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# Highest weight modules and $b$-functions of semi-invariants.

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**0.** In [**31**], S.Suga observed a relation between the irreducibility of certain highest weight modules and the $b$-functions of certain prehomogeneous vector spaces. The purpose of this note is to summarize [**13**] and [**14**], in which we have studied how the observation of Suga should be generalized.

**Convention.** The complex number field (resp. the rational integer ring) is denoted by $\mathbb{C}$ (resp. $\mathbb{Z}$).

## §1. $b$-Functions

Let $f_i(x_1, \cdots, x_n)$ $(1 \leq i \leq l)$ be analytic functions, $D$ the ring of analytic linear differential operators, $0 \leq k < l$, $s = (s_{k+1}, \cdots, s_l)$ independent variables. We know the following.

*For any* $\lambda_1, \cdots, \lambda_k, \mu_{k+1}, \cdots, \mu_l \in \mathbb{Z}_{\geq 0}$, *there exists* $Q(s) \in D[s] = D \otimes_{\mathbb{C}} \mathbb{C}[s]$ *and* $b(s) \in \mathbb{C}[s] \setminus \{0\}$ *such that*

*(1)* $Q(s)(f_1^{\lambda_1} \cdots f_k^{\lambda_k} f_{k+1}^{s_{k+1}+\mu_{k+1}} \cdots f_l^{s_l+\mu_l}) = b(s)(f_1^{\lambda_1} \cdots f_k^{\lambda_k} f_{k+1}^{s_{k+1}} \cdots f_l^{s_l})$,

*(2)* $b(s) = \prod_i (a_{i,k+1}s_{k+1} + \cdots + a_{i,l}s_l + \alpha_i)$ *with some* $(a_{i,k+1}, \cdots, a_{i,l}) \in (\mathbb{Z}_{\geq 0})^{l-k} \setminus \{0\}$ *and*

*(3) with some* $\alpha_i \in \mathbb{Q}_{>0}$.

This type of theorem is first obtained by M. Sato [**27**] for relative invariants of a prehomogeneous vector spaces. When $k = 0$ and $l = 1$, (1) is obtained by I.N.Bernsteim [**3**] for a polynomial $f$, and by J.E.Björk in general; $(2) + (3)$ is obtained by M.Kashiwara [**18**]. When

$k = 0$ and $l > 1$, (1) and (2) are obtained by C.Sabbah [**25**]. See [**11**] for (3). Because of the positivity, we get the general assertion from the case where $k = 0$ by a specialization $s_j \to \lambda_j$ for some $j$'s with $\mu_j = 0$.

We call a polynomial $b(s)$ appearing in (1) *a b-function.* The totality of such $b(s)$'s (not necessarily satisfying (2) or (3)) forms an ideal of $\mathbb{C}[s]$. The author does not know whether, in general, this is a principal ideal or not. (Cf. [**12**, 6.4].) If it is a principal ideal, we call its generator *the b-function.*

If we have *the* $b$-function, we get *a* $b$-function by a specialization as above, but the resulting polynomial is not necessarily *the* $b$-function. In fact, this difference is one of our main concern.

## §2. Generalized Verma modules.

Let $G$ be a simply connected complex simple Lie group, $P$ a parabolic subgroup, $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras, and $U(-)$ the enveloping algebra. A $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ with a finite dimensional irreducible $\mathfrak{p}$-module $E$ is called *a generalized Verma module.* In the special case where $\mathfrak{p}$ is a Borel subalgebra, such a $\mathfrak{g}$-module is called *a Verma module.*

The Verma modules are first introduced by D.N.Verma [**32**]. Since then a considerable progress has been made [**4**], [**5**], [**6**], [**17**], [**20**], [**7**], [**1**], $\cdots$, and our present knowledge is fairly satisfactory.

Concerning the generalized Verma modules, their significance was first recognized by J.Lepowsky. He showed in [**22**] that every irreducible Harish-Chandra module can be obtained as a subquotient of a (non-unitary) principal series representation, which can be constructed from a generalized Verma module. (Cf. [**9**, Chapter 9].) This result was largely improved by W.Casselman, and then by A.Beilinson and J.Bernstein [**2**]. Casselman showed that 'subquotient' may be replaced with 'submodule'. (Cf. [**29**, Introduction].)

