<table>
<thead>
<tr>
<th>Title</th>
<th>A GENERALIZED CUNTZ ALGEBRA $\mathcal{O}^M_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KATAYAMA, YOSHIKAZU</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1994), 858: 87-90</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1994-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/83797">http://hdl.handle.net/2433/83797</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
A GENERALIZED CUNTZ ALGEBRA $O_N^M$

YOSHIKAZU KATAYAMA (片 山 良 一)

Division of Mathematical Science, Osaka kyoiku University

Dedicated to Professor Masamichi Takesaki on his sixty-th birthday

Let $M$ be a von Neumann algebra with a faithful normal tracial state $\tau$ and $N$ be a von Neumann subalgebra of $M$. We construct a tensor algebra $T_N(M)$ relative to $N$;

\[
\begin{align*}
T_N^p(M) &= L^2(M) \otimes L^2(M) \otimes \cdots \otimes L^2(M) \\
T_N^0(M) &= L^2(N)
\end{align*}
\]

where $L^2(M)$ and $L^2(N)$ are Hilbert spaces with respect to the trace $\tau$ and $\otimes$ means the $N$-relative tensor product $\otimes_N$ and

\[
T_N(M) = \sum_{p=0}^{\infty} T_N^p(M).
\]

Then $T_N(M)$ is also $N$-bimodule.

For $x \in M$, a creation operator $o(x)$ is defined by

\[
\begin{align*}
& \left\{ o(x)x_1 \otimes \cdots \otimes x_p = x \otimes x_1 \otimes \cdots \otimes x_p, \quad x_1 \otimes \cdots \otimes x_p \in T_N^p(M) \right. \\
& \left. o(x)x_0 = xx_0, \quad x_0 \in T_N^0(M). \right.
\end{align*}
\]

Then an annihilation operator $o(x)$ is the following;

\[
\begin{align*}
& \left\{ o(x)^*x_1 \otimes x_2 \otimes \cdots \otimes x_p = E(x^*x_1)x_2 \otimes \cdots \otimes x_p \\
& o(x)^*x_0 = 0.
\right.
\end{align*}
\]

where $E$ is the conditional expectation of $M$ onto $N$ with respect to $\tau$.

A $N$-rank one operator $(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q)$ is defined by

\[
\begin{align*}
& \{(x_1 \otimes \cdots \otimes x_p) \otimes (y_1 \otimes \cdots \otimes y_q)\}(z_1 \otimes \cdots \otimes z_r) \\
& = \delta_{q,r}x_1 \otimes \cdots \otimes x_p < z_1 \otimes \cdots \otimes z_r, y_1 \otimes \cdots \otimes y_q > N
\end{align*}
\]

where

\[
< z_1 \otimes \cdots \otimes z_q, y_1 \otimes \cdots \otimes y_q > N = E(y_q^* \cdots (E(y_1^*z_1)z_2) \cdots z_q).
\]

Typeset by AMS-TEX
Let $\{u_i\}_{i=1}^n$ be a Pimsner-Popa bases for $M \supset N$. Then $o(u_i)$ are isometries for $0 \leq i \leq n - 1$ and $o(u_n)$ may be a partially isometry such that
\[
\sum_{i=1}^n o(u_i) o(u_i)^* = 1_{T_N(M)} - 1 \otimes 1
\]
A $N$-compact operator algebra $K_N(M)$ is the $C^\ast$-algebra generated by all of $N$-rank one operators. A $C^\ast$-algebra $P_N^M$ is generated by all creation operators and an identity operator. Then the $N$-compact operator algebra $K_N(M)$ turns out to be a closed ideal of $P_N^M$. A quotient $C^\ast$-algebra $O_N^M$ of $P_N^M$ by $K_N(M)$ is called a generalized Cuntz algebra. The coset of $o(x)$ in $O_N^M = P_N^M / K_N(M)$ is also denoted by $o(x)$ without any confusion. Note that if $M = C^n$ and $N = C$, $O_N^M$ is a Cuntz algebra $O_n([3])$. A gauge action $\alpha$ of the torus $T$ into $Aut(O_N^M)$ can be defined by
\[
\alpha_t(o(x)) = o(e^{it} x), t \in T.
\]
By the use fo Pimsner-Popa bases, the fixed point algebra $(O_N^M)^T$ is isomorphic to a inductive limit algebra of a reduced von Neumann algebra of $M_n(C) \otimes \cdots \otimes M_n(C) \otimes N$.

**Theorem 1.** If $M \supset N$ is a factor-subfactor pair with a finite index $[M \ N]$, then there is a gauge invariant state $\phi$ on $O_N^M$ such that
\[
\phi(o(x_1) \cdots o(x_n) o(y_m)^* \cdots o(y_1)^*) = \delta_{n,m}[M \ N]^{\ast - n} \tau(E(x_1) \cdots E(x_{n-1}) E(x_n y_n^* y_{n-1}^* \cdots y_1^*)
\]
and $\phi$ is a unique KMS-state with respect to the gauge action and inverse temperature $-\log [M \ N]$.

