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The Cauchy problem for a class of hyperbolic operators with double characteristics

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1. Introduction

In [10] we proved that the Cauchy problem for hyperbolic operators is C^∞ well-posed if the operators satisfy some microlocal *a priori* estimates. So, in the studies of C^∞ well-posedness of the hyperbolic Cauchy problem the problems are reduced to obtaining the microlocal *a priori* estimates. In [11] we investigated a class of hyperbolic operators with double characteristics, which contains effectively hyperbolic operators, applying results in [10]. In [11] we imposed some extra conditions on hyperbolic operators. In this article we shall show that the Cauchy problem for hyperbolic operators with double characteristics is C^∞ well-posed under reasonable assumptions. In doing so, we shall use ideas in Kajitani-Wakabayashi-Nishitani [13]. One of chief distinctions of our treatment is the use of 'time functions'. Using 'time functions' we can consider effectively hyperbolic operators and a wide class of non effectively hyperbolic operators with a unified treatment. C^∞ well-podeness of the Cauchy problem for effectively hyperbolic operators was proved by Iwasaki [5] (see, also, [6], [14], [15], [16]). Ivrii [7] studied C^∞ well-posedness of the Cauchy problem for a class of non effectively hyperbolic operators (see, also, [2]).

Let $P(x,\xi)$ be a polynomial of $\xi=(\xi_1,\xi')=(\xi_1,\cdots,\xi_n)$ of degree m whose coefficients are C^∞ functions of $x=(x_1,x')=(x_1,\cdots,x_n)\in\mathbf{R}^n$. We define the operator $P^w(x,D)$ with Weyl symbol $P(x,\xi)$ by

$$P^w(x,D)u(x)=(2\pi)^{-n}\int\left\{\int e^{i(x-y)\cdot\xi}P(\frac{x+y}{2},\xi)u(y)\,dy\right\}d\xi$$

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for $u \in C_0^\infty(\mathbf{R}^n)$. We consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P^w(x,D)u = f & \text{in } \Omega, \\ \text{supp } u \subset \{x_1 \geq 0\} \end{cases}$$

in the C^∞ (or $\mathcal{D}'$) category, where Ω is an open subset of $\mathbf{R}^n$ and contains the origin, and supp $f \subset \{ x_1 \geq 0 \}$. Let $p(x,\xi)$ be the principal part of $P(x,\xi)$. We assume that

(P-1) $p(x,\xi)$ is hyperbolic with respect to $\vartheta = (1,0,\cdots,0) \in\mathbf{R}^n$ for each $x \in \mathbf{R}^n$, *i.e.*, $p(x,\xi - i\vartheta) \neq 0$ for $x \in\mathbf{R}^n$ and $\xi \in\mathbf{R}^n$.

To state our assumptions and results we need the following

Definition 1.1. Let $z^0 = (x^0,\xi^0) \in T^*\mathbf{R}^n \setminus 0$ and assume that (P-1) is satisfied. (i) The localization polynomial $p_{z^0}(\delta z)$ of $p(x,\xi)$ at z^0 is defined by

$$p(z^0 + s\delta z) = s^\mu (p_{z^0}(\delta z) + o(1)) \quad \text{as } s \to 0,$$

