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SOME REMARKS ON COLOMBEAU'S GENERALIZED FUNCTIONS

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Introduction

About ten years ago, J. F. Colombeau introduced a class $\mathcal{G}(\mathbf{R}^n)$ of generalized functions admitting a structure of associative algebra. This new class of generalized functions includes Schwartz' distributions, satisfies Leibniz' rule concerning differentiation of the product of two elements, and it also has a sheaf structure. Instead, for example,

$$x \cdot \delta(x) \neq 0, \quad x \cdot \frac{1}{x+i0} \neq 1 \quad in \quad \mathcal{G}(\mathbf{R}^n). \tag{1}$$

Such inequalities are inevitable if we admit the associative law among functions $\delta(x)$, x and $1/(x+i0)$. However this theory has another equality, $\approx$ (so-called, association), which is an equivalence relationship in $\mathcal{G}(\mathbf{R}^n)$. In fact,

$$x \cdot \delta(x) \approx 0, \quad x \cdot \frac{1}{(x+i0)} \approx 1. \tag{2}$$

Though $\{f \in \mathcal{G}(\mathbf{R}^n);\ f \approx 0\}$ is not an ideal, this kind of equalities in a weak sense is indispensable for the non-linear theory of generalized functions.

Further, Oberguggenberger successfully introduced a subalgebra $\mathcal{G}^\infty(\mathbf{R}^n)$ similar to $C^\infty(\mathbf{R}^n)$ in $\mathcal{D}'(\mathbf{R}^n)$. Indeed, this notion on smoothness for Colombeau genealized functions easily induces the definitions of singular supports and wave front sets similar to those for distributions. Moreover, several mathematicians including Professor Pilipović, who introduced me this magnificent theory, are tackling pseudo-differential calculus for $\mathcal{G}(\mathbf{R}^n)$.

The aim of this note is in giving a simpler definition of generalized functions of Colombeau's type, and in improving the definition of "association" as the new association will just fit Oberguggenberger's $\mathcal{G}^\infty(\mathbf{R}^n)$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Colombeau's definitions

Definition 1. Let $\mathcal{A}_q$ $(q = 0, 1, 2, \dots)$ be a decreasing sequence of subsets of $C_0^\infty(\mathbf{R}^n)$ defined as follows:

$$\mathcal{A}_q(\mathbf{R}^n) = \{\phi \in C_0^\infty(\mathbf{R}^n); \int_{\mathbf{R}^n} x^\alpha \phi(x) dx = \delta_{0,\, |\alpha|},\ \ 0 \le \forall |\alpha| \le q\} \tag{3}$$

for $q = 0, 1, 2, \dots$. Indeed,

$$\phi \in C_0^\infty(\mathbf{R}^n) \ \textit{belongs to}\ \mathcal{A}_q \quad \textit{iff}\ \ \widehat{\phi}(\xi) = 1 + \mathrm{O}(|\xi|^{q+1}) \ \ \textit{at}\ \ \xi = 0,$$

where $\widehat{\phi}(\xi)$ is the Fourier transform of ϕ. Hence, for any $\varphi \in \mathcal{A}_0$, we have

$$(1 + \sum_{|\alpha|=1}^{q} C_\alpha \partial_x^\alpha)\varphi(x) \in \mathcal{A}_q$$

for some suitable constants C_α.

Moreover we define a one-parameter deformation ϕ_ϵ of $\phi \in \mathcal{A}_q$ by

$$\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon}), \tag{4}$$

which also belongs to $\mathcal{A}_q$ for any $\epsilon > 0$. Then we have the following definition due to Colombeau:

Definition 2. Let Ω be an open set in $\mathbf{R}^n$. Then, a mapping

$$R : \mathcal{A}_0 \ni \phi \ \mapsto \ R(\phi, x) \in C^\infty(\Omega) \tag{5}$$

is said to be *moderate* if, for any $K \subset\subset \Omega$, any $\alpha \ge 0$, there exists a positive constant $N_{K,\alpha}$ satisfying an estimate

$$\sup_{x \in K} |\partial_x^\alpha R(\phi_\epsilon, x)| \le C_{K,\alpha,\phi}\epsilon^{-N_{K,\alpha}} \ \textit{for}\ \ \forall \phi \in \mathcal{A}_{N_{K,\alpha}},\ 0 < \forall \epsilon < \epsilon_{K,\alpha,\phi} \tag{6}$$

with some positive constants $C_{K,\alpha,\phi}$ and $\epsilon_{K,\alpha,\phi}$.

