A GENERALIZED CUNTZ ALGEBRA $O_N^M$

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Dedicated to Professor Masamichi Takesaki on his sixty-th birthday

Let $M$ be a von Neumann algebra with a faithful normal tracial state $\tau$ and $N$ be a von Neumann subalgebra of $M$. We construct a tensor algebra $T_N(M)$ relative to $N$;

$$
\begin{align*}
T_N^p(M) &= L^2(M) \otimes L^2(M) \otimes \cdots \otimes L^2(M) \\
T_N^0(M) &= L^2(N)
\end{align*}
$$

where $L^2(M)$ and $L^2(N)$ are Hilbert spaces with respect to the trace $\tau$ and $\otimes$ means the $N$-relative tensor product $\otimes_N$ and

$$
T_N(M) = \sum_{p=0}^{\infty} T_N^p(M).
$$

Then $T_N(M)$ is also $N$-bimodule.

For $x \in M$, a creation operator $o(x)$ is defined by

$$
\begin{align*}
o(x)x_1 \otimes \cdots \otimes x_p &= x \otimes x_1 \otimes \cdots \otimes x_p, & x_1 \otimes \cdots \otimes x_p \in T_N^p(M) \\
o(x)x_0 &= xx_0, & x_0 \in T_N^0(M).
\end{align*}
$$

Then an annihilation operator $\tilde{o}(x)$ is the following;

$$
\begin{align*}
o(x)^*x_1 \otimes x_2 \otimes \cdots \otimes x_p &= E(x^*x_1)x_2 \otimes \cdots \otimes x_p \\
o(x)^*x_0 &= 0.
\end{align*}
$$

where $E$ is the conditional expectation of $M$ onto $N$ with respect to $\tau$.

A $N$-rank one operator $(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q)$ is defined by

$$
\begin{align*}
(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q) := \delta_{q,r}x_1 \otimes \cdots \otimes x_p y_1 \otimes \cdots \otimes y_q,
\end{align*}
$$

where

$$
< z_1 \otimes \cdots \otimes z_q, y_1 \otimes \cdots \otimes y_q >_N = E(y_q^* \cdots (E(y_1^* z_1)z_2) \cdots z_q).
$$

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Let \{u_i\}_{i=1}^{n} be a Pimsner-Popa bases for \(M \supset N\). Then \(o(u_i)\) are isometries for \(0 \leq i \leq n - 1\) and \(o(u_n)\) may be a partially isometry such that

\[
\sum_{i=1}^{n} o(u_i) o(u_i)^* = 1_{T_N(M)} - 1 \otimes 1
\]

A \(N\)-compact operator algebra \(K_N(M)\) is the \(C^*\)-algebra generated by all of \(N\)-rank one operators. A \(C^*\)-algebra \(P_N^M\) is generated by all creation operators and an identity operator. Then the \(N\)-compact operator algebra \(K_N(M)\) turns out to be a closed ideal of \(P_N^M\). A quotient \(C^*\)-algebra \(O_N^M\) of \(P_N^M\) by \(K_N(M)\) is called a generalized Cuntz algebra. The coset of \(o(x)\) in \(O_N^M = P_N^M / K_N(M)\) is also denoted by \(o(x)\) without any confusion. Note that if \(M = C^n\) and \(N = C\), \(O_N^M\) is a Cuntz algebra \(O_n([3])\). A gauge action \(\alpha\) of the torus \(T\) into \(\text{Aut}(O_N^M)\) can be defined by

\[
\alpha_t(o(x)) = o(e^{it}x), \ t \in T.
\]

By the use for Pimsner-Popa bases, the fixed point algebra \((O_N^M)^T\) is isomorphic to a inductive limit algebra of a reduced von Neumann algebra of \(M_n(C) \otimes \cdots \otimes M_n(C) \otimes N\).

**Theorem 1.** If \(M \supset N\) is a factor-subfactor pair with a finite index \([M \ N]\), then there is a gauge invariant state \(\phi\) on \(O_N^M\) such that

\[
\phi(o(x_1) \cdots o(x_n) o(y_m)^* \cdots o(y_1)^*)
= \delta_{n.m} [MN]^{-n} \tau(E(x_1 \cdots E(x_{n-1}E(x_n y_n^*)y_{n-1}^*) \cdots y_1^*)
\]

and \(\phi\) is a unique KMS-state with respect to the gauge action and inverse temperature \(-\log [M \ N]\).