and $p_{z^0}(\delta z) \not\equiv 0$ in $\delta z \in T_{z^0}(T^*\mathbf{R}^n)$ ($\simeq\mathbf{R}^{2n}$). We denote by $\Gamma(p_{z^0},(0,\vartheta))$ the connected component of the set $\{\delta z \in T_{z^0}(T^*\mathbf{R}^n);\ p_{z^0}(\delta z) \neq 0\}$ which contains $(0,\vartheta)$. (ii) Let $t(x,\xi)$ be a real-valued function in $C(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ which is positively homogeneous of degree 0. We say that $t(x,\xi)$ is a time function for p with respect to $(0,\vartheta)$ ($\in\mathbf{R}^{2n}$) at z^0 if $t(z^0) = 0$ and if there are a neighborhood $\mathcal{U}$ of z^0 and $K \subset\subset \Gamma(p_{z^0},(0,\vartheta))$ such that $t(x,\xi)$ is Lipschitz continuous in $\mathcal{U}$ and $-(|\xi|\nabla_\xi t(x,\xi), -\nabla_x t(x,\xi)) \in K$ for *a.e.* $(x,\xi) \in \mathcal{U}$. (iii) We denote by $F_p(z^0)$ the Hamilton map corresponding to $Hess\ p/2$ at z^0, *i.e.*, $F_p(z^0) = \frac{1}{2}\begin{pmatrix} p_{\xi x}(z^0) & p_{\xi\xi}(z^0) \\ -p_{xx}(z^0) & -p_{x\xi}(z^0) \end{pmatrix}$. We define $Tr^+\ F_p(z^0) = \sum \lambda_j$, where $\lambda_j > 0$ and the $i\lambda_j$ are the eigenvalues of $F_p(z^0)$ on the positive imaginary axis. (iv) We denote by $K^\pm_{x^0}$ the sets $\{x(t);\ \pm t \geq 0$, and $\{x(t)\}$ is a Lipschitz continuous curve in $\mathbf{R}^n$ satisfying $\frac{d}{dt}x(t) \in \Gamma(p(x(t),\cdot),\vartheta)^*$ (*a.e.* t) and $x(0) = x^0\}$, where $\Gamma^* = \{x \in\mathbf{R}^n;\ x \cdot \xi \geq 0$ for any $\xi \in \Gamma\}$.

Remark. (i) It can be proved that $p_{z^0}(\delta z)$ is hyperbolic with respect to $(0,\vartheta) \in \mathbf{R}^{2n}$ under (P-1) (see, *e.g.*, [3]). (ii) We can also define 'time functions' for microhyperbolic functions (symbols) (see [8] and [17]). (iii) When $t(x,\xi)$ is a real-valued function in $C^1(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ and positively homogeneous of degree 0, $t(x,\xi)$ is a time function for p with respect to $(0,\vartheta)$ at z^0 if and only if $-H_t(z^0) \in \Gamma(p_{z^0},(0,\vartheta))$, where $H_t(z) = \sum_{j=1}^n ((\partial t/\partial \xi_j)(z)(\partial/\partial x_j) - (\partial t/\partial x_j)(z)(\partial/\partial \xi_j))$.

In addition to (P-1) we impose the following condition on $p(x,\xi)$ for every $z^0 = (x^0,\xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$, where $S^{n-1} = \{\xi \in \mathbf{R}^n;\ |\xi| = 1\}$.

$(\text{P-2})_{z^0}$ There are conic neighborhoods $\mathcal{C}$ and $\tilde{\mathcal{C}}$ of z^0 and (y^0,η^0), respectively, a homogeneous canonical transformation χ: $\tilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$, time functions $t_j(y,\eta)$ ($1 \le j \le d$) for $p \circ \chi$ with respect to $(0,\vartheta)$ at (y^0,η^0), a real-valued symbol $\lambda(y,\eta')$ of positively homogeneous of degree 1, a non-negative symbol $\alpha(y,\eta')$ of positively homogeneous of degree 2, an elliptic symbol $e(y,\eta')$ and $C > 0$ such that $z^0 = \chi(y^0,\eta^0)$, $d\chi_{(y^0,\eta^0)}(0,\vartheta) \in \Gamma(p_{z^0},(0,\vartheta))$,

$$
\begin{aligned}
(1.1) \qquad & p(\chi(y,\eta)) = e(y,\eta)\{\eta_1(\eta_1 - \lambda(y,\eta')) - \alpha(y,\eta')\} \qquad \text{in } \tilde{\mathcal{C}}, \\
& T(y,\eta')\frac{\partial \alpha}{\partial y_1}(y,\eta') \le C\alpha(y,\eta') \qquad \text{for } (y,\eta') \in \tilde{\mathcal{C}}',
\end{aligned}
$$

where $T(y,\eta') = \min_{1\le j\le d} |t_j(y,0,\eta')|$ and $\tilde{\mathcal{C}}' = \{(y,\eta');\ (y,\eta) \in \tilde{\mathcal{C}} \text{ for some } \eta_1\}$.