Concerning the properties of the generalized Verma modules, we have [**24**] (generalization of [**4**] and [**6**]), [**16**] (irreducibility criterion) and [**17**, 2.25] (translation principle). At present, our knowledge about the homomorphisms between generalized Verma modules is very poor, although some results are obtained by J.Lepowsky, B.D.Boe, D.H.Collingwood, R.S.Irving, Hisayosi Matumoto, $\cdots$.

### §3. Observation of Suga.

Let $L$ be a Levi subgoup of $P$, and $\mathfrak{u}$ the nilpotent radical of $\mathfrak{p}$. Assume that $\mathfrak{u}$ is commutative. Then $\mathfrak{u}$ has an open $\mathrm{ad}(L)$-orbit, i.e., $(L, \mathrm{ad}, \mathfrak{u})$ is a prehomogeneous vector space. Assume further that there exists a relatively $\mathrm{ad}(L)$-invariant non-constant irreducible polynomial function $f_0$ on $\mathfrak{u}$. Then $\mathfrak{p}$ is a maximal parabolic subalgebra corresponding to one of the following diagrams.

$(A_{2p-1}, p)$

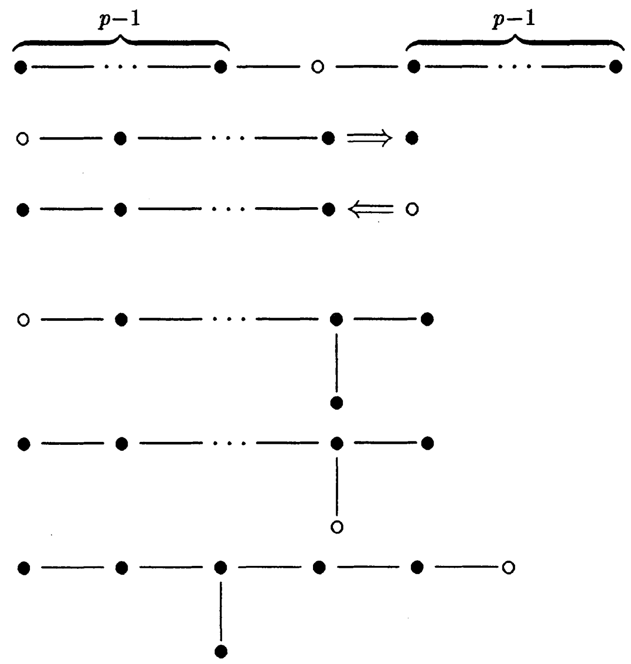
$(B_p, 1)$

$(C_p, p)$

$(D_p, 1)$

$(D_{2p}, 2p)$

$(E_7, 7)$




Let $b(s)$ be the $b$-function of $f$, i.e., the minimal polynomial such that $Q(s)f_0^{s+1} = b_0(s)f^s$ with some $Q(s) \in D[s]$, whose explicit form is given by

$(A_{2p-1}, p)$ $\quad b_0(s) = (s+1)(s+2)\cdots(s+p)$

$(B_p, 1)$ $\quad b_0(s) = (s+1)(s+\frac{2p-1}{2})$

$(C_p, p)$ $\quad b_0(s) = (s+1)(s+\frac{3}{2})(s+\frac{4}{2})\cdots(s+\frac{p+1}{2})$

$(D_p, 1)$ $\quad b_0(s) = (s+1)(s+\frac{2p-2}{2})$

$(D_{2p}, 2p)$ $\qquad b_0(s) = (s+1)(s+3)\cdots(s+2p-1)$

$(E_7, 7)$ $\qquad b_0(s) = (s+1)(s+5)(s+9)$

See [**21**], also [**23**] and [**15**]. (Cf. [**10**, 2.5].)

Let $\varpi$ be the fundamental weight corresponding to the white node of the above diagram. Then $\varpi$ can be extended to a Lie algebra character of $\mathfrak{p}$.

