gl$ by

$$\mathcal{F}gl = \left\{ X = \sum_{n \in \mathbb{Z}} X_n t^n ; X_n \in gl(N + 2, F_n) \right\}. \quad (2.1)$$

Let $^*$ be an anti-involution of $\mathcal{F}gl$ defined by

$$X^* = \sum_{n \in \mathbb{Z}} X_n^*(-1/t)^n$$

for $X = \sum_{n \in \mathbb{Z}} X_n t^n$. This is well-defined by the definition of our filtration $\{F_n\}_{n \in \mathbb{Z}}$.

We define a formal loop group $\mathcal{F}G_0$, following [T], by

$$\mathcal{F}G_0 = \left\{ g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathcal{F}gl; g^* J g = J, \det g = 1, g_0|_{(z, \rho) = (0, 0)} = 1 \right\} \quad (2.2)$$

and its subgroups by

$$\mathcal{F}K = \left\{ k = \sum_{n \in \mathbb{Z}} k_n t^n \in \mathcal{F}G_0; \theta^{(\infty)} k = k \right\} \quad (2.3)$$

$$\mathcal{F}P = \left\{ p = \sum_{n \in \mathbb{Z}} p_n t^n \in \mathcal{F}G_0; p_0 \in A_R N_R, p_n = 0 \text{ if } n < 0 \right\}. \quad (2.4)$$

Since $\mathcal{F}G_0$ is canonically embedded in $\mathcal{F}gl$, we can define an involution $\theta^{(\infty)}$ of $\mathcal{F}G_0$ by

$$\theta^{(\infty)}(g) = (g^*)^{-1} \quad \text{for } g \in \mathcal{F}G_0,$$

which we call Cartan involution of $\mathcal{F}GL$.

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathcal{F}G$ as

$$g = kp \quad (k \in \mathcal{F}K, p \in \mathcal{F}P). \quad (2.5)$$

Let $s$ be another indeterminate. Define an infinite dimensional group $G^{(\infty)}$, which we call Hauser group, by

$$G^{(\infty)} = \left\{ g = \sum_{n \geq 0} g_n s^n \in GL(N + 2, C[[s]]); g^* J g = J, \det g = 1, g_0 = 1 \right\},$$
where $\mathbb{C}[[s]]$ is a ring of formal power series in $s$ over $\mathbb{C}$ and $g^* = \sum g_n s^n$.

Let $j$ be a homomorphism of $GL(N+2, \mathbb{C}[[s]])$ into $\mathcal{F}GL$ given by

$$j : g = \sum_{n \geq 0} g_n s^n \mapsto j(g) = \sum_{n \geq 0} g_n \left( \frac{1}{t} - t + 2z \right)^n.$$  

Then it is easy to see that $j$ is injective and that the image of $G^{(\infty)}$ by $j$ is in $\mathcal{F}G_0$. We denote by $\mathcal{F}H$ the image of $G^{(\infty)}$ by $j$. The following equations characterize the elements of $\mathcal{F}H$ in $\mathcal{F}G$.

**Lemma 2.1.** An element $g \in \mathcal{F}G$ belongs to $\mathcal{F}H$ if and only if $g$ satisfies the following equations:

$$\partial_t g = -\rho \left( \partial_z + \frac{1}{t} \partial_\rho \right) g \quad (2.6)$$

$$\partial_t g = -\frac{\rho}{2} \left( 1 + \frac{1}{t^2} \right) \partial_z g. \quad (2.7)$$

This characterization will play an important role in the proof of our main theorem.

**Definition.** Let $\mathcal{FP}$ be as in (2.4). We define $\mathcal{SP}$ to be a subset of $\mathcal{FP}$ consisting of elements $p = \sum_{n \geq 0} p_n t^n$ which satisfy the following conditions:

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z,\rho)=(0,0)} = 1. \quad (2.8)$$

We call $\mathcal{SP}$ the space of potentials.

It follows from (2.8) that $p_0$ is a solution of the Ernst equation (1.2) for $p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP}$.