Let $z^0 = (x^0,\xi^0) \in \mathbf{R}^n \times S^{n-1}$ satisfy $dp(z^0) = 0$, and let F_1 and F_2 be classical Fourier integral operators corresponding to χ and χ^{-1} which are elliptic at (y^0,η^0) and z^0, respectively. Under the assumption $(\text{P-2})_{z^0}$ we have

$$
\sigma(F_2 P^w(x,D) F_1)(y,\eta) = \tilde{e}(y,\eta)\{\eta_1(\eta_1 - \lambda(y,\eta')) - \alpha(y,\eta') + \beta(y,\eta)\}
$$

in a conic neighborhood $\tilde{\mathcal{C}}_0$ of (y^0,η^0) if $|\eta| \ge 1$, where $\sigma(a^w(y,D))(y,\eta) = a(y,\eta)$, $\tilde{e}(y,\eta)$ is an elliptic classical symbol in $\tilde{\mathcal{C}}_0$ and $\beta(y,\eta)$ is a classical symbol in $S^1_{1,0}$. For the imaginary part of the subprincipal symbol of $P^w(x,D)$ we assume that for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$

$(\text{P-3})_{z^0}$ There are $A(y,\eta') \in S^0_{1,0}$ and $C > 0$ such that

$$
T(y,\eta')|\operatorname{Im}\beta(y,0,\eta') + \frac{1}{2}\frac{\partial\lambda}{\partial y_1}(y,\eta') - A(y,\eta')\lambda(y,\eta')| \le C(\sqrt{\alpha(y,\eta')} + 1) \quad \text{in } \tilde{\mathcal{C}}_0'.
$$

To control $\operatorname{Re}\beta(x,\xi)$ we assume that at least one of the following conditions $(\text{P-4-1})_{z^0}$ and $(\text{P-4-2})_{z^0}$ is satisfied for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$:

$(\text{P-4-1})_{z^0}$ There are $B(y,\eta') \in S^0_{1,0}$ and $C > 0$ such that

$$
T(y,\eta')|\operatorname{Re}\beta(y,0,\eta') - B(y,\eta')\lambda(y,\eta')| \le C(\sqrt{\alpha(y,\eta')} + 1) \quad \text{in } \tilde{\mathcal{C}}_0'.
$$

$(\text{P-4-2})_{z^0}$ Re $P_{m-1}(z^0) < Tr^+ F_p(z^0)$, where $P_{m-1}(x,\xi)$ denotes the homogeneous part of degree $(m-1)$ of $P(x,\xi)$.

Now we can state our main result.

Theorem 1.2. *Assume that Ω is bounded. Under the above assumptions, For any $f \in \mathcal{D}'$ with* supp $f \subset \{x_1 \geq 0\}$ *there is $u \in \mathcal{D}'$ satisfying* (CP). *Moreover, if $x^0 \in \Omega$, $K^-_{x^0} \cap \{x_1 \geq 0\} \subset\subset \Omega$,* supp $u \subset \{x_1 \geq 0\}$ *and $P^w(x,D)u = 0$ (resp. $P^w(x,D)u \in C^\infty$) near $K^-_{x^0}$, then $x^0 \notin$supp u (resp. $x^0 \notin$sing supp u).*

Remark. If the hypothese of Theorem 1.2 are fulfilled, taking $\Omega = \mathbf{R}^n$ (CP) is well-posed in $\mathcal{D}'$ and C^∞, and supp $u \subset \{x \in \mathbf{R}^n;\ x \in K^+_y$ for some $y \in$supp $f\}$.

In [11] we assumed that all time functions $t_j(y,\eta)$ ($1 \leq j \leq d$) in $(\text{P-2})_{z^0}$ do not depend on η under suitable choice of canonical coordinates and belong to $\mathcal{B}^\infty(\mathbf{R}^n)$. Then we could use usual symbol calculus in $S^\infty_{1,1/2}$. Under the assumption $(\text{P-2})_{z^0}$ we need symbol calculus with large parameters in a subclass of $S^\infty_{1/2,1/2}$ which is not included in $S^\infty_{\rho,1/2}$ for $\rho > 1/2$.