**Suga Observation.** *[**31**]. For $\lambda \in \mathbf{C}$, the following conditions are equivalent. (1) The $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\lambda\varpi} \mathbf{C}$ is irreducible. (2) $b_0(\lambda - j) \neq 0$ for any $j = 1, 2, \cdots$.*

This observation can be checked by determining the irreducibility using the criterion of Jantzen [**16**] and by comparing with the explicit form of $b_0(s)$.

§**4.** Roughly speaking

(4.1) $$(L,\ \text{adjoint action},\ \mathfrak{u}) \fallingdotseq (L,\ \text{left action},\ G/P).$$

At one hand, we have the $\mathcal{D}$-module $\mathcal{D}f_0^\lambda$, which is related to the left hand side. (Here $\mathcal{D}$ denotes the sheaf of differential operators.) We can show that $\mathcal{D}f_0^\lambda$ is simple if and only if $b(\lambda - j) \neq 0$ for any $j \in \mathbf{Z}$. On the other hand, we can expect that we get a $\mathcal{D}$-module, say $\mathcal{M}(\lambda)$, on $G/P$ by 'localizing' the generalized Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\lambda\varpi} \mathbf{C}$ as in [**1**]. Then $\mathcal{M}(\lambda)$, which is related to the right hand side of (4.1), would be simple if and only if $M(\lambda)$ is simple. Hence, by showing that $\mathcal{M}(\lambda) \fallingdotseq \mathcal{D}f_0^\lambda$, we would be able to explain the observation of Suga to some extent.

In [**13**], we have tried to realize this idea and get [**13**, 9.13]. In the case considered in §3, this result asserts the following:

**(A)** *Assume that $\lambda \in \mathbf{C}$ satisfies*

(4.2) $$\langle \lambda\varpi + \rho, \alpha^\vee \rangle \neq 0, -1, -2, \cdots \quad \textit{for any positive coroot } \alpha^\vee.$$

*Then the following conditions are equivalent.*

(4.3) $$U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\lambda\varpi} \mathbb{C} \text{ is irreducible.}$$

(4.4) $$b_0(\lambda - j) \neq 0 \text{ for any } j \in \mathbb{Z}.$$

*Here $\rho$ denotes the half of the sum of the positive roots.*

**Remark.** For instance, in the case $(A_{2p-1}, p)$,

$$(4.2) \Leftrightarrow \lambda \neq \cdots, -3, -2, -1,$$

$$(4.3) \Leftrightarrow \lambda \neq -p+1, -p+2, -p+3, \cdots,$$

$$(4.4) \Leftrightarrow \lambda \notin \mathbb{Z}.$$

In general, the totality of the parameter $\lambda$ for which $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\lambda\varpi} \mathbb{C}$ is reducible is a finite union of arithmetic series $\lambda_j + \mathbb{Z}_{\geq 0}$. But as in the case $(A_{2p-1}, p)$, first several terms of these arithmetic series are veiled by the assumption (4.2). By this reason, the above result is unsatisfactory. In our proof, we needed this undesirable assumption in order to use the generality concerning the localization of $\mathfrak{g}$-modules [**1**].

## §5. Semi-invariants.

Although [**13**, 9.13] is unsatisfactory, it gives us an insight. So let us give a slightly more detailed explanation.

Let $T$ be a maximal torus of $G$, $B = B_+$ a Borel subgroup, and $B_-$ the Borel subgroup such that $B_+ \cap B_- = T$. Since $G$ is assumed to be simply connected, any integral weight $\varpi$, i.e., a $\mathbb{Z}$-linear combination of the fundamental weights $\varpi_1, \cdots, \varpi_l$, can be integrated to a character of $T$, which we shall denote by the same letter $\varpi$. We also denote by the same letter the compositions of the projections $B_\pm \to T$ and $\varpi$.

Let $\varpi \in \sum_{i=1}^{l} \mathbb{Z}_{\geq 0} \varpi_i$. It is known that there exists a unique holomorphic function $f^\varpi$ on $G$ such that

$$f^\varpi(e) = 1 \text{ and } f^\varpi(b'gb) = \varpi(b') f^\varpi(g) \varpi(b).$$

for any $b' \in B_-$, $g \in G$, and $b \in B$. This function is called *semi-invariant* associated to $\varpi$.

