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Quasitraces on exact C^*-algebras are traces

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1. draft

Abstract

It is shown that quasitraces (more precisely 2-quasi-traces in the sense of [B+H]) are traces on all exact C^*-algebras. In particular it holds for all nuclear C^*-algebras and for subalgebras of nuclear C^*-algebras. As consequences one gets: (1) Every stably finite exact unital C^*-algebra has a trace state, and (2) If an AW^*-factor of type II_1 is generated (as an AW^* algebra) by an exact C^*-subalgebra, then it is a von Neumann II_1-factor. This is a partial solution to a well known problem of Kaplansky [Kap]. Moreover the present result is crucial for the proof of $RR(A)=0$ for every simple irrational rotation algebra A of any dimension given by Blackadar, Kumjian and Rørdam in [BKR].

List of sections:

2. An application of Voiculescu's semicircular systems

We shall need the following algebraic characterisation of unital C^*-algebras without trace states:

Lemma 2.1

Let A be a unital C^*-algebra. Then the following two conditions are equivalent:

(a) A has no trace state

(b) There is a finite set $\{a_1, \dots, a_n\} \subseteq A$, such that

$$\sum_{i=1}^n a_i^* a_i = 1 \quad \text{and} \quad \left\| \sum_{i=1}^n a_i a_i^* \right\| < 1$$

proof

(b) $\Rightarrow$ (a): Assume (b) and let τ be a trace state on A. Then $\tau(\sum_{i=1}^n a_i a_i^*) = \tau(\sum_{i=1}^n a_i^* a_i) = 1$, which contradicts that $\|\sum_{i=1}^n a_i a_i^*\| < 1$.

(a) $\Rightarrow$ (b). Assuming (a). Then the second dual A^{**} is a von Neumann algebra without normal trace states, i.e. A^{**} is a properly infinite von Neumann algebra. Hence, we can choose two isometries $v_1, v_2 \in A^{**}$ such that $v_1^* v_1 \perp v_2^* v_2$ and $v_1 v_1^* + v_2 v_2^* = 1$. Since $v_1^* v_1 + v_2^* v_2 = 2$. Choose a net $(b_1^{(\alpha)}, b_2^{(\alpha)})_{\alpha \in \Lambda}$ in $A \oplus A$ which converges to (v_1, v_2) in σ-strong*-topology. Then

$$\begin{cases} \sum_{i=1}^2 (b_i^{(\alpha)})^* b_i^{(\alpha)} \to \sum_{i=1}^2 v_i^* v_i = 2 & \sigma\text{-weakly} \\ \sum_{i=1}^2 b_i^{(\alpha)} (b_i^{(\alpha)})^* \to \sum_{i=1}^2 v_i v_i^* = 1 & \sigma\text{-weakly}. \end{cases}$$

Since the restriction of σ-weak topology on A^{**} to A is equal to the $\sigma(A, A^*)$-topology we get

$$\{2,1\} \in \overline{\left\{\left(\sum_{i=1}^{2} b_i^* b_i, \sum_{i=1}^{2} b_i b_i^*\right) \,\middle|\, b_1, b_2 \in A\right\}}^{\sigma(A\oplus A, A^*\oplus A^*)}$$

Since the set

$$\left\{\left(\sum_{i=1}^{n} b_i^* b_i, \sum_{i=1}^{n} b_i b_i^*\right) \,\middle|\, n \in \mathbb{N}, b_1 \ldots b_n \in A\right\}$$

is convex, and since convex sets in Banach spaces have the same closure in norm and weak topology, we get, that for all $\varepsilon > 0$ there is $n \in \mathbb{N}$, and $b_1 \ldots b_n \in A$ such that

$$\left\|\sum_{i=1}^{n} b_i^* b_i - 2\right\| \leq \varepsilon$$

$$\left\|\sum_{i=1}^{n} b_i b_i^* - 1\right\| \leq \varepsilon$$

Assume $\varepsilon = \frac{1}{3}$. Then $\frac{5}{3} \leq \sum_{i=1}^{n} b_i^* b_i \leq \frac{7}{3}$ ~~and $\frac{-2}{3} \leq$~~ $\sum b_i b_i^* \leq \frac{4}{3}$

~~Set~~

$$a_i = b_i \left(\sum_{i=1}^{n} b_i^* b_i\right)^{-1/2}$$

Then $\sum_{i=1}^{n} a_i^* a_i = 1$ and since

$$a_i a_i^* = b_i b_i \left(\sum_{j=1}^{n} b_j^* b_j\right)^{-1} b_i^* \leq \frac{3}{5} b_i b_i^*$$

we have

$$\left\|\sum_{i=1}^{n} a_i a_i^*\right\| \leq \frac{3}{5}\left\|\sum_{i=1}^{n} b_i b_i^*\right\| \leq \frac{4}{5} < 1,$$

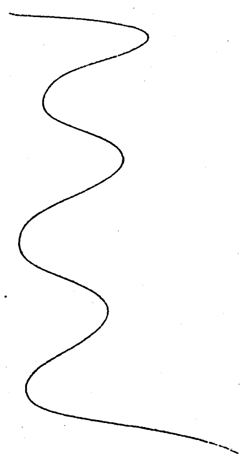
which proves (b).

Remarks 2.2

(1) By using an arbitrary number of isometries $(v_i)_{i=1}^{n}$ in the proof of (a) $\Rightarrow$ (b) one gets easily (a) $\Leftrightarrow$ (b') where:

(b') For all $\varepsilon > 0$ there is a finite set $\{a_1, \ldots a_n\} \subseteq A$, such that
$$\sum_{i=1}^{n} a_i^* a_i = 1 \quad \text{and} \quad \left\|\sum a_i a_i^*\right\| < \varepsilon$$

(2) It is possible to give a direct proof of (a) $\Leftrightarrow$ (c) without passing to A^{**}. See appendix (to be filled in)



2.4

In [V1] Voiculescu introduced the reduced free product $(A,\varphi) = *_{i\in I}(A_i,\varphi_i)$ of a family $(A_i)_{i\in I}$ ~~with~~ of unital C^*-algebras with respect to a specified set of states $(\varphi_i)_{i\in I}$, $\varphi_i \in S(A_i)$. $\varphi = *_{i\in I}\varphi_i$ is a state on A characterised by

$$\varphi(a_{i_1}a_{i_2}\cdots a_{i_n}) = 0$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $a_{i_k} \in A_{i_k}$ and $\varphi_{i_k}(a_{i_k}) = 0$.

A special case of interest is the semicircular system introduced in [V2]. Here

$$\begin{cases} A_i = C([-1,1]) \\ \tau_i(f) = \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt, \quad f \in C[-1,1] \end{cases}$$

for all i, let $x_i \in A_i \subseteq A$ be the identity function on $[-1,1]$. Then A is the C^*-algebra generated by 1 and $(x_i)_{i\in I}$, and $\tau = *_{i\in I}\tau_i$ is a faithful trace on A, and $(A,(x_i)_{i\in I},\tau)$ is a semicircular system in the sense of [V2].

A concrete model for $(A,(x_i)_{i\in I},\tau)$ can be obtained in the following way (cf [V1]):

Let H be a Hilbert space with orthonormal basis $(e_i)_{i\in I}$, and let

$$\mathcal{F}(H) = \mathbb{C} \oplus \Big(\sum_{n=1}^{\infty\oplus} \underbrace{H\otimes\cdots\otimes H}_{n}\Big)$$

be the full Fock space based on H, ~~and~~ let ℓ_i be the isometry $\mathcal{F}(H) \to \mathcal{F}(H)$ obtained by tensoring from the left by e_i on each $H^{\otimes n}$, $n=0,1,2,$ where $H^{\otimes 0} = \mathbb{C}$ for $n=0$.

$$(*)\quad \begin{cases} s_i^*s_i = 1 & \forall i \in I \\ s_is_i^* \perp s_js_j^* & \text{for all } i,j\in I,\ i\neq j \\ 1-\sum_{i\in I} s_is_i^* & \text{of the } \mathbb{C}\text{-component of } \mathcal{F}(H). \end{cases}$$

Then $x_i = \frac{1}{2}(s_i + s_i^*)$ generate a semicircular system and the trace state τ is simply the "vacuum-state", i.e. the vector state given by a unitvector in the $\mathbb{C}$-component of $\mathcal{F}(H)$ on $A = C^*\{(x_i)_{i\in I}, 1\}$.

If $I = \{1,\dots,n\}$ (resp. $I = \mathbb{N}$) we will denote the unital C^*-algebra generated by the x_i's by $\mathcal{V}_n$ (resp $\mathcal{V}_\infty$). By $(*)$ one has natural imbeddings

$$\begin{cases} \mathcal{V}_n \hookrightarrow \mathcal{E}_n, & n \in \mathbb{N} \\ \mathcal{V}_\infty \hookrightarrow \mathcal{O}_\infty \end{cases}.$$

where $\mathcal{E}_n$ denote the compact extension of the Cuntz algebra $\mathcal{O}_n$ given in [ ], and $\mathcal{O}_\infty$ is the usual Cuntz algebra generated by a sequence of isometries $(s_i)_{i=1}^{\infty}$. The trace τ on $\mathcal{V}_n$, $n\in\mathbb{N}$ (resp $\mathcal{V}_\infty$) is the restriction of the unique (pure) state φ on $\mathcal{E}_n$ (resp $\mathcal{O}_\infty$) which satisfies $\varphi(s_is_i^*) = 0$ $\forall i \in I$.