2. Outline of the proof of Theorem 1.2

We assume that (P-1) is satisfied and that $(\text{P-2})_{z^0}$, $(\text{P-3})_{z^0}$ and at least one of the conditions $(\text{P-4-1})_{z^0}$ and $(\text{P-4-2})_{z^0}$ are satisfied for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$. Fix $z^0 = (x^0,\xi^0) \in \mathbf{R}^n \times S^{n-1}$ so that $dp(z^0) = 0$, and let $t_j(x,\xi)$ ($1 \leq j \leq d$) be the time functions in $(\text{P-2})_{z^0}$. Let $\chi(t)$ be a function in $C^\infty(\mathbf{R})$ such that $\chi(t) = 0$ for $|t| \leq 1/2$, $\chi(t) = 1$ for $|t| \geq 1$ and $0 \leq \chi(t) \leq 1$. Let $N \geq 1$, and put

$$W(x,\xi) = \sum_{j=1}^{d} \langle\xi\rangle_N^{1/2}(t_j(x,\xi)^2\chi(|\xi|/N)^2\langle\xi\rangle_N + N)^{-1/2},$$

where $\langle\xi\rangle_N = (N^2 + |\xi|^2)^{1/2}$. We define a metric g in $\mathbf{R}^n \times \mathbf{R}^n$ by

$$g_{x,\xi} = W(x,\xi)^2(|dx|^2 + \langle\xi\rangle_N^{-2}|d\xi|^2).$$

Then g is σ temperate in the sense of Hörmander (see [4]). Here and after we use notations and terminologies in [4]. Define

$$\Phi(x,\xi) = \prod_{j=1}^{d}((t_j(x,\xi)^2\chi(|\xi|/N)^2\langle\xi\rangle_N + t_j(x,\xi)\chi(|\xi|/N)\langle\xi\rangle_N^{1/2}).$$

We can prove that $\Phi(x,\xi)$ is σ, g temperate in the sense of Hörmander (see [4]). Choose $\rho(x,\xi) \in C^\infty(\mathbf{R}^{2n})$ such that supp $\rho \subset \{(x,\xi);\ |x|^2+|\xi|^2 < c(\rho)\}$, $\int \rho(x,\xi)dx\,d\xi = 1$, $\rho(x,\xi) \geq 0$ and

$$|\rho^{(\alpha)}_{(\beta)}(x,\xi)| \leq C(\rho)A(\rho)^{|\alpha|+|\beta|}|\alpha+\beta|!^\kappa$$

for any $(x,\xi) \in \mathbf{R}^{2n}$ and any multi-indices α and β, where $\rho^{(\alpha)}_{(\beta)}(x,\xi) = D^\beta_x\partial^\alpha_\xi\rho(x,\xi)$, $c(\rho)$, $C(\rho)$ and $A(\rho)$ are positive constants and $\kappa > 1$ will be specified later. Taking $c(\rho)$ to be small enough, we put

$$\begin{aligned}\widetilde{W}(x,\xi) &= \int \rho(W(y,\eta)(x-y), \langle\eta\rangle_N^{-1}W(y,\eta)(\xi-\eta)) \\ &\qquad \times \langle\eta\rangle_N^{-n}W(y,\eta)^{2n+1}dy\,d\eta, \\ \widetilde{\Phi}(x,\xi) &= \int \rho(\widetilde{W}(x,\xi)(x-y), \langle\xi\rangle_N^{-1}\widetilde{W}(x,\xi)(\xi-\eta)) \\ &\qquad \times \langle\xi\rangle_N^{-n}\widetilde{W}(x,\xi)^{2n}\Phi(y,\eta)dy\,d\eta.\end{aligned}$$

