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The behavior of solutions with singularities of linear partial differential equations in $C^{n+1}$

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Let $P(z, \partial)$ be a linear partial differential operator with order $m \geq 1$. Its coefficients are holomorphic in a neighbourhood of the origin $z = 0$ in $C^{n+1}$. $K$ is a nonsingular complex hypersurface through $z = 0$, which is characteristic for $P(z, \partial)$. We choose the coordinate so that $K = \{z_{0} = 0\}$. In the present talk we consider

$$P(z, \partial)u(z) = f(z),$$

where $u(z)$ and $f(z)$ are holomorphic except on $K$. In order to state the results we give notations and definitions: $z = (z_{0}, z_{1}, \cdots, z_{n}) = (z, z')$, $\delta_{i} = \partial/\partial_{i}$, $\partial = (\delta_{0}, \delta_{1}, \cdots, \delta_{n}) = (\delta_{0}, \delta')$ and $|z| = \max\{|z_{i}|; 0 \leq i \leq n\}$. We write $P(z, \partial)$ in the form

$$(1) \hspace{1cm} P(z, \partial) = \sum_{k=0}^{m} P_{k}(z, \partial)$$

$$P_{k}(z, \partial) = \sum_{l=s_{k}}^{k} A_{k,l}(z, \delta') \delta_{0}^{k-l}$$

$$A_{k,l}(z, \delta') = (z_{0})^{j(k,l)} a_{k,l}(z, \delta'),$$

where $P_{k}(z, \partial)$ is the homogeneous part of order $k$. If $P_{k}(z, \partial) \not\equiv 0$, $A_{k,s_{k}}(z, \delta') \not\equiv 0$, and if $A_{k,l}(z, \delta') \not\equiv 0$, $a_{k,l}(0, z', \delta') \not\equiv 0$. If $P_{k}(z, \partial) \equiv 0$, put $s_{k} = +\infty$. Let us define the irregularities of $K$, which are closely related to the characteristic indices introduced in \cite{1} and others. Put $d_{k,l} = l + j(k, l)$, $d_{k} = \min\{d_{k,l}; s_{k} \leq l \leq k\}$ and $e_{k} = d_{k} - k$. Put $\Sigma^{*} = \text{the convex hull of} \ U_{k=0}^{m} \Pi(k, e_{k})$, where $\Pi(a, b) = \{(x, y) \in R^{2}; x \leq a, y \geq b\}$. The boundary of $\Sigma^{*}$ consists of a vertical half line $\Sigma_{0}^{*}$, a horizontal half line $\Sigma_{p}^{*}$ and segments $\Sigma_{i}^{*}$ $(1 \leq i \leq p - 1)$. The set of vertices of $\Sigma^{*}$ consists of $p$ points $(k_{i}, e_{k_{i}})$, $0 \leq k_{p-1} < k_{p-2} < \cdots < k_{1} < k_{0} = m$ (see Figure 1). Let $\gamma_{i}$ be the slope of $\Sigma_{i}^{*}$, $0 = \gamma_{p} < \gamma_{p-1} < \cdots < \gamma_{1} < \gamma_{0} = +\infty$.

**Definition 1.** We call $\gamma_{i}$, $(0 \leq i \leq p)$ the irregularities of $K$ for $P(z, \partial)$. In particular $\gamma_{p-1}$ is called the minimal irregularity and denote by $\gamma_{\min,p}$.

Let us define some functions spaces. Let $\Theta = \Theta^{0} \times \Theta'$ be a polydisk: $\Theta^{0} = \{z_{0} \in C^{1}; |z_{0}| \leq R\}$, $\Theta' = \{z' \in C^{n}; |z'| \leq R\}$. Put $\Theta_{0} = \{z_{0} \in \Theta^{0} - \{0\}; |\arg z_{0}| < \theta\}$ and $\Theta_{\theta} = \Theta_{0} \times \Theta'$. $\mathcal{O}(\Theta)$ ($\mathcal{O}(\Theta')$) is the set of all holomorphic functions on $\Theta$ (resp. $\Theta'$). $\mathcal{O}(\Theta_{\theta})$ is the set of all holomorphic functions on $\Theta_{\theta}$, which contains multi-valued functions on $\Theta - K$, if $\theta > \pi$. Now we introduce

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**Figure 1.** Characteristic polygon

**Definition 2.** \( \text{Asy}_\{\kappa\}(\Omega_\theta) \) \((0 < \kappa \leq +\infty)\) is the set of all \( f(z) \in \mathcal{O}(\Omega_\theta) \) with the following asymptotic expansion: for any \( \theta' \) with \( 0 < \theta' < \theta \) and any \( N \)

\[
|f(z) - \left( \sum_{k=0}^{N-1} a_k(z')z_0^k \right) | \leq A_{\theta'}B_{\theta}^{N}\Gamma(N/\kappa + 1)|z_0|^N \quad \text{in } \Omega_{\theta'},
\]

where \( a_k(z') \in \mathcal{O}(\Omega') \).

