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Kyoto University
A GENERALIZED CUNTZ ALGEBRA $\mathcal{O}_N^M$

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Dedicated to Professor Masamichi Takesaki on his sixty-th birthday

Let $M$ be a von Neumann algebra with a faithful normal tracial state $\tau$ and $N$ be a von Neumann subalgebra of $M$. We construct a tensor algebra $T_N(M)$ relative to $N$;

\[
\left\{
\begin{array}{l}
T_N^p(M) = L^2(M) \otimes L^2(M) \otimes \cdots \otimes L^2(M) \\
T_N^0(M) = L^2(N)
\end{array}
\right.
\]

where $L^2(M)$ and $L^2(N)$ are Hilbert spaces with respect to the trace $\tau$ and $\otimes$ means the $N$-relative tensor product $\otimes_N$ and

\[
T_N(M) = \sum_{p=0}^{\infty} T_N^p(M).
\]

Then $T_N(M)$ is also $N$-bimodule.

For $x \in M$, a creation operator $o(x)$ is defined by

\[
\left\{
\begin{array}{l}
o(x)x_1 \otimes \cdots \otimes x_p = x \otimes x_1 \otimes \cdots \otimes x_p, \quad x_1 \otimes \cdots \otimes x_p \in T_N^p(M) \\
o(x)x_0 = xx_0, \quad x_0 \in T_N^0(M).
\end{array}
\right.
\]

Then an annihilation operator $o(x)$ is the following;

\[
\left\{
\begin{array}{l}
o(x)^* x_1 \otimes x_2 \otimes \cdots \otimes x_p = E(x^*x_1)x_2 \otimes \cdots \otimes x_p \\
o(x)^* x_0 = 0.
\end{array}
\right.
\]

where $E$ is the conditional expectation of $M$ onto $N$ with respect to $\tau$.

A $N$-rank one operator $(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q)$ is defined by

\[
\{(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q)\}(z_1 \otimes \cdots \otimes z_r)
= \delta_{q,r}x_1 \otimes \cdots \otimes x_p \langle z_1 \otimes \cdots \otimes z_r, y_1 \otimes \cdots \otimes y_q \rangle_N
\]

where

\[
\langle z_1 \otimes \cdots \otimes z_q, y_1 \otimes \cdots \otimes y_q \rangle_N = E(y_q^* \cdots (E(y_1^*z_1)z_2) \cdots z_q).
\]

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Let \( \{u_i\}_{i=1}^n \) be a Pimsner-Popa bases for \( M \supset N \). Then \( o(u_i) \) are isometries for \( 0 \leq i \leq n - 1 \) and \( o(u_n) \) may be a partially isometry such that
\[
\sum_{i=1}^n o(u_i)o(u_i)^* = 1_{TN(M)} - 1 \otimes 1
\]

A \( N \)-compact operator algebra \( K_N(M) \) is the \( C^* \)-algebra generated by all of \( N \)-rank one operators. A \( C^* \)-algebra \( P^M_N \) is generated by all creation operators and an identity operator. Then the \( N \)-compact operator algebra \( K_N(M) \) turns out to be a closed ideal of \( P^M_N \). A quotient \( C^* \)-algebra \( O^M_N \) of \( P^M_N \) by \( K_N(M) \) is called a generalized Cuntz algebra. The coset of \( o(x) \) in \( O^M_N = P^M_N / K_N(M) \) is also denoted by \( o(x) \) without any confusion. Note that if \( M = C^n \) and \( N = C \), \( O^M_N \) is a Cuntz algebra \( O_n([3]) \). A gauge action \( \alpha \) of the torus \( T \) into \( Aut(O^M_N) \) can be defined by
\[\alpha_t(o(x)) = o(e^{it}x), t \in T.\]

By the use of Pimsner-Popa bases, the fixed point algebra \( (O^M_N)^T \) is isomorphic to a inductive limit algebra of a reduced von Neumann algebra of \( M_n(C) \otimes \cdots \otimes M_n(C) \otimes N \).

**Theorem 1.** If \( M \supset N \) is a factor-subfactor pair with a finite index \([M \ N] \), then there is a gauge invariant state \( \phi \) on \( O^M_N \) such that
\[
\phi(o(x_1) \cdots o(x_n)o(y_m)^* \cdots o(y_1)^*) = \delta_{n,m}[MN]^{-n}\tau(E(x_1 \cdots E(x_{n-1}E(x_n y_m^*)y_{n-1}^*) \cdots y_1^*)
\]
and \( \phi \) is a unique KMS-state with respect to the gauge action and inverse temperature \(-\log [M N]\).