Then we have the following

Lemma 2.1. *There are positive constants C_1, C_2 and A such that*

$$\begin{aligned}&C_1^{-1}W(x,\xi) \leq \widetilde{W}(x,\xi) \leq C_1W(x,\xi), \\ &C_1^{-1}\Phi(x,\xi) \leq \widetilde{\Phi}(x,\xi) \leq C_1\Phi(x,\xi), \\ &|\widetilde{W}^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_2A^{|\alpha|+|\beta|}|\alpha+\beta|!^\kappa W(x,\xi)^{1+|\alpha|+|\beta|}\langle\xi\rangle_N^{-|\alpha|} \\ &|\widetilde{\Phi}^{(\alpha)}_{(\beta)}(x,\xi)| \leq C_2A^{|\alpha|+|\beta|}|\alpha+\beta|!^\kappa\Phi(x,\xi)W(x,\xi)^{|\alpha|+|\beta|}\langle\xi\rangle_N^{-|\alpha|}\end{aligned}$$

Moreover, there are a conic neighborhood $\mathcal{C}_1$ of z^0, a closed convex cone Γ in $T^*\mathbf{R}^n\setminus 0$, $c > 0$ and $\gamma(N) > 0$ such that $\Gamma \subset\subset \Gamma(p_z,(0,\vartheta))$ for $z = (x,\xi) \in \mathcal{C}_1$ with $|\xi| = 1$ and

$$\begin{aligned}&(-\langle\xi\rangle_N\nabla_\xi\widetilde{\Phi}(x,\xi), \nabla_x\widetilde{\Phi}(x,\xi)) \in \Gamma, \\ &|(-\langle\xi\rangle_N\nabla_\xi\widetilde{\Phi}(x,\xi), \nabla_x\widetilde{\Phi}(x,\xi))| \geq cW(x,\xi)\Phi(x,\xi)\end{aligned}$$

if $(x,\xi) \in \mathcal{C}_1$ and $|\xi| \geq 2\gamma(N)$.

Define a metric g_0 by

$$g_{0,x,\xi} = |dx|^2 + \langle\xi\rangle_h^{-2}|d\xi|^2,$$

where $h \geq N$. Following the arguments in §18.4 of [4] and in [13], we can prove the following

Proposition 2.2. *Let $m(x,\xi)$ be σ, g_0 temperate and $a(x,\xi) \in S(m, g_0)$. Fix k, κ and δ so that $k \geq 2$, $1 < \kappa < 3/2$ and $0 < \delta < 3 - 2\kappa$. Then there are $C_{\alpha,\beta} > 0$, $M_0 > 1$, $s_{\alpha,\beta}(x,\xi)$, $\widetilde{s}_{\alpha,\beta}(x,\xi)$ and $r_k(x,\xi)$ such that $\widetilde{s}_{\alpha,\beta}(x,\xi)$ is real-valued,*

$$\widetilde{\Phi}^{\mp M} \# a \# \widetilde{\Phi}^{\pm M} =$$

$$\sum_{|\alpha|+|\beta|\leq k-1} (-1)^{|\alpha|}(\pm M\nabla_\xi \widetilde{\Phi}/\widetilde{\Phi})^\alpha (\mp iM\nabla_x \widetilde{\Phi}/\widetilde{\Phi})^\beta a^{(\beta)}_{(\alpha)}(x,\xi)/(\alpha!\beta!)$$

$$+ \sum_{|\alpha|+|\beta|\leq k-1} s_{\alpha,\beta}(x,\xi) a^{(\beta)}_{(\alpha)}(x,\xi) + \sum_{2\leq|\alpha|+|\beta|\leq k-1} \widetilde{s}_{\alpha,\beta}(x,\xi) a^{(\beta)}_{(\alpha)}(x,\xi) + r_k(x,\xi),$$

$$|s_{\alpha,\beta}{}^{(\tilde{\alpha})}_{(\tilde{\beta})}(x,\xi)| \leq C_{\alpha,\beta} M^{|\alpha|+|\beta|} W(x,\xi)^{2+|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}|} \langle\xi\rangle_N^{-1-|\alpha|-|\tilde{\alpha}|},$$

$$|\widetilde{s}_{\alpha,\beta}{}^{(\tilde{\alpha})}_{(\tilde{\beta})}(x,\xi)| \leq C_{\alpha,\beta} M^{|\alpha|+|\beta|-1} W(x,\xi)^{|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}|} \langle\xi\rangle_N^{-|\alpha|-|\tilde{\alpha}|},$$

$$r_k(x,\xi) \in S(m(W/\langle\xi\rangle_N)^k, g)$$

if $N = M^{2-\delta}$ and $M \geq M_0$.

