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Holomorphic and Singular Solutions of Non Linear Singular Partial Differential Equations (Complex Analysis and Differential Equations)

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Holomorphic and Singular Solutions of Non Linear Singular Partial Differential Equations

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In this note, I will report some results on holomorphic and singular solutions of singular partial differential equations of the following three cases:

1. linear case;
2. non linear first order case;
3. non linear higher order case.

1 Linear case

First of all, let us survey my result in the case of linear Fuchsian case. Let \((t, x) = (t, x_1, \cdots, x_n) \in C_t \times C_x^n\) and let us consider

\[
(E_1) \quad (t \frac{\partial}{\partial t})^m u = \sum_{j+|\alpha| \leq m \atop j < m} a_{j,\alpha}(t, x)(t \frac{\partial}{\partial t})^j (\frac{\partial}{\partial x})^\alpha u + f(t, x),
\]

where \(m \in \mathbb{N}^* (= \{1, 2, \cdots\})\), \(\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n (= \{0, 1, 2, \cdots\}^n)\), \(|\alpha| = |\alpha_1| + \cdots + |\alpha_n|\) and

\[
(\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}.
\]
Assume the following conditions:

A1) \( a_{j,\alpha}(t, x) \) and \( f(t, x) \) are holomorphic near the origin;
A2) \( a_{j,\alpha}(0, x) \equiv 0 \) if \( |\alpha| > 0 \).

Then, \((E_1)\) is called a Fuchsian type equation with respect to \( t \). The indicial polynomial \( C(\rho, x) \) is defined by

\[
C(\rho, x) = \rho^m - \sum_{j < m} a_{j,0}(0, x) \rho^j
\]

and the characteristic exponents \( \rho_1(x), \ldots, \rho_m(x) \) are defined by the roots of \( C(\rho, x) = 0 \).

**Definition of \( \mathcal{O} \).** \( \mathcal{O} \) is the set of all functions \( u(t, x) \) satisfying the following: there are \( \varepsilon > 0 \) and \( r > 0 \) such that \( u(t, x) \) is holomorphic in \( \{(t, x) \in \mathcal{R}(C \setminus \{0\}) \times C^n ; \; 0 < |t| < \varepsilon \) and \( |x| \leq r \} \), where \( \mathcal{R}(C \setminus \{0\}) \) is the universal covering space of \( C \setminus \{0\} \).

**THEOREM 1** (Tahara [1]). Denote by \( \mathcal{S} \) the set of all \( \mathcal{O} \)-solutions of \((E_1)\). Then, if \( \rho_i(0) \notin \mathbb{N} \) (1 \( \leq i \leq m \)) and \( \rho_i(0) - \rho_j(0) \notin \mathbb{Z} \) (1 \( \leq i \neq j \leq m \)) hold, we have

\[
\mathcal{S} = \{U(\varphi_1, \cdots, \varphi_m) ; (\varphi_1, \cdots, \varphi_m) \in (C\{x\})^m\},
\]

where \( U(\varphi_1, \cdots, \varphi_m) \) is an \( \mathcal{O} \)-solution of \((E_1)\) depending on \((\varphi_1, \cdots, \varphi_m) \in (C\{x\})^m \) which can be taken arbitrarily and having an expansion of the following form:

\[
U(\varphi_1, \cdots, \varphi_m) = \sum_{i=0}^{\infty} u_i(x) t^i + \sum_{i=1}^{m} \sum_{j=0}^{\infty} \sum_{k=0}^{mj} \phi_{i,j,k}(x) t^{\rho_i(x)+j} (\log t)^{k}
\]

with \( \phi_{i,0,0}(x) = \varphi_i(x) \) (i = 1, \( \cdots, m \)).
2 Non linear first order case

Next, I will report a result for non linear first order equation of the following form:

\[(E_2) \quad t \frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}),\]

where \((t, x) \in C_t \times C_x^n\) and \(\frac{\partial u}{\partial x} = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})\).

Put \(v = (v_1, \ldots, v_n)\) and assume the following:

B1) \quad F(t, x, u, v)\) is holomorphic near the origin ;
B2) \quad F(0, x, 0, 0) \equiv 0 near x = 0 ;
B3) \quad \frac{\partial F}{\partial v_i}(0, x, 0, 0) \equiv 0 for i = 1, \ldots, n.

Then, \((E_2)\) is called an equation of Briot-Bouquet type with respect to \(t\) (in [3]). Put

\[\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0).\]