**Theorem 2.2.** Let $p \in \mathcal{FP}$. Then $p \in \mathcal{SP}$ if and only if $p^* p \in \mathcal{FH}$.

Let $p \in \mathcal{SP}$ and $g \in G^{(\infty)}$. By (2.5) there exist $k \in \mathcal{FK}$ and $p_g \in \mathcal{FP}$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g. \quad (2.9)$$

Then, it follows immediately from Theorem 2.2 that $p_g$ is in $\mathcal{SP}$. Thus we can define an action of the Hauser group $G^{(\infty)}$ on $\mathcal{SP}$ to the right by

$$\mathcal{SP} \times G^{(\infty)} \to \mathcal{SP} \quad (p, g) \mapsto p_g, \quad (2.10)$$

where $p_g$ is given by (2.9).

From the fact that an element $g = \sum_{n \geq 0} g_n s^n \in G^{(\infty)}$ such that $g^* = g$ and such that $g_0$ is positive definite decomposes as $g = h^* h$ for some $h \in G^{(\infty)}$, we have
Corollary 2.3. The action of $G^{(\infty)}$ on $SP$ given by (2.10) is transitive.

Remark. As we mentioned in [S], our group $G^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (1.2) through the action (2.10).

3. 2-Cocycle on $\mathcal{F}G_0$

The formal loop algebra $\mathcal{F}\mathfrak{g}l$ becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map

$$\exp : \mathcal{F}\mathfrak{g}l \rightarrow \mathcal{F}GL$$

given by

$$\exp X = e^X = \sum_{n \geq 0} \frac{X^n}{n!}$$

is called the formal exponential map. Note that for any $g \in \mathcal{F}G_0$ we can find a unique element $X$ in $\mathcal{F}\mathfrak{g}l$ such that $g = e^X$, since the logarithm given by

$$\log (1 + A) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} A^n$$

is well-defined and satisfies

$$e^{\log(1+A)} = 1 + A$$

for $A = \sum_{n \in \mathbb{Z}} a_n t^n \in \mathcal{F}\mathfrak{g}l$ with $a_0 \in \mathfrak{g}(N + 2, m)$, where $m$ is the maximal ideal of $R$.

For $X, Y$ in $\mathcal{F}\mathfrak{g}l$, let $c_n(X, Y)$ $(n = 1, 2, \cdots)$ be the elements in $\mathcal{F}\mathfrak{g}l$ which are determined by

$$\exp vX \exp vY = \exp \sum_{n \geq 0} c_n(X, Y) v^n,$$

where $v$ is an indeterminate. Furthermore $c_n$'s are uniquely determined by the following recursion formulas (see [V]):

$$c_1(X, Y) = X + Y$$

$$(n + 1)c_{n+1}(X, Y) = \frac{1}{2} [X - Y, c_n(X, Y)]$$

$$+ \sum_{p \geq 1, 2p \leq n} K_{2p} \sum_{k_1, \cdots, k_{2p} > 0} [c_{k_1}(X, Y), \cdots, [c_{k_{2p}}(X, Y), X + Y] \cdots] \quad (n \geq 1),$$

where $K_{2p}$'s are determined by

$$\frac{x}{1 - e^{-x}} - \frac{1}{2} x = 1 + \sum_{p \geq 1} K_{2p} x^{2p}.$$

We set $C(X, Y) = \sum_{n \geq 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathcal{F}\mathfrak{g}l$ for $X, Y$ such that $X_0, Y_0 \in \mathfrak{gl}(N + 2, m)$. 

Lemma 3.1. For $n \geq 2$, there exists a $\mathcal{F}\mathfrak{g}l$-valued function $L_n(\cdot, \cdot)$ which satisfies
\begin{equation}
 c_n(X, Y) = [X, L_n(X, Y)] + [Y, L_n(-Y, -X)].
\end{equation}
for $X, Y \in \mathcal{F}\mathfrak{g}l$.

Note that $L_n$'s are not uniquely determined, however, we fix $L_n$'s so that there holds
\begin{equation}
 L(X, vY) = \left( \frac{e^{-\text{ad}X} - 1 + \text{ad}X}{\text{ad}X(1 - e^{-\text{ad}X})} - \frac{1}{4} \right) vY + O(v^2),
\end{equation}
where we put $L(X, Y) = \sum_{n \geq 2} L_n(X, Y)$ for $X, Y \in \mathcal{F}\mathfrak{g}l$ such that $X_0, Y_0 \in \mathfrak{g}(N+2, m)$. Thus, we obtain
\begin{equation}
 C(X, Y) = X + Y + [X, L(X, Y)] + [Y, L(-Y, -X)].
\end{equation}