2.6.

<u>Lemma 2.3</u>

Let A be a unital C^*-algebra without trace states. Then $A \otimes \mathcal{V}_\infty$ contains a non-unitary isometry. ($A \otimes \mathcal{V}_\infty$ is the spatial (= minimal) C^*-tensorproduct)

<u>proof</u>.

By lemma 2.1 we can choose $a_1, \dots a_n \in A$, such that

$$\sum_{i=1}^n a_i^* a_i = 1 \quad \text{and} \quad \| \sum_{i=1}^n a_i a_i^* \| < 1$$

and let $(s_i)_{i=1}^\infty$ be as in (*), with $1 \in \mathbb{N}$. Then

$$\mathcal{O}_\infty = C^*((s_i)_{i=1}^\infty) \quad \text{and with } x_i = \tfrac{1}{2}(s_i + s_i^*),\ i \in \mathbb{N}$$

$$\mathcal{V}_\infty = C^*((x_i)_{i=1}^\infty, 1)$$

and $(x_i)_{i=1}^\infty$ is a semicircular system with respect to a faithful trace state τ on $\mathcal{V}_\infty$.

With the above notation

$$A \otimes \mathcal{V}_\infty \subseteq A \otimes \mathcal{O}_\infty$$

Set $\quad y = 2 \sum_{i=1}^n a_i \otimes x_i \in A \otimes \mathcal{V}_\infty$

Then $\quad y = v + w$, where $v, w \in A \otimes \mathcal{O}_\infty$ are given by

$$v = \sum_{i=1}^n a_i \otimes s_i, \quad w = \sum_{i=1}^n a_i \otimes s_i^* .$$

Since

$$s_i^* s_j = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases},$$

we have $\quad v^* v = \sum_{i=1}^n a_i^* a_i \otimes 1 = 1_{A \otimes \mathcal{O}_\infty}$.

i.e. v is an isometry. The range projection of v is clearly bounded by $1 \otimes \sum_{i=1}^n s_i s_i^*$. Thus v is a non-unitary isometry in $A \otimes \mathcal{O}_\infty$. Since

$$w^* = \sum_{i=1}^n a_i^* \otimes s_i .$$

we get as above

$$w w^* = \sum_{i=1}^n a_i a_i^* \otimes 1$$

so by the choice of $(a_i)_{i=1}^n$, we have $\|w\| < 1$.

We may assume that $A \otimes \mathcal{O}_\infty \subseteq B(K)$ for some Hilbert space K (unitpreserving imbedding).

For $\xi \in K$

$$\|y\xi\| = \|(v+w)\xi\| \geq \|v\xi\| - \|w\xi\| \geq (1 - \|w\|)\|\xi\|$$

Hence $y^* y$ is invertible. However

$$y = (1 + w v^*) v$$

and since $1 + wv^*$ is invertible, because $\|wv^*\| < 1$, while v is not invertible, it follows that y is not invertible. Set $u = y(y^*y)^{-1/2} \in A \otimes \mathcal{V}_\infty$, then

$$y = u(y^*y)^{1/2}$$

is the polar-decomposition of y, and $u^*u = 1$ while $uu^* \neq 1$. This completes the proof.

Theorem 2.4

Let A be a unital C^*-algebra without trace states. Then $A \otimes C_r^*(\mathbb{F}_\infty)$ contains a non-unitary isometry. (Here $\mathbb{F}_\infty$ = Free group on ~~infinitely~~ (countably) many generators)

proof

~~Note that $\mathbb{F}_\infty$ can be imbedded in $\mathbb{F}_2$: if a, b are generators for $\mathbb{F}_2$ then $\{a^n b a^{-n}, n \in \mathbb{N}\}$ are free generators of a subgroup $\simeq \mathbb{F}_\infty$). Hence $C_r^*(\mathbb{F}_\infty)$ can be imbedded in $C_r(\mathbb{F}_2)$ as a unital subalgebra.~~ Let $(u_n)_{n=1}^\infty$ be the unitary generators of $C_r^*(\mathbb{F}_\infty)$, and set

$$y_n = \tfrac{1}{2}(u_n + u_n^*)$$

Then $(y_n)_{n=1}^\infty$ is a free system in the sense of Voiculescu, $sp(y_n) = [-1,1]$ and the measure on $sp(y_n)$ given by the trace have density

$$g(t) = \frac{1}{\pi\sqrt{1-t^2}}, \quad t \in (-1,1)$$

obtained by projecting the uniform distribution on the unitcircle $\mathbb{T}$ onto the real axis. Let

$$G(t) = \int_{-1}^{t} g(t)\,dt$$

be the distribution function for $g(t)$, and let

$$F(t) = \int_{-1}^{t} \frac{2}{\pi}\sqrt{1-t^2}\,dt$$

be the distribution function for the semi-circular distribution. Then $\Phi = F^{-1} \circ G$ is a homeomorphism of $[-1,1]$ onto itself which transforms the measure given by density $g(t)$ onto the measure with density

$$f(t) = \frac{2}{\pi}\sqrt{1-t^2}, \quad -1 < t < 1$$

Hence $x_n = \Phi(y_n)$ form a semicircular system in the sense of [V2], so

$$\mathcal{V}_\infty \simeq C^*(1, x_1, x_2, \dots) \subseteq C_r^*(\mathbb{F}_\infty),$$

~~Altogether $\mathcal{V}_\infty$ can be imbedded as a unital C^*-subalgebra of $C_r^*(\mathbb{F}_2$~~). This extends to a unital imbedding of $A \otimes \mathcal{V}_\infty$ into $A \otimes C_r(\mathbb{F}_\infty)$ and the theorem now follows from lemma 2.3.

Remark 2.5

a) Since any free group $\mathbb{F}_n$ with at least 2 generators contains a copy of $\mathbb{F}_\infty$, $C_r^*(\mathbb{F}_\infty)$ has a unital imbedding in $C_r^*(\mathbb{F}_n)$, $n \geq 2$, so in cor. 2.4, $C_r^*(\mathbb{F}_\infty)$ can be exchanged by $C_r^*(\mathbb{F}_n)$ for any $n \geq 2$.

b) By choosing a continuous function $[-1,1] \to \mathbb{T}$ which transforms the semicircular distribution into the uniform distribution on $\mathbb{T}$ one gets that $C_r^*(\mathbb{F}_n)$ can be imbedded in $\mathcal{V}_n$ for any $n \geq 2$. Hence in lemma 2.3 $\mathcal{V}_\infty$ can be exchanged by $\mathcal{V}_n$ for any $n \geq 2$.

Remark 2.6

The algebras $\mathcal{V}_n$, $n \geq 2$ (including $n=\infty$) are simple with unique normalized traces. This can be proved as follows: $\mathcal{V}_n$ has a trace τ_1 for which $x_1, y_2, x_3, \dots$ is a semicircular system. The function $\Phi : [-1,1] \to \mathbb{T}$ in remark 2.5 can be chosen such that Φ is one-to-one except at the endpoints. In this way $C_r^*(\mathbb{F}_n)$ imbeds in $\mathcal{V}_n$ in a way, such that the G.N.S-representation given by τ of $C^*(\mathbb{F}_n)$ and $\mathcal{V}_n$ generates the same v.N. algebra $\mathcal{L}(\mathbb{F}_n)$. ~~Particularly~~

$$C^*(\mathbb{F}_n) \subseteq \mathcal{V}_n \subseteq \mathcal{L}(\mathbb{F}_n).$$

Now the Dixmier - averaging argument of Powers [ ] and Akemann-Ostrand [ ] works, and with minor modifications ~~of~~ their proofs one get the desired conclusion (details will be filled in later).

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3. Quasitraces on C^*-algebras and AW^*-algebras

Throughout this section A denotes a unital C^*-algebra. It has become customary to rename the 2-quasitraces of Blackadar and Handelman [BH] to quasitraces (see f.inst [R], ([BKR]):

Definition 3.1 A quasitrace τ on A is a function $\tau : A \to \mathbb{C}$ which satisfies.

(i) $\tau(x^*x) = \tau(xx^*) \geq 0$ for all $x \in A$

(ii) τ is linear on abelian C^*-subalgebras of A.

(iii) If $x = a+ib$, $a,b \in A_{s.a}$, then $\tau(x) = \tau(a) + i\tau(b)$

(iv) There is a function $\tau_2 : M_2(A) \to \mathbb{C}$ satisfying (i), (ii), (iii) such that

$$\tau(x) = \tau_2(x \otimes e_{11}), \quad x \in A.$$

A quasitrace is normalized if $\tau(1) = 1$, and the set of normalized quasitraces on A is denoted $QT(A)$.