Remark. Proposition 2.2 was essentially proved in [13].

Let $t_0(x,\xi)$ be real-valued functions in $S^0_{1,0}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ such that $t_0(x,\xi)$ are positively homogeneous of degree 0, $t_0(x,\xi) = x_1 - x_1^0 + |x - x^0|^2 + |\xi/|\xi| - \xi^0|^2$ near z^0. Put

$$\Lambda(x,\xi) = (at_0(x,\xi) - b)\log\langle\xi\rangle(1 - \Theta_{h/4}(\xi))\psi(x,\xi),$$

where $\Theta(t) \in C_0^\infty(\mathbf{R})$ satisfies $\Theta(t) = 1$ if $|t| \leq 1$ and supp $\Theta \subset (-2,2)$, $\Theta_h(\xi) = \Theta(|\xi|/h)$, $\psi(x,\xi) \in C^\infty(T^*\mathbf{R}^n \setminus 0)$ is positively homogeneous of degree 0, $\psi(x,\xi) = 1$ in a conic neighborhood of z^0, $a \geq 1$, $b \in \Omega$ and $h \geq 1$. Roughly speaking, by virtue of results in [10], in order to prove Theorem 1.2 it suffices to obtain uniform microlocal *a priori* estimates in $\gamma \geq \gamma_0$ for $P^w_\Lambda(x, D - i\gamma\vartheta) \equiv (e^{-\Lambda})^w(x,D)P^w(x, D - i\gamma\vartheta)(e^\Lambda)^w(x,D)$, where $h = K\gamma$, and γ_0 and K are positive constants. In doing so, we put

$$Q^w_\Lambda(y, D; \gamma) = F_2 P^w_\Lambda(x, D - i\gamma\vartheta)F_1,$$

where F_1 and F_2 are the Fourier integral operators given in the assumptions. In order to get *a priori* estimates for $Q^w_\Lambda(y,D;\gamma)$ we use norms $\|(\widetilde{\Phi}^{-M})^w(x,D)u\|_{L^2}$, *i.e.*, we study $Q^w(y,D;\gamma) \equiv (\widetilde{\Phi}^{-M})^w(x,D)Q^w_\Lambda(y,D;\gamma)(\widetilde{\Phi}^M)^w(x,D)$. Then Proposition 2.2 admits us to calculate the symbol $Q(y,\eta;\gamma)$ of $Q^w(y,D;\gamma)$. After the calculation the procedure to prove Theorem 1.2 is almost the same as in [9] and [11]. However, the symbols appearing in the proof are not so good as in [9] and [11]. To apply Fefferman-Phong's inequality [1] we need more complicate discussions. For a detail of the proof we refer to [12].

3. Some remarks

We remarked that C^∞ well-posedness of the Cauchy problem for hyperbolic operators can be proved if microlocal *a priori* estimates are proved (see [10]). So, one can also prove well-posedness of the Cauchy problem if one can prove microlocal *a priori* estimates under other microlocal assumptions. For example, the Cauchy problem for $P^w(x, D)$ is C^∞ well-posed if $P^w(x, D)$ satisfies at least one of the conditions given in [11], [13] and here for every $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$.