Further, R is said to be *null* if, for any $K \subset\subset \Omega$, any α, there exist a positive constant $N_{K,\alpha}$ and an increasing sequence $\{a_q(K,\alpha)\}_{q=0}^\infty$ of positive numbers with $\lim_{q\to\infty} a_q = \infty$ such that

$$\sup_{x \in K} |\partial_x^\alpha R(\phi_\epsilon, x)| \le C_{K,\alpha,\phi}\epsilon^{a_q(K,\alpha) - N_{K,\alpha}} \tag{7}$$

$$\textit{for}\ \forall q \ge N_{K,\alpha}, \forall \phi \in \mathcal{A}_q,\ 0 < \forall \epsilon \le \epsilon_{K,\alpha,\phi}$$

with some positive constants $C_{K,\alpha,\phi}$, $\epsilon_{K,\alpha,\phi}$.

It is easy to see that the totality of null maps constitutes an ideal in the algebra of the totality of moderate maps. Here the product RR' of maps R, R' is defined by its value at ϕ:

$$(RR')(\phi, x) = R(\phi, x)R'(\phi, x).$$

Hence we have

Definition 3.

$$\mathcal{G}(\Omega) = \{moderate\ maps\}/\{null\ maps\}.$$

Then the distributions on $\mathbf{R}^n$ are imbedded by

$$\text{Cd} : \mathcal{D}'(\mathbf{R}^n) \ni f \mapsto [(\text{Cd}f)(\phi, x)] \in \mathcal{G}(\mathbf{R}^n) \tag{8}$$

with

$$(\text{Cd}f)(\phi, x) = \int_{\mathbf{R}^n} f(x+y)\phi(y)dy. \tag{9}$$

It is easy to see that "Cd" has a local property, and so that "Cd" induces a sheaf imbedding

$$\text{Cd} : \mathcal{D}' \to \mathcal{G}.$$

The following theorem is the most basic in the theory of Colombeau, which is directly derived from the moment property of $\mathcal{A}_q$:

Theorem 4.

$$(\text{Cd}f)(\phi, x) - f(x) : \mathcal{A}_0 \to C^\infty(\Omega')$$

is a null map for any $f \in C^\infty(\Omega)$ *and any* $\Omega' \subset\subset \Omega$.

As a direct consequence of this theorem, we have

Theorem 5. $C^\infty(\Omega)$ *is imbedded by* Cd *in* $\mathcal{G}(\Omega)$ *as a subalgebra.*
Here we introduce Oberguggenberger's $\mathcal{G}^\infty$:

Definition 6. $[R(\phi, x)]$ belongs to $\mathcal{G}^\infty(\Omega)$ iff $R(\phi_\epsilon, x)$ satisfies the following estimates: For any $K \subset\subset \Omega$, there exists a positive constant N_K such that

$$\sup_{x\in K} |\partial_x^\alpha R(\phi_\epsilon, x)| \le C_{K,\alpha,\phi}\epsilon^{-N_K}\ ,\ \forall\alpha \ge 0,\ \forall\phi \in \mathcal{A}_{N_K},\ 0 < \forall\epsilon \le \epsilon_{K,\alpha,\phi}. \tag{10}$$

Corollary 7. $\mathcal{G}^\infty(\Omega)$ *is a subalgebra of* $\mathcal{G}(\Omega)$ *closed under differentiation satisfying that*

$$\mathcal{D}'(\Omega) \cap \mathcal{G}^\infty(\Omega) = C^\infty(\Omega). \tag{11}$$