**Definition 3.** \( \tilde{\mathcal{M}} - \text{Asy}_\{\kappa\}(\Omega_\theta) \) \((0 < \kappa \leq +\infty)\) is the set of all \( f(z) \in \mathcal{O}(\Omega_\theta) \) with the following asymptotic expansion: for any \( \theta' \) with \( 0 < \theta' < \theta \) and any \( N \)

\[
|f(z) - \left( \sum_{k=0}^{N-1} a_k(z')z_0^k \right) \log z_0 - \left( \sum_{k=-H}^{N-1} b_k(z')z_0^k \right) | \leq A_{\theta'}B_{\theta}^{N}\Gamma(N/\kappa + 1)|z_0|^N| \log z_0| \quad \text{in } \Omega_{\theta'},
\]

and

\[
|f(z) - \left( \sum_{k=0}^{N} a_k(z')z_0^k \right) \log z_0 - \left( \sum_{k=-H}^{N-1} b_k(z')z_0^k \right) | \leq A_{\theta'}B_{\theta}^{N}\Gamma(N/\kappa + 1)|z_0|^N \quad \text{in } \Omega_{\theta'},
\]
where $H \in \mathcal{N}$ and $a_k(z'), b_k(z') \in \mathcal{O}(\Omega')$.

**Definition 4.** $\mathcal{M}(\Omega)$ is the set of all $f(z) \in \mathcal{O}(\Omega_{+\infty})$ with the form $f(z) = a(z) \log z_0 + b(z) z_0^{-H}$, where $H \in \mathcal{N}$ and $a(z), b(z) \in \mathcal{O}(\Omega)$.

Definition 5. $\mathcal{O}_{(\kappa)}(\Omega_{\theta})(\kappa > 0)$ is the set of all $f(z) \in \mathcal{O}(\Omega_{\theta})$ with the following bound: for any $\theta'$ with $0 < \theta' < \theta$ and any $\epsilon > 0$

$$|f(z)| \leq C_{\epsilon} \exp(\epsilon|z_0|^{-\kappa}) \quad \text{in} \quad \Omega_{\theta'}.$$

Now we suppose that $P(z, \partial)$ satisfies the following condition:

**Condition**

$$\begin{cases} 
(a) \gamma_{\min,P} \neq +\infty \\
(b) \; d_{k_{p-1}} = 0 \\
(c) \; d_{k_i} = s_k_i \; \text{for} \; 0 \leq i \leq p - 1.
\end{cases}$$

Put $\gamma = \gamma_{\min,P}$. Then the main results are the following.

**Theorem 6.** If $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_{\theta})$ is a solution of

$$P(z, \partial)u(z) = f(z) \in \text{Asy}_{(\kappa)}(\Omega_{\theta}) \quad (0 < \kappa \leq \gamma),$$

then $u(z) \in \text{Asy}_{(\kappa)}(\Omega_{\theta})$.

**Corollary 7.** Suppose that $f(z) \in \mathcal{O}(\Omega)$ and $\theta > (\pi/2\gamma) + \pi$ in Theorem 6. Then $u(z) \in \mathcal{O}(\Omega)$.

**Theorem 8.** If $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_{\theta})$ is a solution of

$$P(z, \partial)u(z) = f(z) \in \tilde{\mathcal{M}} - \text{Asy}_{(\kappa)}(\Omega_{\theta}) \quad (0 < \kappa \leq \gamma),$$

then $u(z) \in \tilde{\mathcal{M}} - \text{Asy}_{(\kappa)}(\Omega_{\theta})$.

**Corollary 9.** Suppose that $f(z) \in \tilde{\mathcal{M}}(\Omega)$ and $\theta > (\pi/2\gamma) + 2\pi$ in Theorem 8. Then $u(z) \in \tilde{\mathcal{M}}(\Omega)$.

Condition (a) means that $K$ is an irregular characteristic surface in the sense in [1] and it is equivalent to $p > 1$. Condition (b) means that the $(p-1)$-th localization on $K$ of $P(z, \partial)$, which is defined in [1], is a function. Condition (c) is an assumption imposed on the vertices of the characteristic polygon $\Sigma^*$. Theorems 6 and 8 are shown by the detailed analysis of the integral representation of solutions singular on $K ([2, 3])$ and for this purpose we assume (c).

A simple example satisfying the conditions in Theorems is

$$P(z, \partial) = a(z) \partial_1^m + b(z) \partial_1^{k''} \partial_0^{k'-l'} + \partial_0^k, \quad z = (z_0, z_1) \quad m > k' > k,$$
where we assume $a(0)b(0) \neq 0$ and $l' > k' - k$. For this $P(z, \partial)$, we have $\gamma = \min\{k/(m - k), (l' - k' + k)/(k' - k)\}$. We can also obtain results similar to Theorems 6 and 8 of the following type for other $\mathcal{F}(\Omega_\theta) \subset \mathcal{O}(\Omega_\theta)$:

$$\begin{cases}
  u(z) \in \mathcal{O}_{\gamma}(\Omega_\theta) \\
P(z, \partial)u(z) = f(z) \in \mathcal{F}(\Omega_\theta)
\end{cases} \implies u(z) \in \mathcal{F}(\Omega_\theta).$$

**REFERENCES**