Note that (i), (ii), (iii) corresponds to the quasitraces of [BH]. If A is an AW^*-algebra (i), (ii) and (iii) implies (iv), but it is not known whether it is true in general.

By [BH, Thm II.2.2] there is a bijection between $QT(A)$ and the set $\{SCDF(A)\}$ of lower continuous semicontinuous dimension function D on A (in the sense of Cuntz). The correspondance is given by

$$D(x) = \sup_{\varepsilon>0} \tau(f_\varepsilon(|x|)) \quad , x \in A$$

where

$$f_\varepsilon(t) = \begin{cases} 0 & 0 \le t \le \varepsilon/2 \\ \frac{2}{\varepsilon}t - 1, & \frac{\varepsilon}{2} < t < \varepsilon \\ 1 \quad , & t \ge \varepsilon \end{cases}$$

This correspandance, together with [BH, Thm. I.1.17] gives:

<u>proposition 3.2</u>
Let τ be a quasitrace on A, then

$$I = \{ x \in A \mid \tau(x^*x) = 0 \}$$

is a closed two-sided ideal in A and there is a (unique) quasitrace $\bar{\tau}$ on A/I, such that

$$\tau(x) = \bar{\tau}(g(x)) \quad , x \in A$$

where g denotes the quotient map.

By an ultrapower construction based on dimension functions Blackadar and Handelman shows that all quasitraces comes from AW^*-algebras in the following sense:

<u>proposition 3.3</u> ([BH, cor. II.2.4])
Let τ be a quasitrace on A. There exists a unital $*$-homomorphism Θ of A into a finite AW^*-algebra and a quasitrace $\bar{\tau}$ on M, such that

$$\tau(a) = \Theta \circ \bar{\tau}(a) \quad , a \in A.$$

By well known properties for quasitraces of AW^*-algebras, it follows that

<u>corollary 3.4</u> (BH, Sect II)
Let τ be a quasitrace on A. Then
(1) τ is orderpreserving on A_{sa}.
(2) τ extends uniquely to a quasitrace τ_n on $M_n(A)$, s.t. $\tau_n(x \otimes e_{11}) = \tau(x)$, $x \in A$ ($\forall n \in \mathbb{N}$).

<u>lemma 3.5</u> Let τ be a quasitrace on A, and set

$$\|x\|_2 = \tau(x^*x)^{1/2} \quad , x \in A.$$

Then
(1) $\tau(a+b)^{1/2} \le \tau(a)^{1/2} + \tau(b)^{1/2}$, $a, b \in A_+$
(2) $\|x+y\|_2^{2/3} \le \|x\|_2^{2/3} + \|y\|_2^{2/3}$, $x, y \in A$.
(3) $\|xy\|_2 \le \|x\| \|y\|_2$ and $\|xy\|_2 \le \|x\|_2 \|y\|$, $x, y \in A$.

<u>proof</u> (1) follows by a slight modification of the proof of [B+H, Cor. II 1.11]: Set

$$x = a^{1/2} \otimes e_{11} + b^{1/2} \otimes e_{21} \in M_2(A)$$

Then

$$x^*x = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad xx^* = \begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix}$$

Moreover, for $\lambda > 0$, set

$$x_\lambda = \lambda^{1/2} a^{1/2} \otimes e_{11} - \lambda^{-1/2} b^{1/2} \otimes e_{21}$$

Then

$$xx^* \le xx^* + x_\lambda x_\lambda^* = \begin{pmatrix} (1+\lambda)a & 0 \\ 0 & (1+\lambda^{-1})b \end{pmatrix}$$

Hence by (i), (iv) of def 3.1 and corollary 3.4,

$$\tau(a+b) = \tau_2(x^*x) = \tau_2(xx^*) \le \tau_2\begin{pmatrix} (1+\lambda)a & 0 \\ 0 & (1+\lambda^{-1})b \end{pmatrix}$$

Since $a \otimes e_{11}$ and $b \otimes e_{22}$ commutes in $M_2(A))$, we get by (ii) and (iv) of def 3.1,

$$(*) \quad \tau(a+b) \le (1+\lambda)\tau_2(a\otimes e_{11}) + (1+\tfrac{1}{\lambda})\tau_2(b\otimes e_{22})$$
$$= (1+\lambda)\tau(a) + (1+\tfrac{1}{\lambda})\tau(b).$$

The last equally follows because $y b\otimes e_{22} = y^*y$ and $b\otimes e_{11} = yy^*$ for $y = b^{1/2}\otimes e_{12}$. ~~If~~

If $\tau(a)>0$ and $\tau(b)>0$, the right side of $(*)$ has minimum at $\lambda = (\tau(b)/\tau(a))^{1/2}$ and the minimum value is $(\tau(a)^{1/2}+\tau(b)^{1/2})^2$ proving (1) in this case. If $\tau(a)=0$ (resp. $\tau(b)=0$), then (1) follows trivial by letting $\lambda\to\infty$ (resp. $\lambda\to 0$).

(2) Let $x,y\in A$. For $\lambda>0$

$$(x+y)^*(x+y) \le (x+y)^*(x+y) + (\lambda^{\frac12}x-\lambda^{-\frac12}y)^*(\lambda^{\frac12}x-\lambda^{-\frac12}y)$$
$$= (1+\lambda)x^*x + (1+\tfrac{1}{\lambda})y^*y$$

Hence by (1):

$$\|x+y\|_2 \le (1+\lambda)^{\frac12}\|x\|_2 + (1+\tfrac{1}{\lambda})^{\frac12}\|y\|_2$$

If $\|x\|_2>0$ and $\|y\|_2>0$ the right side has minimum at $\lambda = (\|y\|_2/\|x\|_2)^{2/3}$ and the minimum value is $(\|x\|_2^{2/3}+\|y\|_2^{2/3})^{3/2}$ proving (2) in this case. The remaining cases $\|x\|_2=0$ or $\|y\|_2=0$ follows by letting $\lambda\to\infty$ or $\lambda\to 0$.

(3) Since $y^*x^*xy \le \|x\|^2y^*y$ the first inequality follows from corollary 3.4, and the second now follows by using $\|z\|_2 = \|z^*\|_2$, $z\in A$.

<u>Definition 3.6</u>

If τ is a faithful quasitrace on A we set

$$d_\tau(x,y) = \|x-y\|_2^{2/3}, \quad x,y\in A$$

Then d_τ is a metric on A by lemma 3.5 (2).

<u>Lemma 3.7</u>

Let τ be a faithful quasitrace on A. Then

(1) The involution $x\to x^*$ is continuous in d_τ-metric

(2) The sum is continuous in d_τ-metric on A

(3) The product is continuous in d_τ-metric on bounded sets of A

(4) $x\to\tau(x)$ is continuous in d_τ-metric on A_+.

<u>proof</u>

(1) Clear since $\|x\|_2 = \|x^*\|_2$, $x\in A$.

(2) Clear from lemma 3.5 (2)

(3) For $x,y,x_0,y_0\in A$, lemma 3.5 (2), and (3) give

$$\|xy-x_0y_0\|_2^{2/3} \le \|x(y-y_0)\|_2^{2/3} + \|(x-x_0)y_0\|_2^{2/3}$$
$$\le \|x\|^{2/3}\|y-y_0\|_2^{2/3} + \|y_0\|^{2/3}\|x-x_0\|_2^{2/3}$$

This proves (3)

(4). For $a, b \in A_+$

$$a \le b + |a-b| \quad \text{and} \quad b \le a + |a-b|$$

Thus by lemma 3.4 (ii)

$$|\tau(a)^{1/2} - \tau(b)^{1/2}| \le \tau(|a-b|)^{1/2},$$

and since τ is linear on $C^*(|a-b|, 1)$ we get

$$|\tau(a)^{1/2} - \tau(b)^{1/2}| \le \tau(|a-b|^2)^{1/4}\tau(1)^{1/4} = \|a-b\|_2^{1/2}\tau(1)^{1/4}$$

This proves (4).

<u>Lemma 3.8</u>

Let τ be a faithful quasitrace on A. Then the unitball of A is closed in the d_τ-metric

<u>proof</u>

Assume x_n be a sequence in the unitball of A, and that $x_n \to x \in A$ in d_τ-metric. Set $a_n = x_n^* x$ and $a = x^* x$. By lemma 3.7,

$$(*) \qquad \tau(a_n^p) \to \tau(a^p), \qquad p = 0, 1, 2, \ldots$$

Let μ_n (resp μ) be the measure on $sp(a_n)$, resp $sp(a)$ given by the linear functional $\tau|_{C^*(a_n,1)}$ (resp $\tau|_{C^*(a,1)}$). All the measures can be considered as measures in the interval $J = [0, \max\{1, \|a\|\}]$, because $\|a_n\| \le 1$ for all n. Hence by $(*)$ $\mu_n \to \mu$ in the w^*-topology on $C(J)^*$. Since μ_n has support in $[0,1]$, μ also have support in $[0,1]$, and since τ is faithful $\operatorname{supp}(\mu) = sp(a)$. Hence $\|x\|^2 = \|a\| \le 1$ ∎

<u>Lemma 3.9</u>

Let τ be a faithful quasitrace on A. If the unitball of A is complete in d_τ-metric, then A is an AW^*-algebra and τ is a normal quasitrace on A, i.e.

$$\tau(\text{LUB}\, p_i) = \sum_{i \in I} \tau(p_i)$$

for any orthogonal set of projections $(p_i)_{i \in I}$ in A.