For every $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$, choosing a suitable homogeneous canonical transformation χ from a conic neighborhood $\tilde{\mathcal{C}}$ of $(y^0, \eta^0) = (0, 0, \cdots, 0, 1)$ to a conic neighborhood $\mathcal{C}$ of z^0 and representing $p(\chi(y, \eta))$ in the form of (1.1), we shall give some examples which satisfy the condition $(\text{P-2})_{z^0}$ when χ satisfies $d\chi_{(y^0,\eta^0)}(0, \vartheta) \in \Gamma(p_{z^0}, (0, \vartheta))$. We note that $dp(z^0) = 0$ implies that $\lambda(y^0, \eta^{0\prime}) = \alpha(y^0, \eta^{0\prime}) = 0$.

Example 3.1. Let $f(s)$ be a function in $C^\infty(\mathbf{R}^d)$ such that $f(0) = 0$ and $f(s) \geq 0$, $\partial f/\partial s_j(s) \geq 0$ and $\sum_{j=1}^d s_j \partial f/\partial s_j(s) \leq Cf(s)$ if $0 \leq s_j \leq 1$ ($1 \leq j \leq d$), where $s = (s_1, s_2, \cdots, s_d)$ and $C \geq 0$. If $f(s)$ is a polynomial of s with non-negative coefficient, then $f(s)$ satisfies the above conditions. Let $t_j(y, \eta)$ ($1 \leq j \leq d$) be real-valued functions in $C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ which are positively homogeneous of degree 0 and satisfy $t_j(y^0, \eta^0) = 0$. Choose symbols $\alpha_j(y', \eta')$, $q_j(y, \eta')$ and $r_j(y, \eta')$ ($1 \leq j \leq d$) so that these are positively homogeneous of degree 0, $\alpha_j(y', \eta') \geq 0$, $q_j(y, \eta') > 0$, $r_j(y, \eta') \geq 0$ and $\alpha_j(y^{0\prime}, \eta^{0\prime}) r_j(y^0, \eta^{0\prime}) = 0$. Put

$$s_j(y, \eta') = \alpha_j(y', \eta')(q_j(y, \eta') t_j(y, 0, \eta')^2 + r_j(y, \eta')),$$
$$\alpha(y, \eta') = f(s_1(y, \eta'), \cdots, s_d(y, \eta'))\eta_n^2.$$

Then $(\text{P-2})_{z^0}$ is satisfied if

$$\frac{\partial t_j}{\partial y_1}(y^0, \eta^0) > 0 \quad \text{and} \quad q_{(y^0,\eta^0)}(-H_{t_j}(y^0, \eta^0)) > 0 \quad (1 \leq j \leq d), \tag{3.1}$$

where

$$q_{(y^0,\eta^0)}(\delta y,\delta\eta) = \delta\eta_1(\delta\eta_1 - \nabla_y\lambda(y^0,\eta^{0\prime})\cdot\delta y - \nabla_{\eta'}\lambda(y^0,\eta^{0\prime})\cdot\delta\eta')$$
$$-\sum_{j=1}^{d}\frac{\partial f}{\partial s_j}(0)\{\alpha_j(y^{0\prime},\eta^{0\prime})q_j(y^0,\eta^{0\prime})(\nabla_y t_j(y^0.\eta^0)\cdot\delta y + \nabla_{\eta'}t_j(y^0,\eta^0)\cdot\delta\eta')^2$$
$$+ r_j(y^0,\eta^{0\prime})(Hess\ \alpha_j(y^{0\prime},\eta^{0\prime}))(\delta y',\delta\eta')/2$$
$$+ \alpha_j(y^0,\eta^{0\prime})(Hess\ r_j(y^0,\eta^{0\prime}))(\delta y,\delta\eta')/2\},$$
$$(Hess\ r_j(y^0,\eta^{0\prime}))(\delta y,\delta\eta') = \sum_{k,\ell=1}^{n}\frac{\partial^2 r_j}{\partial y_k\partial y_\ell}(y^0,\eta^{0\prime})\delta y_k\delta y_\ell$$
$$+2\sum_{k=1}^{n}\sum_{\ell=2}^{n}\frac{\partial^2 r_j}{\partial y_k\partial\eta_\ell}(y^0,\eta^{0\prime})\delta y_k\delta\eta_\ell + \sum_{k,\ell=2}^{n}\frac{\partial^2 r_j}{\partial\eta_k\partial\eta_\ell}(y^0,\eta^{0\prime})\delta\eta_k\delta\eta_\ell.$$

Here (3.1) implies that $t_j(y,\eta)$ ($1\le j\le d$) are time functions for $p\circ\chi$ with respect to $(0,\vartheta)$ at (y^0,η^0).