As above, the theory seems to be successful, however, as stated at (1), there is a heavy difficulty; that is, two functions in $\mathcal{G}(\Omega)$ are hardly equal to each other. For example,

Example 8. For any $\varphi \in C^\infty(\mathbf{R}^n)$ we have

$$\varphi(x)\delta(x) = 0 \; in \;\; \mathcal{G}(\Omega) \;\; iff \;\; \varphi(x) = \mathrm{O}(|x|^\infty) \;\; at \; x = 0.$$

To solve such a difficulty Colombeau introduced an equivalence relationship $\approx$:

Definition 9. For a function $f(x) = [R(\phi, x)] \in \mathcal{G}(\Omega)$ we define $f \approx 0$ by

$$\lim_{\epsilon \to 0} R(\phi_\epsilon, x) = 0 \;\; in \;\; \mathcal{D}'(\Omega), \tag{12}$$

that is, for any fixed $\psi \in C_0^\infty(\Omega)$ we have

$$\lim_{\epsilon \to 0} \int_\Omega R(\phi_\epsilon, x)\psi(x)dx = 0. \tag{13}$$

Remark 10. The totality $\{f \in \mathcal{G}(\Omega); f \approx 0\}$ of null functions in the sense of association is not an ideal of $\mathcal{G}(\Omega)$, but is closed under the operations by linear differential operators with C^∞-coefficients.

Example 11. For two continuous functions $f(x), g(x)$ on Ω, we have

$$(\mathrm{Cd}f)(\mathrm{Cd}g) \approx \mathrm{Cd}(fg). \tag{14}$$

Further (14) also holds for two distributions f, g which admit a normal product fg as distributions.

Example 12.

$$\delta(x) \cdot \left(\frac{1}{x + i0} + \frac{1}{x - i0} \right) \approx -\delta'(x). \tag{15}$$

(15) is interesting for nonlinear theory of generalized functions.

2. New definitions

We introduce a new definition of generalized functions of Colombeau's type, which is much simpler because no sequences like $\{a_q\}_{q=0}^{\infty}$ are necessary. Though the relationship between those definitions is not clear now, many notions concerning regularity for generalized functions are easily defined under the new definition.

Definition 13. Let $\mathcal{S}$ be the space of all rapidly decreasing smooth functions on $\mathbf{R}^n$. Then, a subset $\mathcal{S}_0 \subset \mathcal{S}$ is defined as follows:

$$\mathcal{S}_0 = \{\phi \in \mathcal{S};\ \widehat{\phi}(\xi) = 1 + \mathrm{O}(|\xi|^{\infty})\ \ as\ \ \xi \to 0\}. \tag{16}$$

It is clear that for an element $\phi \in \mathcal{S}_0$ every moment other than of degree 0 vanishes. Further the one-parameter family $\{\phi_\epsilon; \epsilon > 0\}$ is contaied in $\mathcal{S}_0$. Hence we can introduce the following definition of generalized functions:

Definition 14. Let Ω be an open set in $\mathbf{R}^n$. Then, a mapping

$$R : \mathcal{S}_0 \ni \phi \mapsto R(\phi, x) \in C^{\infty}(\Omega) \tag{17}$$

is said to be *moderate* if, for any $K \subset\subset \Omega$ any $\alpha \geq 0$, there exists a positive number $N_{K,\alpha}$ satisfying an estimate

$$\sup_{x\in K} |\partial_x^{\alpha} R(\phi_\epsilon, x)| \leq C_{K,\alpha,\phi}\epsilon^{-N_{K,\alpha}}\ \ for\ \ \forall\phi \in \mathcal{S}_0,\ 0 < \forall\epsilon \leq \epsilon_{K,\alpha,\phi} \tag{18}$$

with some positive constants $C_{K,\alpha,\phi}$, $\epsilon_{K,\alpha,\phi}$.