<u>proof</u>

Let B be a maximal abelian C^*-subalgebra of A. ~~By def 3.1 (ii), τ_B is a positive linear faithful functional on B~~ By lemma 3.8, the unitball of B is closed in d_τ-metric and hence also complete in d_τ-metric by the assumption on A. Since τ_B is a positive linear functional on B, $\|x-y\|_2 = d_\tau(x,y)^{3/2}$ is an equivalent metic on B, and completeness of unitball B in the $\| \ \|_2$-norm associated with the ~~faithful~~ positive functional implies that B is a W^*-algebra and that τ is a normal on B. This clearly implies that A is an AW^*-algebra, and that

$$\tau(\text{LUB}\, p_i) = \sum_{i \in I} \tau(p_i)$$

for every orthogonal set of projections $(p_i)_{i \in I}$ in A

The converse of lemma 3.9 is also true:

Proposition 3.10

Let M be a finite AW^*-algebra with a normal faithful quasitrace τ. Then the unitball of M is complete in d_τ-metric.

proof

Let D be the central valued dimension function on M. Since $\tau = \tau_{|Z(M)} \circ D$, $\tau(e \vee f) \leq \tau(e) + \tau(f)$ projections $e, f \in M$, and the normality of τ ensures that also

$$\tau\left(\bigvee_{n=1}^{\infty} e_n\right) \leq \sum_{n=1}^{\infty} \tau(e_n)$$

for any sequence of projections in M.

We prove first that the unitary group $U(M)$ is complete in d_τ-metric. Let $(u_n)_{n=1}^{\infty}$ be a Cauchy sequence of unitaries in d_τ-metric. By passing to a subsequence we may assume that

$$d_\tau(u_n, u_{n+1})^{3/2} = \|u_n - u_{n+1}\|_2 < 4^{-n}, \quad n \in \mathbb{N}.$$

Set

$$e_n = \chi_{[0, 2^{-n}]}(|u_n - u_{n+1}|)$$

Then $\|(u_n - u_{n+1}) e_n\| \leq 2^{-n}$

and since $e_n^{\perp} \leq 2^n |u_n - u_{n+1}|$ we have

$$\tau(e_n^{\perp}) \leq 2^n \tau(|u_n - u_{n+1}|)$$
$$\leq 2^n \tau(1)^{1/2} \|u_n - u_{n+1}\|_2$$
$$< 2^{-n} \tau(1)^{1/2}$$

Here we have used the linearity of τ on $C^*(|u_n - u_{n+1}|, 1)$.

Set $f_n = \bigwedge_{k \geq n} e_k$. Then

$$\tau(f_n^{\perp}) \leq \sum_{k=n}^{\infty} \tau(e_k^{\perp}) < 2^{1-n} \tau(1)^{1/2}.$$

For all $k \geq n$

$$\|(u_k - u_{k+1}) f_n\| \leq \|(u_k - u_{k+1}) e_k\| \leq 2^{-k}$$

Hence, $\sum_{k=n}^{\infty} (u_{k+1} - u_k) f_n$ converges in C^*-norm, i.e. Equivalently

$$(*) \qquad v_n = \lim_{k \to \infty} u_k f_n$$

exists in C^*-norm for all n. Moreover

$$v_n^* v_n = \lim_{k \to \infty} f_n u_k^* u_k f_n = f_n$$

Therefore v_n is a partial isometry, and since $f_1 \leq f_2 \leq \dots$ we have from $(*)$

$$v_n f_m = v_m, \quad n \geq m.$$

Set $v_0 = 0$ and $f_0 = 0$. Then $w_n = v_n - v_{n-1}$ is sequence of partial isometries with ortogonal supports and ortogonal ranges. By [Kap], there is a partial isometry $w \in M$, s.t.

$$w_n = w(f_n - f_{n-1}) \quad \text{for all } n \in \mathbb{N}$$

and such that

$$w^* w = \mathrm{LUB}_n(w_n^* w_n), \quad w w^* = \mathrm{LUB}_n(w_n w_n^*)$$

Since $w_n^* w_n = f_n - f_{n-1}$, and since $\tau(f_n^{\perp}) \to 0$ for $n \to \infty$ $w^* w = 1$, so also $w w^* = 1$ by finiteness of M.

Note that $v_n = \sum_{k=1}^{n}(v_k - v_{k-1}) = w f_n$ for all $n \in \mathbb{N}$.

Hence by (*)

$$\lim_{k\to\infty} \|(u_k - w) f_n\| = 0, \quad n \in \mathbb{N}.$$

so also

$$\lim_{k\to\infty} \|(u_k - w) f_n\|_2 = 0, \quad n \in \mathbb{N}$$

Let $\varepsilon > 0$ and choose n, such that $\tau(f_n^{\perp}) < \varepsilon$.
By lemma 3.5 (3)

$$\|(u_k - w) f_n^{\perp}\|_2 \leq 2\|f_n^{\perp}\|_2 < 2\varepsilon^{1/2}.$$

so by lemma 3.5 (2)

$$\limsup_{k\to\infty} \|u_k - w\|_2 < 2\varepsilon^{1/2}.$$

Hence u_k converges to w in d_τ-metric, proving the completeness of $U(M)$ in d_τ-norm.

We next prove, that the selfadjoint part of the unitball $(M_{s.a})_1$ is complete in d_τ-metric. Let a_n be a d_τ-Cauchy sequence in $(M_{s.a})_1$. Let u_n be the Cayley transformed of a_n:

$$u_n = (a_n + i1)(a_n - i1)^{-1} \in U(M)$$

Then

$$u_n - u_m = 2(a_n - i1)^{-1}(a_m - a_n)(a_m - i1)^{-1}$$

so by lemma 3.5 (3)

$$\|u_n - u_m\|_2 \leq 2\|a_m - a_n\|_2.$$

so u_n converges in d_τ to a unitary $u \in U(H)$.
Since $sp(a_n) \subseteq [-1,1]$, $sp(u_n) \subseteq \{t \in \mathbb{T} \mid \mathrm{Re}\, t \leq 0\}$
Hence $\|1 + u_n\| \leq \sqrt{2}$, and so $\|1 + u\| \leq \sqrt{2}$ by lemma 3.8,
i.e. $sp(u) \subseteq \{t \in \mathbb{T} \mid \mathrm{Re}\, t \leq 0\}$. Let

$$a = i(u+1)(u-1)^{-1}.$$

be the inverse Cayley transformed of u. Then

$$sp(a) \subseteq [-1,1]$$

Hence $a \in (M_{s.a})_1$. Since $a_n = i(u_n+1)(u_n-1)^{-1}$, we have

$$a_n - a = 2i(u_n - 1)^{-1}(u - u_n)(u-1)^{-1}$$

By the conditions on $sp(u_n)$ and $sp(u)$,

$$\|(u_n - 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad \|(u-1)^{-1}\| \leq \frac{1}{\sqrt{2}}$$

Hence, using lemma 3.5 (3)

$$\|a_n - a\|_2 \leq \|u - u_n\|_2 \to 0 \quad \text{for } n \to \infty$$

proving d_τ-completeness of $(M_{s.a})_1$. Finally if x_n is a d_τ-Cauchy net in M_1, the closed unitball of M, then $a_n = \frac{1}{2}(x_n + x_n^*)$ and $b_n = \frac{1}{2i}(x_n - x_n^*)$ are d_τ-Cauchy nets in $(M_{s.a.})_1$ by lemma 3.5 (2). Hence by the d_τ-completeness of $(M_{s.a})_1$ and by lemma 3.6, $x_n = a_n + i b_n$ is convergent in d_τ-norm. Moreover by lemma 3.8 the limit is also in M_1. This completes the proof. ■

We need the following version of "Kaplansky's Density Theorem"

Lemma 3.11

Let A be a unital C^*-algebra with a faithful quasitrace τ, and let B be a unital C^*-subalgebra. Then, the following two conditions are equivalent

(1) B is dense in A in d_τ-metric.

(2) B_1 is dense in A_1 in d_τ-metric.