Let \(G\) be a finite group. We consider the two following cases

\[
M = L^\infty(G) \rtimes_\alpha G, \quad N = L^\infty(G), \quad \text{cannonical trace } \tau \text{ on } M
\]

and

\[
M = W^*(G) \rtimes_\delta G, \quad N = W^*(G), \quad \text{cannonical trace } \tau \text{ on } M
\]

where \(\alpha\) is translation on \(G\) and \(\delta\) is a canonical co-action of \(G\).

The Cuntz algebra \(O_{[G]}\) is generated by isometries \(S_g, g \in G\). A canonical co-action \(\delta_1\) of \(G\) on \(O_{[G]}\) is defined by \(\delta_1(S_g) = S_g \otimes \lambda(g)([1])\). A canonical action \(\alpha^1\) of \(G\) on \(O_{[G]}\) is defined by \(\alpha^1_h(S_g) = S_{hg}\).

**Proposition 2.** The generalized Cuntz algebras \(O_{L^\infty(G) \rtimes_\alpha G}^L\) and \(O_{W^*(G) \rtimes_\delta G}^W\) are isomorphic to \(O_{[G]} \rtimes_{\delta_1} G\) and \(O_{[G]} \rtimes_{\alpha^1} G\) respectively.

**Proposition 3.** ([2]) The two crossed products \(O_{[G]} \rtimes_{\delta_1} G\) and \(O_{[G]} \rtimes_{\alpha^1} G\) are isomorphic to \(O_{[G]}\).

For Coxeter graph \(A_l\), we construct finite dimensional von Neumann algebras \(M\) and \(N\) as follows. For \(l=2m+1\) (resp. \(l=2m\)) let \(N\) be the \(m\)-direct sum of \(C\) and \(M\) be \(m+1\)-direct sum \(C \oplus M_2 \oplus \cdots \oplus M_2 \oplus C\) (resp. \(M\) be \(m\)-direct sum \(C \oplus M_2 \oplus \cdots \oplus M_2\)). The inclusion of \(M \supset N\) is given by the bicolored graph of \(A_l\) and a trace \(\tau\) on \(M\) is defined by Perron-Frobenius eigen vector. An element of \(N\) with only \(i\)-th component 1 and otherwise 0 is denoted by \(e_i\).
Proposition 4. Let $M$ and $N$ be finite dimensional algebras associated with Coxeter graph $A_l$. Then we have

(1) for odd $l$, (resp even $l$), we obtain quasi Pimsner-Popa base $\{u_i\}_{i=1}^{m+1}$ such that

$$E(u_i^*u_j) = \begin{cases} \delta_{i,j} & i=1,m+1 \text{ (resp. } 1-e_m \text{ only for } i=m+1) \\ \delta_{i,j}(e_{i-1} + e_i) & \in N, \text{ otherwise} \end{cases}$$

where $\lambda$ is the Perron-Frobenius eigen value of $X^tX$ where $X$ is adjacent matrix of $A_l$

(2) all non zero elements of $\{e_j o(u_i)\}_{i=1}^{m+1}$ are non zero partially isometries which generate $O_N^M$.

(3) $(O_N^M)^T$ is AF-algebra which Bratteli diagram is the repetition of labelled bicolored graph associated with $X^tX$.

(4) $O_N^M$ is isomorphic to Cuntz-Krieger algebra $O_A$ ([4]) where $A$ is the adjacent matrix of the line graph $\ell(A_l)$ of $A_l$.

C. Sutherland pointed out to us the above relation between the line graph of $A_l$ and the matrix $A$.

In the case of $A_5$, we define the base $\{u_i\}$ by

$$\begin{cases} u_1 = \sqrt{3} \oplus \begin{pmatrix} 0 \\ \sqrt{3/2} \end{pmatrix} \oplus 0 \\ u_2 = 0 \oplus \begin{pmatrix} 0 & \sqrt{3/2} \\ \sqrt{3/2} & 0 \end{pmatrix} \oplus 0 \\ u_3 = 0 \oplus \begin{pmatrix} \sqrt{3/2} & 0 \\ 0 & 0 \end{pmatrix} \oplus \sqrt{3}. \end{cases}$$

Bratteli diagram of the fixed point algebra $(O_N^M)^T$ with unique tracial state is

and $O_N^M = O_A$ is as follows

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

the line graph $\ell(A_5)$ the adjacent matrix of $\ell(A_5)$. 


The $K$-group $K_0(O^M_N)$ for $A_5$ is integer which show that it is different from Cuntz algebras.

Remark 5. When von Neumann algebras $M \supset N$ have quasi-Pimzner-Popa base ($E(u_i^*u_i)$ may be a projection in $N$ instead of $E(u_i^*u_i) = 1$) Theorem 1 holds true even for non-factor $N$ if the fixed point algebra has a unique tracial state.

REFERENCES