Example 3.2. Let $n\ge 3$, and put

$$\alpha(y,\eta') = \left(y_1+\sqrt{y_2^2+y_n^2}\right)^2\left(y_1-\sqrt{y_2^2+y_n^2}\right)^2\eta_n^2\ (=(y_1^2-y_2^2-y_n^2)^2\eta_n^2).$$

Then $(\text{P-2})_{z^0}$ is satisfied if

$$|\frac{\partial\lambda}{\partial\eta_2}(y^0,\eta^{0\prime})|^2 + |\frac{\partial\lambda}{\partial\eta_n}(y^0,\eta^{0\prime})|^2 < 1. \tag{3.2}$$

Here we have chosen $t_1(y,\eta) = (y_1+\sqrt{y_2^2+y_n^2})\eta_n$ and $t_2(y,\eta) = (y_1-\sqrt{y_2^2+y_n^2})\eta_n$ which are Lipschitz continuous, and (3.2) implies that $t_j(y,\eta)$ ($j = 1,2$) are time functions for $p\circ\chi$ with respect to $(0,\vartheta)$ at (y^0,η^0).

Finally we shall give meaning of time functions. Applying the same arguments as in [9], we have the following

Theorem 3.3. *Let $z^0 = (x^0,\xi^0)\in\mathbf{R}^n\times S^{n-1}$, and let $P(x,\xi)$ be a symbol in S^m such that $p(x,\xi)$ is microhyperbolic with respect to $(0,\vartheta)\in\mathbf{R}^{2n}$ at z^0, where $p(x,\xi)$ denotes the principal symbol of $P(x,\xi)$. Assume that $(\text{P-2})_{z^0}$, $(\text{P-3})_{z^0}$ and at least one of the conditions $(\text{P-4-1})_{z^0}$ and $(\text{P-4-2})_{z^0}$ are satisfied. If $t(x,\xi)$ is a smooth time function for $p(x,\xi)$ with respect to $(0,\vartheta)$ at z^0, $z^0\notin WF(P^w(x,D)u)$ and $WF(u)\cap\{t(x,\xi)<0\}\cap C=\emptyset$ with some conic neighborhood C of z^0, then $z^0\notin WF(u)$.*

Remark. (i) Theorem 3.3 is a microlocal version of Hölmgren's uniqueness theorem. (ii) $(0, \vartheta)$ can be replaced by any non-zero vector in $\mathbf{R}^{2n}$. (iii) We can give the theorem in the form of Theorem 1.3 in [9].

Assume that the hypothese of Theorem 3.3 are satisfied for z^0 replaced by $z = (x, \xi) \in \Omega \cap \{|\xi| = 1\}$, where Ω is an open conic set in $T^*\mathbf{R}^n \setminus 0$ and contains z^0. Let $t(x, \xi)$ be a smooth time function for $p(x, \xi)$ with respect to $\widetilde{\vartheta} \in \mathbf{R}^{2n}$ in Ω, *i.e.*, $t(x, \xi)$ is a real-valued smooth function in $T^*\mathbf{R}^n \setminus 0$ and positively homogeneous of degree 0 and $-H_t(z) \in \Gamma(p_z, \widetilde{\vartheta})$ for $z \in \Omega$. If $WF(P^w(x, D)u) \cap \Omega = \emptyset$, and if u is not smooth at the present time (*i.e.*, $WF(u) \cap \{t(x, \xi) = 0\} \cap \Omega \neq \emptyset$), then u was not smooth in the past (*i.e.*, $WF(u) \cap \{t(x, \xi) < 0\} \cap \Omega \neq \emptyset$). So time functions give measure of time concerning propagation of singularities.

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