Here A_1 and B_1 denotes the norm-closed unitballs of A and B respectively.

proof

(2) $\Rightarrow$ (1) trivial

(1) $\Rightarrow$ (2) This follows essentially the proof of the "classical" Kaplansky theorem:

Consider the real function

$$f(t) = \frac{2t}{1+t^2}, \quad t \in \mathbb{R}$$

Then $|f(t)| \leq 1$ for all $t \in \mathbb{R}$, and the restriction of f to $[-1,1]$ is a homeomorphism of $[-1,1]$. Let

$$g : [-1,1] \to [-1,1]$$

be the inverse of this function. Note that

$$f(-t) = -f(t), \quad t \in \mathbb{R}$$
$$g(-t) = -g(t), \quad t \in [-1,1]$$

Assume (1), and let $x \in A_1$. Set

$$a = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(A)_{sa})_1$$

Since g is an odd function $b = g(a)$ is of the form

$$b = g(a) = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}$$

for some $y \in A_1$. Moreover, since $a = f(b)$,

$$x = 2y(1+y^*y)^{-1} = 2(1+yy^*)^{-1}y$$

Choose a sequence $y_n \in A$, s.t. $\|y_n - y\|_2 \to 0$, and set

$$x_n = 2y_n(1+y_n^*y_n)^{-1} \in B.$$

Then $x_n^*x_n = 4y_n^*y_n(1+y_n^*y_n)^{-1} \leq 1$

because $\sup_{s \geq 0} 4s(1+s)^{-2} = 1$. Hence $x_n \in B_1$. Moreover

$$x_n - x = 2(1+yy^*)^{-1}((1+yy^*)y_n - y(1+y_n^*y_n))(1+y_n^*y_n)^{-1}$$
$$= 2(1+yy^*)^{-1}(y_n - y)(1+y_n^*y_n)^{-1} +$$
$$2(1+yy^*)^{-1}y(y^* - y_n^*)y_n(1+y_n^*y_n)^{-1}$$

Since $(1+yy^*)^{-1}$, $(1+y_n^*y_n)^{-1}$, $2(1+yy^*)^{-1}y$ and $2y_n(1+y_n^*y_n)$ all have C^*-norm at most 1, Lemma 3.5 yields

$$\|x_n - x\|_2^{2/3} \leq 2^{2/3}\|y_n - y\|_2^{2/3} + 2^{-2/3}\|y_n^* - y^*\|_2^{2/3}$$
$$= (2^{2/3} + 2^{-2/3})\|y_n - y\|_2^{2/3}$$
$$\to 0 \quad \text{for } n \to \infty$$

Hence x is in the d_τ-closure of B_1.

Proposition 3.12

Let M be a finite AW^*-algebra with a faithful normal quasitrace τ, and let A be a unital C^*-subalgebra of M. Then the d_τ-closure of A in M is the smallest AW^*-subalgebra of M containing A.

proof

Let B be the d_τ-closure of A. By lemma 3.7, B is a unital C^*-subalgebra of M (note that norm-convergence implies τ-convergence). By lemma 3.8 and lemma 3.11, B_1 is the d_τ-closure of A_1. Hence by proposition 3.10 applied to M, B_1 is complete in d_τ-metric, so by lemma 3.9, B is an AW^*-algebra in its own right. To be a AW^*-subalgebra however we also require, that if $p = LUB(p_i)$ of a set of ortogonal projections $(p_i)_{i \in I}$ in B it is contained in B when the LUB is computed the projection lattice of M.*) However this is clearly true, because p is the d_τ-limit of the net $\left(\sum_{i \in J} p_i\right)_{J \in \mathcal{J}}$ where $\mathcal{J}$ is the family of finite subsets of I (cf. proof of lemma 3.11). Hence B is an AW^*-subalgebra of M. Conversely if C is an AW^*-subalgebra of M containing A, then by prop. 3.10, C_1 is d_τ-complete so by lemma 3.11 C is d_τ-closed. Hence $C \supseteq B$. ∎

*) cf [Ber, ...]

4. Ultraproducts and AW^*-completions

The following lemma is probably well known. For completeness we include a proof.

Lemma 4.1

Let $(X_n, d_n)_{n=1}^{\infty}$ be a sequence of metric spaces with a uniform bound on $\operatorname{diam}(X_n)$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Define an equivalens relation $\sim$ on $X = \prod_{n=1}^{\infty} X_n$ by

$$x \sim y \iff \lim_{\mathcal{U}} d_n(x_n, y_n) = 0$$

Then $X/\sim$ is a complete metric space in the metric

$$d([x],[y]) = \lim_{\mathcal{U}} d_n(x_n, y_n)$$

proof

Define

$$\bar{d}(x,y) = \lim_{\mathcal{U}} d_n(x_n, y_n), \quad x, y \in X.$$

then $\bar{d}$ induces a metric on $X/\sim$ by

$$d([x],[y]) = \bar{d}(x,y).$$

Let $(z_i)_{i=1}^{\infty}$ be a Cauchy sequence in $X/\sim$. To prove convergence of $(z_i)_{i=1}^{\infty}$ is suffices to prove, that $(z_i)_{i=1}^{\infty}$ has a convergent subsequence. Hence, we may assume

$$d(z_i, z_{i+1}) < 2^{-i}, \quad i \in \mathbb{N}$$

Choose $x^{(i)} = (x_n^{(i)})_{n \in \mathbb{N}}$ in X, such that $z_i = [x^{(i)}]$. Since

$$\lim_{n,\mathcal{U}} d(x_n^{(i)}, x_n^{(i+1)}) < 2^{-i}$$

we can choose sets

$$F_1 \supset F_2 \supset \dots \supset F_i \supset$$

in $\mathcal{U}$, such that

$$d(x_n^{(i)}, x_n^{(i+1)}) < 2^{-i} \quad \forall n \in F_i .$$

Since $\mathcal{U}$ is free we can exchange F_i by $F_i \cap \{i, i+1, \dots\}$ to obtain that also

$$\bigcap_{i=1}^{\infty} F_i = \emptyset$$

Set $F_0 = \mathbb{N}$, and note that $\mathbb{N}$ is the disjoint union of $(F_{i-1} \setminus F_i)_{i=1}^{\infty}$. Hence we can define $x = (x_n)_{n=1}^{\infty} \in X$ by

$$x_n = x_n^{(i)}, \quad n \in F_{i-1} \setminus F_i$$

Let $n \in F_i$. Then $n \in F_{j-1} \setminus F_j$ for some $j > i$. For this j,

$$d_n(x_n^{(i)}, x_n) = d_n(x_n^{(i)}, x_n^{(j)}) \leq \sum_{k=i}^{j-1} d(x_n^{(k)}, x_n^{(k+1)}) < 2^{1-i} .$$

Since $F_i \in \mathcal{U}$, $d([x^{(i)}], [x]) \leq \sup_{n \in F_i} d_n(x_n^{(i)}, x_n) \leq 2^{1-i}$.

Therefore $z_i = [x^i]$ converges to $[x]$ in $X/\sim$.

If $(A_n)_{n=1}^{\infty}$ is a sequence of C^*-algebras, we set $\ell^{\infty}\{A_n\} = \{(x_n)_{n=1}^{\infty} \mid x_n \in A, \sup \|x_n\| < \infty\}$. If $A_n = A$ (fixed) for all n, we write $\ell^{\infty}(A)$ instead.

<u>Proposition 4.2</u>
Let $(A_n, \tau_n)_{n=1}^{\infty}$ be a sequence of unital C^*-algebras with normalized quasi-traces τ_n, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Set

$$J_{\mathcal{U}} = \{(x_n)_{n=1}^{\infty} \in \ell^{\infty}\{A_n\} \mid \lim_{\mathcal{U}} \tau_n(x_n^* x_n) = 0\}.$$

Then $J_{\mathcal{U}}$ is a norm-closed two-sided ideal in $\ell^{\infty}\{A_n\}$, and $\ell^{\infty}\{A_n\}/J_{\mathcal{U}}$ is a finite AW^*-algebra with normal faithful quasitrace $\tau_{\mathcal{U}}$ given by

$$(*) \quad \tau_{\mathcal{U}}([x]) = \lim_{\mathcal{U}} \tau_n(x_n), \quad x = (x_n)_{n=1}^{\infty} \in \ell^{\infty}\{A_n\}.$$

<u>proof</u>
It is no loss of generality to assume that each τ_n is faithful. Otherwise we can exchange A_n by A_n/I_n, where

$$I_n = \{x \in A_n \mid \tau_n(x^* x) = 0\}$$

(cf. prop. 3.2). It is clear, that

$$\tilde{\tau}_{\mathcal{U}}(x) = \lim_{\mathcal{U}} \tau_n(x_n), \quad x = (x_n)_{n=1}^{\infty} \in \ell^{\infty}\{A_n\}$$

defines a quasitrace on $\ell^{\infty}\{A_n\}$, so by proposition 3.2, $I_{\mathcal{U}}$ is a norm-closed two sided ideal in $\ell^{\infty}\{A_n\}$, and there is a faithful quasi-trace $\tau_{\mathcal{U}}$ on $\ell^\infty\{A_n\}/I_{\mathcal{U}}$, such that (*) holds. Since $*$-homomorphism of a C^*-algebra onto a C^*-algebra B maps the closed unitball of A onto the closed unitball of B, get from def. 3.6 and lemma 4.1, that the unitball of $\ell^\infty\{A_n\}/I_{\mathcal{U}}$ is complete in the metric associated with $\tau_{\mathcal{U}}$. Hence lemma 3.9 completes the proof of proposition 4.2 ∎