Definition of \(\bar{O}_+\). We denote by \(\bar{O}_+\) the set of all \(u(t, x)\) satisfying the following i) and ii):

i) There are \(r > 0\) and a positive-valued continuous function \(\varepsilon(s)\) on \(R_s\) such that \(u(t, x)\) is a holomorphic function on

\[\{(t, x) \in \mathcal{R} \subset \mathbb{C} \times \mathbb{C}^n ; 0 < |t| < \varepsilon(\arg t), |x| \leq r\};\]

ii) There is an \(a > 0\) such that for any \(\theta > 0\) we have

\[\max_{|x| \leq r} |u(t, x)| = O(|t|^a)\]

as \(t \to 0\) under the condition \(|\arg t| < \theta\).
THEOREM 2 (Gérard-Tahara [4]). Denote by $S_+$ the set of all $\mathcal{O}_+$-solutions of $(E_2)$. Then, if $\rho(0) \notin N^*$ holds, we have:

$$S_+ = \begin{cases} 
\{u_0\}, & \text{when } \operatorname{Re}\rho(0) \leq 0, \\
\{u_0\} \cup \{U(\varphi); \ 0 \neq \varphi(x) \in C\{x\}\}, & \text{when } \operatorname{Re}\rho(0) > 0,
\end{cases}$$

where $u_0$ is the unique holomorphic solution of $(E_2)$ and $U(\varphi)$ is an $\mathcal{O}_+$-solution of $(E_2)$ having an expansion of the following form:

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{i+j \geq k+2} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)$$

$\forall \varphi_{0,1,0}(x) = \varphi(x)$ which can be taken arbitrarily.

3 Non linear higher order case

Lastly, I will report a generalization of the result in section 2 to higher order case.

Let us consider

$$\begin{align*}
(E_3) \quad (t \frac{\partial}{\partial t})^m u &= F(t, x, \{(t \frac{\partial}{\partial t})^j (\frac{\partial}{\partial x})^{\alpha} u\}_{j+|\alpha| \leq m}), \\
\text{where } (t, x) &\in C_t \times C^n_x \text{ and } m \in N^*. \text{ Put } \\
z &= \{z_{j,\alpha}\}_{j+|\alpha| \leq m}^{j<m}
\end{align*}$$

and assume the following conditions:

$C_1)$ $F(t,x,z)$ is holomorphic near the origin ;

$C_2)$ $F(0,x,0) \equiv 0$ near $x = 0$;

$C_3)$ $\frac{\partial F}{\partial z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

Note the following: 1) if $m = 1$, $(E_3)$ is nothing but $(E_2)$; 2) if $(E_3)$ is linear, $(E_3)$ is nothing but $(E_1)$. Thus, $(E_3)$ includes both cases $(E_1)$ and $(E_2)$. 

Put
\[
C(\rho, x) = \rho^m - \sum_{j < m} \frac{\partial F}{\partial z_{j,0}}(0, x, 0)\rho^j
\]
and denote by \(\rho_1(x), \cdots, \rho_m(x)\) the roots of \(C(\rho, x) = 0\) in \(\rho\). Set
\[
\mu = \text{the cardinal of } \{i; \text{Re}\rho_i(0) > 0\}.
\]
If \(\mu = 0\), this implies that \(\text{Re}\rho_i(0) \leq 0\) for all \(i = 1, \cdots, m\). When \(\mu \geq 1\), by a renumberation we may assume
\[
\begin{cases}
\text{Re}\rho_i(0) > 0, & \text{for } 1 \leq i \leq \mu,
\text{Re}\rho_i(0) \leq 0, & \text{for } \mu + 1 \leq i \leq m.
\end{cases}
\]
Then we have:

**THEOREM 3** (Gérard-Tahara [5]). Denote by \(S_+\) the set of all \(\widetilde{O}_+\)-solutions of \((E_3)\). Then we have:

(I) If \(\mu = 0\), we have
\[
S_+ = \{u_0\},
\]
where \(u_0\) is the unique holomorphic solution of \((E_3)\).

(II) If \(\mu \geq 1\), under the additional conditions:
\[
\begin{align*}
1) & \quad \rho_i(0) \neq \rho_j(0) \text{ for } 1 \leq i \neq j \leq \mu, \\
2) & \quad C(1, 0) \neq 0, \\
3) & \quad C(i + j_{1}\rho_1(0) + \cdots + j_{\mu}\rho_{\mu}(0), 0) \neq 0 \text{ for any } (i, j) \in \mathbb{N} \times \mathbb{N}^\mu \text{ satisfying } i + |j| \geq 2,
\end{align*}
\]
we have
\[
S_+ = \{U(\varphi_1, \cdots, \varphi_{\mu}); (\varphi_1, \cdots, \varphi_{\mu}) \in (C\{x\})^{\mu}\},
\]
where \(U(\varphi_1, \cdots, \varphi_{\mu})\) is an \(\widetilde{O}_+\)-solution of \((E_3)\) depending on \((\varphi_1, \cdots, \varphi_{\mu}) \in (C\{x\})^{\mu}\) which can be taken arbitrarily and having an expansion of the following form:
\[
U(\varphi_1, \cdots, \varphi_{\mu}) = \sum_{i \geq 1} u_i(x)t^i
\]
\[
+ \sum_{i+2m|j| \geq k+2m} \phi_{i,j,k}(x) t^{i+j_1 \rho_1(x)+\cdots+j_\mu \rho_\mu(x)} (\log t)^k
\]

with \( \phi_{0,e_p,0}(x) = \varphi_p(x) \) (\( p = 1, \cdots, \mu \)) where \( e_1 = (1,0,\cdots,0), \cdots, e_\mu = (0,\cdots,0,1) \in N^\mu \).

参考文献