Let \( G \) be a finite group. We consider the following two cases
\[
M = L^\infty(G) \rtimes_{\alpha} G, \quad N = L^\infty(G), \quad \text{cannonical trace } \tau \text{ on } M
\]
and
\[
M = W^*(G) \rtimes_{\delta} G, \quad N = W^*(G), \quad \text{cannonical trace } \tau \text{ on } M
\]
where \( \alpha \) is translation on \( G \) and \( \delta \) is a canonical co-action of \( G \).

The Cuntz algebra \( O_{|G|} \) is generated by isometries \( S_g, g \in G \). A canonical co-action \( \delta_1 \) of \( G \) on \( O_{|G|} \) is defined by \( \delta_1(S_g) = S_g \otimes \lambda(g)([1]) \). A canonical action \( \alpha_1 \) of \( G \) on \( O_{|G|} \) is defined by \( \alpha_1(S_g) = S_{gh} \).

**Proposition 2.** The generalized Cuntz algebras \( O_{L^\infty(G)} \rtimes_\alpha G \) and \( O_{W^*(G)} \rtimes_\delta G \) are isomorphic to \( O_{|G|} \rtimes_{\delta_1} G \) and \( O_{|G|} \rtimes_{\alpha} G \) respectively.

**Proposition 3.** ([2]) The two crossed products \( O_{|G|} \rtimes_{\delta} G \) and \( O_{|G|} \rtimes_{\alpha} G \) are isomorphic to \( O_{|G|} \).

For Coxeter graph \( A_l \), we construct finite dimensional von Neumann algebras \( M \) and \( N \) as follows. For \( l=2m+1 \) (resp. \( l=2m \)) we let \( N \) be the \( m \)-direct sum of \( C \) and \( M \) be \( m+1 \)-direct sum \( C \oplus M_2 \oplus \cdots \oplus M_2 \oplus C \) (resp. \( M \) be \( m \)-direct sum \( C \oplus M_2 \oplus \cdots \oplus M_2 \)). The inclusion of \( M \supset N \) is given by the bicolored graph of \( A_l \) and a trace \( \tau \) on \( M \) is defined by Perron-Frobenius eigen vector. An element of \( N \) with only \( i \)-th component 1 and otherwise 0 is denoted by \( e_i \).
Proposition 4. Let $M$ and $N$ be finite dimensional algebras associated with Coxeter graph $A_l$. Then we have

1. For odd $l$ (resp. even $l$), we obtain quasi Pimsner-Popa base $\{u_i\}_{i=1}^{m+1}$ such that

   (a) $E(u_i^*u_j) = \begin{cases} \delta_{i,j}, & i=1,m+1 \text{ (resp. } 1-e_m \text{ only for } i=m+1) \\ \delta_{i,j}(e_{i-1} + e_i), & i \in N, \text{ otherwise} \end{cases}$

   

   $M = \sum_{i=1}^{m+1} u_i N, \quad \text{and} \quad \sum_{i=1}^{m+1} u_i u_i^* = \lambda I$

   

   where $\lambda$ is the Perron-Frobenius eigenvalue of $X^tX$ where $X$ is adjacent matrix of $A_l$

   (b) $\begin{cases} e_ju_i = u_i e_j, & \text{for } i=1,\ldots,m+1, j=1,\ldots,m \\ e_{i-1}u_i = u_i e_i, & i=2,\ldots,m \end{cases}$

2. All non zero elements of $\{e_j\circ(u_i)\}_{i=1,j=1}^{m,1}$ are non zero partially isometries which generate $O_N^M$.

3. $(O_N^M)^T$ is AF-algebra which Bratteli diagram is the repetition of labelled bicolored graph associated with $X^tX$.

4. $O_N^M$ is isomorphic to Cuntz-Krieger algebra $O_A$ ([4]) where $A$ is the adjacent matrix of the line graph $\ell(A_l)$ of $A_l$.

C. Sutherland pointed out to us the above relation between the line graph of $A_l$ and the matrix $A$.

In the case of $A_5$, we define the base $\{u_i\}$ by

\[
\begin{aligned}
    u_1 &= \sqrt{3} \oplus \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix} \oplus 0 \\
    u_2 &= 0 \oplus \begin{pmatrix} 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 \end{pmatrix} \oplus 0 \\
    u_3 &= 0 \oplus \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \sqrt{3}.
\end{aligned}
\]

Bratteli diagram of the fixed point algebra $(O_N^M)^T$ with unique tracial state is

and $O_N^M = O_A$ is as follows

the line graph $\ell(A_5)$ the adjacent matrix of $\ell(A_5)$. 


The K-group $K_0(\mathcal{O}_V^M)$ for $A_5$ is integer which show that it is different from Cuntz algebras.

*Remark 5.* When von Neumann algebras $M \supset N$ have quasi-Pimzner-Popa base ($E(u_i^*u_i)$ may be a projection in $N$ instead of $E(u_i^*u_i) = 1$) Theorem 1 holds true even for non-factor $N$ if the fixed point algebra has a unique tracial state.

**References**