The following is a slight extension of [BH, Cor II.2.4]

Corollary 4.3

Let A be unital C^*-algebra with a faithful quasitrace τ. Then there is a one-to-one $*$-homomorphism π of A into a finite AW^*-algebra M with a faithful normal quasitrace, $\bar{\tau}$ such that

$$\tau(x) = \bar{\tau}\circ\pi(x), \quad x \in A.$$

proof

Set $A_n = A$ for all n, and apply prop. 4.2. The $*$-homomorphism π is given by

$$\pi(x) = [(x)_{n=1}^\infty]$$

Let A and M be as in corollary 4.3. Then by prop. 3.12 the closure B of $\pi(A)$ in d_τ-metric is the smallest AW^*-subalgebra of M containing A. Moreover by lemma 3.11, every element of B is the d_τ-limit of a bounded sequence in $\pi(A)$. Since for every $t>0$, the t-ball of B is d_τ-complete by prop. 3.10 B is equal to the quotient C^*-algebra

$$B = \tilde{A}/\tilde{I}$$

where

$$\tilde{A} = \{(x_n)_{n=1}^\infty \in \ell^\infty(A) \mid x_n \text{ is a } d_\tau\text{-Cauchy sequence}\}$$

and

$$\tilde{I} = \{(x_n)_{n=1}^\infty \in \ell^\infty(A) \mid x_n \to 0 \text{ in } d_\tau\text{-metric}\}.$$

and the restriction of $\bar{\tau}$ to $B = \tilde{A}/\tilde{I}$ is given by

$$(*) \quad \bar{\tau}(\dot{x}) = \lim_{n\to\infty} \tau(x_n) \quad \text{for } x = (x_n) \in \tilde{A}.$$

Indeed (*) follows from lemma 3.7(4) when $x \geq 0$ and by def 3.1 (ii) and (iii) for general $x \in \tilde{A}$. In particular we have:

4.6

Proposition 4.4

Let A be a unital C^*-algebra with a faithful quasitrace τ. Let $(\pi, M, \bar\tau)$ and $(\pi^1, M^1, \bar\tau^1)$ be two triples satisfying the conditions of corollary 3.7 and let B (resp B^1) be the AW^*-subalgebra of M (resp M^1) generated by $\pi(A)$ (resp $\pi^1(A)$). Then there is a unique $*$-isomorphism

$$\rho : B \xrightarrow{\text{onto}} B^1$$

such that $\pi^1 = \rho \circ \pi$ and $\bar\tau = \bar\tau^1 \circ \rho$.

Proof

With the notation preceding prop. 4.4 both B and B^1 are naturally isomorphic to $\tilde A / \tilde I$.

Definition 4.5

Let A be a unital C^*-algebra with a faithful quasitrace τ and let $B = \tilde A/\tilde I$ be the finite AW^*-algebra described prior to prop. 4.4 with normal faithful quasitrace

$$\bar\tau([x]) = \lim_{n\to\omega} \tau(x_n)$$

We call $(B, \bar\tau)$ the AW^*-completion of (A, τ).

1.7

Proposition 4.6

Let τ be a faithful normalized quasitrace on a unital C^*-algebra A. If τ is an extreme point in $QT(A)$, then the AW^*-completion of (A,τ) is a finite AW^*-factor.

Proof

The W^*-version of this is well known, and the proof for the above case is the same: Indeed if the AW^*-completion $(B, \bar\tau)$ is not a factor, then choose a central projection $p \in B$, $p \neq 0$, $p \neq 1$. Let π be the imbedding of A in B. Since $\pi(A)$ is $d_{\bar\tau}$-dense in B it follows easily that $\tau = \tau_1 + \tau_2$, where τ_1, τ_2 are the quasitraces

$$\left.\begin{aligned} \tau_1(x) &= \bar\tau(p\,\pi(x)), \\ \tau_2(x) &= \bar\tau((1-p)\,\pi(x)) \end{aligned}\right\} \quad x \in A$$

and $\tau_1 \neq 0$, $\tau_2 \neq 0$. By normalizing τ_1 and τ_2 we get that τ is a non-trivial convex combination of elements from $QT(A)$, which contradicts that τ is extreme ∎

5.1

1. The main result.

Recall that a C^*-algebra A is exact if for all pairs (B, J) of a C^*-algebra B and a closed two sided ideal J in B,

$$0 \to A \otimes J \to A \otimes B \to A \otimes B/J \to 0$$

is exact. Here the tensorproduct $\otimes$ denotes the spatial (= minimal) tensor product of the C^*-algebras involved (cf. [K1]). It is well known, that nuclear C^*-algebras and subalgebras of nuclear algebras are exact. Recently Kirchberg proved, that the class of exact C^*-algebras coincide with the class of quotients of subalgebras of exact C^*-algebras. In particular:

Proposition 5.1 [K2]

Any C^*-quotient of an exact C^*-algebra is exact.

Proposition 5.2

$C^*_r(\mathbb{F}_n)$ is an exact C^*-algebra for any $n \in \mathbb{N}, n \geq 2$ and for $n = \infty$.

proof

This is well known. The case $n=2$ is in [EH,] and the general case follows easily because $\mathbb{F}_n$ can be imbedded in $\mathbb{F}_2$ for all $n \geq 2$ including $n = \infty$. One can also use [DCH, §6] to get that $C^*_r(\Gamma)$ is exact for any discrete subgroup Γ of $SL(2, \mathbb{R})$, in particular for $\Gamma = \mathbb{F}_n$.

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Remark 5.3

A. Connes has shown that $C^*_r(\Gamma)$ is exact for any discrete subgroup of a connected Lie group. (unpublished). This was brought to our attention by G. Skandalis.

Definition 5.4

For any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, set

$$I_{\mathcal{U}} = \{(x_n)_{n=1}^{\infty} \in \ell^{\infty}\{M_n(\mathbb{C})\} \mid \lim_{\mathcal{U}} tr_n(x_n^* x_n) = 0\},$$

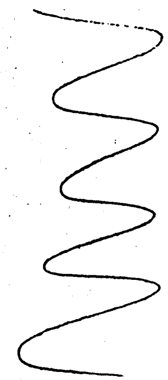
where tr_n is the normalized trace on $M_n(\mathbb{C})$, and set

$$M_{\mathcal{U}} = \ell^{\infty}\{M_n(\mathbb{C})\} / I_{\mathcal{U}} .$$

It is well known (see f.inst. [Mc.D]), that $M_{\mathcal{U}}$ is a II_1-factor with normal trace:

$$\tau_{\mathcal{U}}([x]) = \lim_{\mathcal{U}} \tau_n(x_n) \quad , \quad x = (x_n)_{n=1}^{\infty} \in \ell^{\infty}\{M_n(\mathbb{C})\}$$

Here ...



We shall need the following result of Wassermann

Proposition 5.5 [W]

Let Γ be a residually finite countable discrete ICC-group. Then the II_1-factor $\mathcal{M}(\Gamma)$ associated with the left regular representation of Γ is isomorphic to a subfactor of $M_{\mathcal{U}}' = \ell^\infty\{M_n(\mathbb{C})\}/I_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Particular $\mathcal{M}(\mathbb{F}_n)$ has this property for $n=2,3,\dots$ and $n=\infty$. (As usual, $\mathbb{F}_n$ denotes the free group on n generators)

Lemma 5.6

Let τ be a normalized quasitrace on a unital C^*-algebra A and let τ_n be the (unique) quasitrace on $M_n(A)$, for which $\tau_n(x) = \tau(x \otimes e_{11})$; $x \in A$. Set

$$\tau_n'(x) = \frac{1}{n}\tau(x), \quad x \in A.$$

Then (1) $\tau_n'(x \otimes 1_n) = \tau(x)$, $x \in A$.

(2) $\tau_n'(1 \otimes y) = tr_n(y)$, $y \in M_n(\mathbb{C})$

Moreover, if τ is faithful, so is τ_n'.

proof.