For a series $f = \sum_{n \in \mathbb{Z}} f_n t^n \in R[[t, t^{-1}]]$, we write
\begin{equation}
 \text{Res}_t f = f_{-1} \in R.
\end{equation}

Let $R_0 = R[[z, \rho]] \subset R$, the formal power series in $z$ and $\rho$ over $R$. We define a $R_0$-valued 2-cocycle $\omega$ on $\mathcal{F}\mathfrak{g}l$ by
\begin{equation}
 \omega(X, Y) = \text{Res}_t \text{Re tr} X \partial_t Y
\end{equation}
for $X, Y \in \mathcal{F}\mathfrak{g}l$. Note that
\begin{equation}
 \omega(X^*, Y^*) = -\omega(X, Y)
\end{equation}
for $X, Y \in \mathcal{F}\mathfrak{g}l$.

Now we introduce a group 2-cocycle on $\mathcal{F}\mathcal{G}_0$, following [BM]. Note that, from (3.3), any element $g \in \mathcal{F}\mathcal{G}_0$ can be uniquely written as $g = e^X$ for $X \in \mathcal{F}\mathfrak{g}l$ with $X_0 \in \mathfrak{g}(N+2, m)$.

**Definition.** Let $\Xi$ be a $R_0$-valued function on $\mathcal{F}\mathcal{G}_0 \times \mathcal{F}\mathcal{G}_0$ defined by
\begin{equation}
 \Xi(e^X, e^Y) = \omega(X, L(X, Y)) + \omega(Y, L(-Y, -X)).
\end{equation}
Then $\Xi$ defines a 2-cocycle on $\mathcal{F}\mathcal{G}_0$, i.e. satisfies the cocycle condition:
\begin{equation}
 \Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z)
\end{equation}
for $X, Y, Z \in \mathcal{F}\mathfrak{g}l$. 

4. Central Extension

For any $p \in \mathcal{SP}$, we can find an element $g \in \mathcal{FH}$ which sends the identity element $1 \in \mathcal{SP}$ to $p$ by Corollary 2.2. Then we have $p = kg$ for some $k \in \mathcal{FK}$.

**Proposition 4.1.** For $p = \sum_{n \geq 0} p_n t^n \in \mathcal{SP}$, let $g \in \mathcal{FH}$ and $k \in \mathcal{FK}$ be such that $p = kg$. Let $\tau$ be a solution of (1.3) and (1.4) corresponding to $P = p_0$. Then we have the following relations:

\[
\tau^{-1} \partial_z \tau = \partial_z \Xi(kg, g^{-1})
\]

(4.1)

\[
\tau^{-1} \partial_p \tau = \partial_p \Xi(kg, g^{-1}).
\]

(4.2)

Now we define a central extension of $\mathcal{FG}_0$ in terms of the cocycle $\Xi$.

**Definition.** Let $(\mathcal{FG}_0)^\sim$ be the set given by

\[
(\mathcal{FG}_0)^\sim = \{(g, e^\mu) ; g \in \mathcal{FG}_0, \mu \in R_0\}.
\]

Define a product of any two elements of $(\mathcal{FG}_0)^\sim$ by

\[
(g_1, e^{\mu_1}) \cdot (g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1 + \mu_2 + \Xi(g_1, g_2)})
\]

(4.3)

for $(g_1, e^{\mu_1}), (g_2, e^{\mu_2}) \in (\mathcal{FG}_0)^\sim$. Since $\Xi$ satisfies the cocycle condition (3.7), $(\mathcal{FG}_0)^\sim$ forms a group with group multiplication given by (4.3). Namely, $(\mathcal{FG}_0)^\sim$ is a central extension of $\mathcal{FG}_0$.

Let $\tilde{\theta}^{(\infty)}$ be an involution of $(\mathcal{FG}_0)^\sim$ given by

\[
\tilde{\theta}^{(\infty)}(g, e^\mu) = (\theta^{(\infty)}(g), e^{-\mu}).
\]

If we denote by $(\mathcal{FK})^\sim$ the subgroup of $(\mathcal{FG}_0)^\sim$ consisting of elements which are fixed by $\tilde{\theta}^{(\infty)}$, then we have

\[
(\mathcal{FK})^\sim = \{(k, 1) \in (\mathcal{FG}_0)^\sim ; k \in \mathcal{FK}\}.
\]