Now let us explain [**13**, 9.13] in the case where $\mathfrak{p}$ is a maximal parabolic subalgebra whose nilpotent radical is not necessarily commutative.

**(B)** *Let $\varpi$ be the unique fundamental weight which can be extended to a Lie algebra character of $\mathfrak{p}$, and $b(s)$ the $b$-function of $f^{\varpi}$. Under the condition (4.2), the conditions (4.3) and (4.4) are equivalent.*

**Remark.** (1) Strictly speaking, we need to assume some other conditions. But the author conjectures that the remaining conditions are automatically satisfied, and has proved it in several cases.

(2) In the case considered in §4, we can show that $b_0 = b$ [**14**, proof of (4.2.1)]. Therefore (B) is a generalization of (A).

§**6.** What is important concerning (B) is that it gives us an insight into the significance of the $b$-functions of semi-invariants. In fact, we conjecture the following.

**Conjecture.** *Let $\mathfrak{p}$ be a maximal parabolic subalgebra, $\varpi$ the unique fundamental weight which can be extended to a Lie algebra character of $\mathfrak{p}$, and $b(s)$ the $b$-function of the semi-invariant $f^{\varpi}$. For $\lambda \in \mathbb{C}$, the following conditions are equivalent. (1) $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\lambda\varpi} \mathbb{C}$ is irreducible. (2) $b(\lambda - j) \neq 0$ for any $j = 1, 2, \cdots$.*

**Remark.** In this note, we restrict ourselves to the simplest cases. See [**14**, §3 and §9] for our conjectures in their full generalities, where $b(s)$ is replaced with multi-variable $b$-functions (cf. §1).

§**7.** In the individual cases, once we know the explicit form of the $b$-function, we can check the Conjecture using [**16**]. Let us explain what is known about the $b$-functions.

Except for the cases studied by Suga (cf. §3), our knowledge was almost nothing. In fact, usually the determination of the $b$-function is very difficult and in many cases seems actually impossible with bare hands.

In [**26**], an algorithm to calculate the $b$-function of a relative invariant of a prehomogeneous vector space is given, based on the microlocal analysis, which is an analysis on the cotangent bundle of the base space. If the base space is a vector space, say $V$, then its cotangent bundle $T^*V$ is nothing but the direct product of $V$ and its dual space $V^*$. Hence $T^*V$ can also be regarded as the cotangent bundle of $V^*$. The essential part of the calculation of the $b$-function according to [**26**] is to go backward and forward between these two ways of looking.

Thus, in modifying this algorithm to handle the $b$-functions of the semi-invariants, our disadvantage mainly comes from the fact that our base space $G$ is not a vector space. On the other hand, the following points are of our advantage. Recall that the semi-invariants are relative invariants with respect to the natural $B_- \times B$-action on $G$.

(1) The orbits of this action are the Bruhat cells, whose property is well understood. Especially there are only a finite number of orbits, and hence the orbit decomposition gives a Whitney stratification.

(2) The closures of each orbit is normal [**8**, Corollay 1 in p.85] (cf. [**14**, Remark following (5.11.2)]). Thus we can use the Zariski's main theorem.

(3) As is naturally expected, we need the geometry of the flag manifold $G/B$. Concerning the flag manifold, results of R.Steinberg [**30**] and N.Spaltenstein [**28**] are at our disposal.

Because of these advantages, we can give an algorithm to calculate the $b$-functions of the semi-invariants [**14**], based on the microlocal analysis as in the case of prehomogeneous vector spaces, and we have calculated some examples by this procedure. Conjecture in §6 is formulated partly based on this calculation. The author hopes to discuss the other basis of our Conjecture in a different place.

§**8.** As we have explained, the $b$-functions of semi-invariants should control the irreducibility. Moreover, it seems that there exists an intimate relation of these $b$-functions to intertwining operators and unitarizability.

Before concluding this note, let us remark that the microlocal analysis of semi-invariants is also interesting in its own sake. For example, in the case where $\mathfrak{g} = A_l$ and $\mathfrak{p}$ is a Borel subalgebra, the holonomy diagram and the $W$-graph of the regular representation constructed in [**20**] should be the same, if we assume a conjecture of Kazhdan-Lusztig.

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