From def 3.1(1) we have

(*) $\quad \tau(uau^*) = \tau(a)$, $a \in A_+$, $u \in \mathcal{U}(A)$

where $\mathcal{U}(A)$ is the unitary group of A. By def 3.2(2) to all $a \in A_{sa}$. Hence

$$\tau_n(a \otimes e_{kk}) = \tau_n(a \otimes e_{11}) = \tau(a), \quad a \in A_{sa}$$

and since $(a \otimes e_{kk})_{k=1}^n$ are contained in abelian C^*-subalgebra of $M_n(\mathbb{C})$

$$\tau_n'(a \otimes 1_n) = \frac{1}{n}\sum_{k=1}^n \tau_n(a \otimes e_{kk}) = \tau(a), \quad a \in A_{s.a}.$$

By definition 3.1(3) this can be extended to all $a \in A$, proving (1). (2) holds, because tr_n is the unique normalized quasitrace on $M_n(\mathbb{C})$. Assume next that τ is faithful on A, and let

$$x = \sum_{i=1}^n x_{ij} \otimes e_{ij}$$

be an element of $M_n(A)$ for which $\tau_n'(x^*x) = 0$. By lemma 3.5 (3) also

$$\|(1 \otimes e_{ii})\, x\, (1 \otimes e_{jj})\|_2 = 0$$

where $\|z\|_2 = \tau_n'(z^*z)^{1/2}$. Hence

$$\tau(x_{ij}^* x_{ij}) = n\,\tau_n'(x_{ij}^* x_{ij} \otimes e_{jj}) = 0, \quad 1 \le i,j \le n$$

and so $x_{ij} = 0$ for all i,j, proving $x = 0$. Hence τ_n' is faithful.

Lemma 5.7

Let A be a unital exact C^*-algebra with a faithful normalized quasitrace τ. Then for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the spatial C^*-tensor product

$$A \otimes_{min} M_{\mathcal{U}}$$

can be imbedded in a finite AW^*-algebra N with a faithful normal quasitrace $\bar{\tau}$ for which

$$\bar{\tau}(x \otimes 1) = \tau(x), \quad x \in A$$

$$\bar{\tau}(1 \otimes y) = \tau_{\mathcal{U}}'(y), \quad y \in M_{\mathcal{U}}.$$

Set $N = \ell^\infty\{M_n(A)\}/J_\mathcal{U}$, where

$$J_\mathcal{U} = \{(x_n)_{n=1}^\infty \in \ell^\infty\{M_n(A)\} \mid \lim_{\mathcal{U}} \tau_n'(x_n^* x_n) = 0\}.$$

By proposition 4.2, N is a finite AW^*-algebra with faithful normal quasitrace $\bar{\tau}$ given by,

$$\bar{\tau}([x]) = \lim_{\mathcal{U}} \tau_n'(x), \quad x = (x_n)_{n=1}^\infty \in \ell^\infty(A).$$

Define a unital $*$-homomorphism $\pi: A \to N$ by

$$\pi(x) = [(x \otimes 1_n)_{n=1}^\infty]$$

where $z \to [z]$ is the quotient map from $\ell^\infty\{M_n(A)\}$ to N. By lemma 5.4(1)

$$\bar{\tau} \circ \pi(x) = \tau(x), \quad x \in A$$

so in particular, π is one-to-one. Since by lemma 5.4(2),

$$\tau_n'(1 \otimes y) = \mathrm{tr}_n(y)$$

One has a one-to-one $\overbrace{*}^{\text{unital}}$-homomorphism $\rho: M_\mathcal{U} \to N$ such that

$$\rho([(x_n)_{n=1}^\infty]) = [(1 \otimes x_n)_{n=1}^\infty],$$

for $(x_n)_{n=1}^\infty \in \ell^\infty\{M_n(\mathbb{C})\}$, and moreover

$$\bar{\tau} \circ \rho = \tau_\mathcal{U}.$$

It is clear that $\pi(A)$ and $\rho(M_\mathcal{U})$ are commuting subalgebras of N. The map

$$\beta: \sum_{i=1}^{\ell} x_i \otimes z_i \to \left\| \sum_{i=1}^{\ell} \pi(x_i)\rho(z_i) \right\|$$

defines a C^*-seminorm on the algebraic tensor product $A \odot M_\mathcal{U}$. To prove that β is a norm it suffices to prove that $\beta(x \otimes z) = 0$ implies $x = 0$ or $z = 0$ (see the tensor product section of Sakai's book [S]). But $M_\mathcal{U}$ is a II_1-factor and therefore a simple unital C^*-algebra. Assume $x \in A$, $z \in M_\mathcal{U}$ and $\beta(x \otimes z) = 0$. Since

$$I = \{z \in M_\mathcal{U} \mid \sum_{i=1}^{n} \pi(x)\rho(z) = 0\}$$

is a two sided ideal in $M_\mathcal{U}$, either $I = \{0\}$ or $I = M_\mathcal{U}$. In the first case $z = 0$ and in the second case $x = 0$ proving that β is a C^*-norm on $A \odot M_\mathcal{U}$, so with standard notation for C^*-norms on tensor products,

$$\min \le \beta \le \max.$$

To prove $\beta = \min$, we need the condition, that A is exact:

Let $\ell \in \mathbb{N}$, $x_1, \dots x_\ell \in A$ and $y_1, \dots y_\ell \in \ell^\infty(M_n(\mathbb{C}))$ and let $[y_\ell]$ be the range of y_i in $M_\mathcal{U}$ by the quotient map. Write $y_i = ((y_i)_n)_{n=1}^\infty$. Then

$$\sum_{i=1}^{n} \pi(x_i)\rho([y_i]) = \left[\left(\sum_{i=1}^{n} x_i \otimes (y_i)_n\right)_{n=1}^\infty\right]$$

where $[\cdot]$ on the right side denote the quotient map $\ell^\infty(M_n(A)) \to N$.

Hence $\beta\left(\sum_{i=1}^{l} x_i \otimes [y_i]\right) \leq \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{l} x_i \otimes (y_i)_n \right\|_{min}$

$$= \left\| \sum_{i=1}^{l} x_i \otimes y_i \right\|_{min}$$

Hence the map

$$\sum_{i=1}^{l} x_i \otimes y_i \to \sum_{i=1}^{l} \pi(x_i)\, \varrho([y_i]), \quad x_i \in A,\ y_i \in \ell^\infty\{M_n(\mathbb{C})\}$$

extends to a $*$-homomorphism

$$\varphi: A \otimes \ell^\infty\{M_n(\mathbb{C})\} \to C^*(\pi(A), \varrho(M_{\mathcal{U}})).$$

Note that $\beta(z) = \|\varphi(z)\|$, $z \in A \otimes \ell^\infty\{M_n(\mathbb{C})\}$. For $x \in A$ and $y \in \ell^\infty\{M_n(\mathbb{C})\}$

$$\bar{\tau} \circ \varphi((x\otimes y)^*(x\otimes y)) = \bar{\tau}\left(\varrho([y])^* \pi(x^*x)\, \varrho([y])\right)$$

$$\leq \|x\|^2\, \bar{\tau}\left(\varrho([y^*y])\right)$$

$$= \|x\|^2 \lim_{\mathcal{U}} tr_n(y_n^* y_n)$$

Since $\bar{\tau}$ is faithful, it follows that $\ker(\varphi)$ contains $A \otimes I_{\mathcal{U}}$. Therefore the C^*-tensor norm β on $A \otimes M_{\mathcal{U}}$ is less ore equal to the norm on $A \otimes M_{\mathcal{U}}$ coming from the quotient

$$A \otimes \ell^\infty\{M_n(\mathbb{C})\} / A \otimes I_{\mathcal{U}}.$$

However, exactness of A implies that the latter norm is the minimal C^*-tensor norm. Hence $\beta \leq \min$, so altogether $\beta = \min$. This shows that the map

$$\sum_{i=1}^{l} x_i \otimes z_i \to \sum_{i=1}^{l} \pi(x_i)\, \varrho(z_i), \quad x_i \in A,\ z_i \in M_{\mathcal{U}}$$

extends to a one-to-one $*$-homeomorphism of $A \otimes M_{\mathcal{U}}$ into N with the desired properties ∎

Lemma 5.8

Let N be a finite AW^*-algebra with a faithful normal quasitrace τ and let A and C be two commuting unital C^*-subalgebras of N. Let B be the AW^*-subalgebra of N generated by A. If

(i) $\left\| \sum_{i=1}^{l} a_i c_i \right\| \leq \left\| \sum_{i=1}^{l} a_i \otimes c_i \right\|_{min}$ for all $l \in \mathbb{N}$ and all $a_1, \dots, a_l \in A$, $c_1, \dots, c_l \in C$, and

(ii) C is an exact C^*-algebra,

then $\left\| \sum_{i=1}^{l} b_i c_i \right\| \leq \left\| \sum_{i=1}^{l} b_i \otimes c_i \right\|_{min}$ for all $l \in \mathbb{N}$ and all $b_1, \dots, b_l \in B$, $c_1, \dots, c_l \in C$.

proof

Note first, that by prop 3.12 and lemma 3.11 every element of B is the d_τ-limit of a bounded sequence in A, so by lemma 3.7 B and C also commutes. By the remarks prior to proposition 4.5,

$$B = \tilde{A}/\tilde{I},$$

where

$$\tilde{A} = \{(x_n) \in \ell^\infty(A) \mid x_n \text{ is a } d_\tau\text{-Cauchy sequence}\}$$

$$\tilde{I} = \{(x_n) \in \ell^\infty(A) \mid x_n \to 0 \text{ in } d_\tau\text{-metric}\}.$$

and the quotient map $\varphi: \tilde{A} \to B$ is given by

$$\varphi((a_n)_{n=1}^\infty) = d_\tau\text{-}\lim_{n\to\infty} a_n$$

Since B and C commutes, we can define a $*$-homomorphism

$$\psi : \tilde{A} \odot C \to C^*(B,C) \subseteq N$$

by

$$\psi\left(\sum_{i=1}^{\ell} a_i \otimes c_i\right) = \sum_{i=1}^{\ell} \varphi(a_i) c_i$$