Let $(\mathcal{FP})^\sim$ be a subgroup of $(\mathcal{FG}_0)^\sim$ given by

\[
(\mathcal{FP})^\sim = \{(p, e^\mu) \in (\mathcal{FG}_0)^\sim ; p \in \mathcal{FP}, \mu \in R_0\}.
\]

It follows immediately from the decomposition (2.5) of $\mathcal{FG}$ that $(\mathcal{FG}_0)^\sim$ has a unique decomposition:

\[
(\mathcal{FG}_0)^\sim = (\mathcal{FK})^\sim \cdot (\mathcal{FP})^\sim.
\]

(4.4)

Furthermore, we put

\[
(\mathcal{FH})^\sim = \{(g, e^\gamma) \in (\mathcal{FG}_0)^\sim ; g \in \mathcal{FH}, \gamma \in R\}.
\]
It follows from Lemma 3.2, [HS2] that $\mathcal{F}\mathcal{H}$ can be regarded as a subgroup of $(\mathcal{F}\mathcal{H})\sim$ by

$$\mathcal{F}\mathcal{H} \longrightarrow (\mathcal{F}\mathcal{H})\sim, \quad g \longmapsto (g, 1).$$

Let $(\mathcal{S}\mathcal{P})\sim$ be the subset of $(\mathcal{F}\mathcal{P})\sim$ given by

$$(\mathcal{S}\mathcal{P})\sim = \left\{ (p, e^\mu) \in (\mathcal{F}\mathcal{P})\sim; \ p = \sum_{n \geq 0} p_n t^n \in \mathcal{S}\mathcal{P}, \ \tau = e^{-\mu} \text{ satisfies (1.3) and (1.4) with } P = p_0 \right\}. \quad (4.5)$$

We call $(\mathcal{S}\mathcal{P})\sim$ the space of potentials with conformal factor.

Proposition 4.2. For $p \in \mathcal{S}\mathcal{P}$, let $k \in \mathcal{F}\mathcal{K}$ and $g \in \mathcal{F}\mathcal{H}$ be as above, i.e. $p = kg$. Then we have

$$\Xi(p^*, p) = 2\Xi(kg, g^{-1}). \quad (4.6)$$

Therefore, any element of $(\mathcal{S}\mathcal{P})\sim$ can be written as $(p, e^{-\frac{1}{2}(\Xi(p^*, p) + \gamma)})$ for $p \in \mathcal{S}\mathcal{P}, \gamma \in \mathbb{R}$.

Define an action of $(\mathcal{F}\mathcal{H})\sim$ on the space of potentials with conformal factor $(\mathcal{S}\mathcal{P})\sim$ to the right through the decomposition (4.4):

$$(\mathcal{S}\mathcal{P})\sim \times (\mathcal{F}\mathcal{H})\sim \longrightarrow (\mathcal{S}\mathcal{P})\sim, \quad ((p, e^\mu), (g, e^\gamma)) \longmapsto (pg, e^\alpha). \quad (4.7)$$

Namely, we can find a unique element $(k, 1) \in (\mathcal{F}\mathcal{K})\sim$ and $(pg, e^\alpha) \in (\mathcal{F}\mathcal{P})\sim$ such that

$$(p, e^\mu)(g, e^\gamma) = (k, 1)^{-1}(pg, e^\alpha),$$

where $k$ and $pg$ are the elements given in (2.9). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^\mu)(g, e^\gamma))^{-1} \cdot (p, e^\mu)(g, e^\gamma) = (g^* p^* pg, e^{2(\mu + \gamma) + \Xi(p^*, p)})$$

and

$$\tilde{\theta}^{(\infty)}(pg, e^\alpha)^{-1} \cdot (pg, e^\alpha) = (pg^* pg, e^{2\alpha + \Xi(p^*, p)})$$

we obtain

$$\alpha = \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p^*_g, p_g))$$

$$= \gamma' - \frac{1}{2}\Xi(p^*_g, p_g)$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus $(pg, e^\alpha)$ belongs to $(\mathcal{S}\mathcal{P})\sim$, i.e. the action (4.7) of $(\mathcal{F}\mathcal{H})\sim$ is well-defined.

Now we state our main theorem:

**Theorem 4.3.** The group $(\mathcal{F}\mathcal{H})\sim$ acts transitively on the space of potentials with conformal factor $(\mathcal{S}\mathcal{P})\sim$ by (4.7).
REFERENCES