Further, R is said to be *null* if, for any $K \subset\subset \Omega$, any α, any $\phi \in \mathcal{S}_0$, and any positive integer l, there exist some positive constants $C_{K,\alpha,\phi,l}$, $\epsilon_{K,\alpha,\phi,l}$ satisfying

$$\sup_{x\in K} |\partial_x^{\alpha} R(\phi_\epsilon, x)| \leq C_{K,\alpha,\phi,l}\epsilon^{l}\ \ for\ \ 0 < \forall\epsilon \leq \epsilon_{K,\alpha,\phi,l}. \tag{19}$$

Hence we have our new definition of generalized functions:

Definition 15. Under Definition 14, we introduce a new class $\mathcal{G}_*$ of algebras of generalized functions by

$$\mathcal{G}_*(\Omega) = \{moderate\ \ maps\}/\{null\ \ maps\}. \tag{20}$$

Though no element of $\mathcal{S}_0$ has compact support, the map

$$\mathcal{E}'(\mathbf{R}^n) \ni f \mapsto [(\mathrm{Cd}f)(\phi, x)] \in \mathcal{G}_*(\mathbf{R}^n) \tag{21}$$

is well-defined and has locality. Hence we obtain a sheaf imbedding

$$\mathrm{Cd} : \mathcal{D}' \to \mathcal{G}_*.$$

Theorems 4 and 5 hold also for our $\mathcal{G}_*$, further the definition of smoothness is given in the following way:

Definition 16. $[R(\phi, x)]$ belongs to $\mathcal{G}_*^\infty(\Omega)$ iff, for any $K \subset\subset \Omega$, there exists a positive constant N_K such that

$$(22) \quad \sup_{x \in K} |\partial_x^\alpha R(\phi_\epsilon, x)| \leq C_{K,\alpha,\phi}\epsilon^{-N_K} \ \ \mathit{for}\ \forall\ \alpha \geq 0,\ \forall\ \phi \in \mathcal{S}_0,\ 0 < \forall\epsilon \leq \epsilon_{K,\alpha,\phi}.$$

Under this definition of smoothness we obtain Corollary 7. However the most important result for us is that we can justify $\mathcal{G}_*^\infty$-class in the sense of association if we modify the definition of $\approx$ as follows:

Definition 17. For a function $f(x) = [R(\phi, x)] \in \mathcal{G}_*(\Omega)$ we define $f \approx 0$ by the following: For any $K \subset\subset \Omega$ and any $\phi \in \mathcal{S}_0$, there exist positive constants $\theta_{K,\phi}$, $C_{K,\phi}$ such that, for $\forall\phi \in \mathcal{S}_0$ and $\forall\psi \in C_0^\infty(K)$, we have an estimate

$$(23) \qquad |\int_K R(\phi_\epsilon, x)\psi(x)dx| \leq C_{K,\phi}\epsilon^{\theta_{K,\phi}} \sup\{|\partial_x^\alpha \psi(x)|; x \in K, |\alpha| \leq C_{K,\phi}\}$$

for $0 < \forall\epsilon \leq \theta_{K,\phi}$.

Remark 18. Clearly to see, condition (23) is stronger than (13). Indeed, in the new association , only convergences of positive power speed w.r.t. ϵ are admitted. However we have the facts similar to Remark 10, Example 11, Example 12 though one must replace continuity by Hölder continuity.

As the main consequence from our new definitions we have the following theorem, which shows that generalized smoothness is in harmony with association in our theory.

Theorem 19. *Let $f(x)$ be a distribution on Ω and $g(x) \in \mathcal{G}_*^\infty(\Omega)$. Suppose that*

$$f \approx g \ \ in\ \ \mathcal{G}_*(\Omega).$$

Then, $f \in C^\infty(\Omega)$.

Remark 20. According to Professor Pilipović, the fact in Theorem 19 is false in the original theory of Colombeau and Oberguggenberger.