Since $\varphi(a_i) = d_\tau\text{-}\lim_{n\to\infty} (a_i)_n$ we get from lemma 3.7, that,

$$\sum_{i=1}^{\ell} \varphi(a_i) c_i = d_\tau\text{-}\lim_{n\to\infty} \sum_{i=1}^{\ell} (a_i)_n c_i$$

By lemma 3.8 the t-ball of N;

$$N_t = \{x \in N \mid \|x\| \le t\}$$

is closed in d_τ-metric for all $t > 0$. Hence

$$\left\|\sum_{i=1}^{\ell} \varphi(a_i) c_i\right\| \le \sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{\ell} (a_i)_n c_i\right\|$$

and therefore by condition (i) in the lemma,

$$\left\|\sum_{i=1}^{\ell} \varphi(a_i) c_i\right\| \le \sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{\ell} (a_i)_n \otimes c_i\right\|_{\min,C}$$

$$= \left\|\sum_{i=1}^{\ell} a_i \otimes c_i\right\|_{\min,C}$$

where the first $\|\ \|_{\min}$ is in $A \otimes C$ and the second in $\tilde{A} \otimes C$. The last equality follows from the inclusions

$$\tilde{A} \otimes C \subseteq \ell^\infty(A) \otimes C \subseteq \ell^\infty(A \otimes C).$$

This shows that ψ extends to a $*$-homomorphism

$$\psi : \tilde{A} \otimes C \to C^*(B,C)$$

The kernel of ψ clearly contains

$$\ker(\varphi) \otimes C = \tilde{I} \otimes C$$

Since $C^*(B,C) = (\tilde{A} \otimes C)/\ker\varphi$, the C^*-seminorm on $B \odot C$ inherited from $C^*(B,C)$ is dominated by the C^*-norm on $B \odot C$ coming from

$$(\tilde{A} \otimes C)/(\tilde{I} \otimes C)$$

However, by exactness of C, the latter norm is equal to the minimal tensor norm on $B \odot C$. This proves lemma 5.8.

__Remark 5.9__

Kirchberg recently proved, that exactness for a C^*-algebra is equivalent to the properties C and C' of Archbold and Batty (see [K2] and [AB]). Lemma 5.8 can be considered as an AW^*-analogue of the implication exact $\Rightarrow$ property P'.

__lemma 5.10__

let A be a unital exact C^*-algebra with a faithful quasitrace τ. let M_τ be the AW^*-completion of A with respect to τ. Then

$$M_\tau \otimes C_r^*(\mathbb{F}_\infty)$$

can be imbedded in a finite AW^*-algebra.

5.11

proof Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. By lemma 5.7, $A \otimes M_{\mathcal{U}}$ can be imbedded in a finite AW^*-algebra N with a faithful trace $\bar{\tau}$, such that

$$\bar{\tau}(x \otimes 1) = \tau(x), \quad x \in A.$$

Since $C_r^*(\mathbb{F}_\infty) \subseteq \mathcal{M}(\mathbb{F}_\infty)$, the von Neumann algebra associated with the left regular representation of $\mathbb{F}_\infty$, and since $\mathcal{M}(\mathbb{F}_\infty)$ has a unital imbedding in $M_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (prop. 5.4), we get for this $\mathcal{U}$, that $A \otimes C_r^*(\mathbb{F}_\infty)$ imbeds in a finite AW^*-algebra N s.t.

$$\tau(x) = \bar{\tau}(x \otimes 1), \quad x \in A$$

where $\bar{\tau}$ is a faithful normal quasitrace on N. But the AW^*-completion M_τ of A with respect to τ consider the the smallest AW^*-subalgebra of N containing A, (cf. prop. 4.4 and def. 4.5). Since $C_r^*(\mathbb{F}_\infty)$ is exact it follows from lemma 5.8, that

$$\left\| \sum_{i=1}^{\ell} a_i b_i \right\| \le \left\| \sum_{i=1}^{\ell} a_i \otimes b_i \right\|_{min}$$

for all $a_1, \dots a_\ell \in M_\tau$ and $b \in C_r^*(\mathbb{F}_\infty)$. Since $C_r^*(\mathbb{F}_\infty)$ is simple by [AO], we get as in the proof of lemma 5.7, that $\| \sum a_i b_i \|$ defines a C^*-norm on $M_\tau \odot C_r^*(\mathbb{F}_\infty)$. Hence

$$\left\| \sum_{i=1}^{\ell} a_i b_i \right\| = \left\| \sum_{i=1}^{\ell} a_i \otimes b_i \right\|_{min}$$

proving lemma 5.10.

5.12

Theorem 5.11
Quasitraces on exact unital C^*-algebras are traces.

proof
Let τ be an extreme point the compact convex set $QT(A)$ of normalized quasitraces, and set

$$I = \{ x \in A \mid \tau(x^* x) = 0 \}$$

Then I is a norm-closed two-sided ideal (prop. 3.2) and

$$\tau(x) = \tau_0([x])$$

for a faithful extremal quasitrace τ_0 on A/I. Moreover by prop. 4.6 the AW^*-completion of A/I with respect to τ_0 is a II_1-AW^*-factor M_{τ_0} with a (unique) normal faithful quasitrace $\bar{\tau}_0$ extending τ_0. Assume τ is not linear. Then $\bar{\tau}_0$ fails to be linear. But uniqueness of the dimension function on a II_1-AW^*-factor shows that $\bar{\tau}_0$ is the only normalized quasitrace on M_{τ_0}. Particularly M_{τ_0} has no trace states. Then by Theorem 2.4

$$M_{\tau_0} \otimes C_r^*(\mathbb{F}_\infty)$$

has a non-unitary isometry. Since A/I is also exact (prop. 5.1), this contradicts lemma 5.11. Hence τ is linear. By Krein-Milman's theorem it now follows, that all $\tau \in QT(A)$ are linear ∎

5.13

Corollary 5.12

Every stably finite unital exact C^*-algebra A has a trace state.

proof By [BH], A has a normalized quasitrace.

Corollary 5.13

If an AW^*-II_1-factor M is generated (as a AW^*-algebra) by an exact unital C^*-subalgebra A, then M is a von Neumann algebra.

proof.

Let τ be the unique quasitrace on M. Then τ coincide with the dimension function on projection, so τ is normal. By prop. 3.12 A is d_τ-dense in M, so by Thm. 5.11 and Lemma 3.7(4), τ is additive on M_+ and thus linear on M. Hence by [CP] M is a von Neumann II_1-factor. (Note that the last conclusion also follows from prop. 5.10 because completeness of the unitball of M in the $\| \|_2$-norm associated with τ implies that the range of M by the G.N.S.-representation is a von Neumann algebra). ∎

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6.1

6. References

[BH] B. Blackadar, D. Handelman, Dimension Functions and traces on C^*-algebras. J. Funct. Analysis 45, (1982), 297-340.

[C] J. Cuntz. Dimension functions on simple C^*-algebras Math. Ann 233 (1978) 145-153.

[H] D. Handelman, Homomorphism of C^*-algebras to finite AW^*-algebras, Michigan Math. J. 28 (1981) 229-240

[V1] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, ..., Lecture Notes in Math. 1132 (1985) 556-588

[V2] D. Voiculescu, Circular and semicircular systems and free products, preprint Berkeley 1989.

[R] M. Rørdam, On the structure of simple C^* algebras, tensored with a UHF-algebra. II, preprint 1989, to appear in J. funct. analysis.

[BKR] B. Blackadar, A. Kumjian, M. Rørdam, Approximately Central Matrix Units and the structure of Noncommutative tori, preprint 1991.

[K1] E. Kirchberg The Fubini Theorem for exact C^*-algebras

[K2] E. Kirchberg, Preprint(s), Heidelberg 1991.

[AB] R.J. Archbold and C.J.K. Batty, C^*-tensor norms and slice maps, J. London Math. Soc. 22 (1980) 127-138.

[Aa] J.F. Aarnes, Quasi-Measures and Quasi-States, Preprint, Trondheim 1989.

6.2

[Kap] I. Kaplansky, Projections in Banach algebras, Ann. Math. 53 (1951) 235–249.

[P]. R. Powers

[A+O] C. Akemann, Ostrand,

[Ber] S.K. Berberian, Baer *-rings, Springer Verlag 1972.

[Mc.D] D. McDuff,

[W]. S. Wassermann,

[S]. S. Sakai, C^*-algebras and W^*-algebras, Springer Verlag 1971

// [EH] E. Effros and U. Haagerup, Lifting problems and local reflexivity, Duke Math. J (?)

[DcH] DeCanniere and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, Amer. Math. J ... (?)