Let $G$ be a finite group. We consider the two following cases
\[
M = L^\infty(G) \rtimes_\alpha G, \quad N = L^\infty(G), \quad \text{cannonical trace} \ \tau \ \text{on} \ M
\]
and
\[
M = W^*(G) \rtimes_\delta G, \quad N = W^*(G), \quad \text{cannonical trace} \ \tau \ \text{on} \ M
\]
where $\alpha$ is translation on $G$ and $\delta$ is a canonical co-action of $G$.

The Cuntz algebra $O_{|G|}$ is generated by isometries $S_g, g \in G$. A canonical co-action $\delta_1$ of $G$ on $O_{|G|}$ is defined by $\delta_1(S_g) = S_g \otimes \lambda(g)([1])$. A canonical action $\alpha^1$ of $G$ on $O_{|G|}$ is defined by $\alpha^1_k(S_g) = S_{g^k}$.

**Proposition 2.** The generalized Cuntz algebras $O_{L^\infty(G)}^{L^\infty(G) \rtimes_\alpha G}$ and $O_{W^*(G)}^{W^*(G) \rtimes_\delta G}$ are isomorphic to $O_{|G|} \rtimes_{\delta_1} G$ and $O_{|G|} \rtimes_{\alpha^1} G$ respectively.

**Proposition 3.** ([2]) The two crossed products $O_{|G|} \rtimes_{\delta_1} G$ and $O_{|G|} \rtimes_{\alpha^1} G$ are isomorphic to $O_{|G|}$.

For Coxeter graph $A_l$, we construct finite dimensional von Neumann algebras $M$ and $N$ as follows. For $l=2m+1$ (resp. $l=2m$) let $N$ be the $m$-direct sum of $C$ and $M$ be $m+1$-direct sum $C \oplus M_2 \oplus \cdots \oplus M_2 \oplus C$. (resp. $M$ be $m$-direct sum $C \oplus M_2 \oplus \cdots \oplus M_2$). The inclusion of $M \supset N$ is given by the bicolored graph of $A_l$ and a trace $\tau$ on $M$ is defined by Perron-Frobenius eigen vector. An element of $N$ with only $i$-th component 1 and otherwise 0 is denoted by $e_i$. 
Proposition 4. Let $M$ and $N$ be finite dimensional algebras associated with Coxeter graph $A_l$. Then we have

(1) for odd $l$, (resp. even $l$), we obtain quasi Pimsner-Popa base $\{u_i\}_{i=1}^{m+1}$ such that

(a) $E(u_i^* u_j) = \begin{cases} 1, & i=1, m+1 \text{ (resp. } 1-e_m \text{ only for } i=m+1) \\ \delta_{i,j}(e_{i-1} + e_i), & \in N, \text{ otherwise} \end{cases}$

$$M = \sum_{i=1}^{m+1} u_i N, \quad \text{and} \quad \sum_{i=1}^{m+1} u_i u_i^* = \lambda I$$

where $\lambda$ is the Perron-Frobenius eigen value of $X^t X$ where $X$ is adjacent matrix of $A_l$

(b) $\begin{cases} e_j u_i = u_i e_j, \text{ for } i=1, \ldots, m+1, j=1, \ldots, m \\ e_{i-1} u_i = u_i e_i, \text{ for } i=2, \ldots, m \end{cases}$

(2) all non zero elements of $\{e_j o(u_i)\}_{i=1,j=1}^{m+1,m}$ are non zero partially isometries which generate $\mathcal{O}_N^M$.

(3) $(\mathcal{O}_N^M)^T$ is AF-algebra which Bratteli diagram is the repetition of labelled bicolored graph associated with $X^t X$.

(4) $\mathcal{O}_N^M$ is isomorphic to Cuntz-Krieger algebra $O_A$ ([4]) where $A$ is the adjacent matrix of the line graph $\ell(A_l)$ of $A_l$.

C. Sutherland pointed out to us the above relation between the line graph of $A_l$ and the matrix $A$.

In the case of $A_5$, we define the base $\{u_i\}$ by

$$\begin{align*}
    u_1 &= \sqrt{3} \oplus \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix} \oplus 0 \\
    u_2 &= 0 \oplus \begin{pmatrix} 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 \end{pmatrix} \oplus 0 \\
    u_3 &= 0 \oplus \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \sqrt{3}.
\end{align*}$$

Bratteli diagram of the fixed point algebra $(\mathcal{O}_N^M)^T$ with unique tracial state is

and $\mathcal{O}_N^M = O_A$ is as follows

\[ \ell(A_5) \quad \ell(A_5) \]

the line graph $\ell(A_5)$, the adjacent matrix of $\ell(A_5)$.
The K-group $K_0(\mathcal{O}_N^M)$ for $A_5$ is integer which show that it is different from Cuntz algebras.

Remark 5. When von Neumann algebras $M \supset N$ have quasi-Pimzner-Popa base ($E(u_i^*u_i)$ may be a projection in $N$ instead of $E(u_i^*u_i) = 1$) Theorem 1 holds true even for non-factor $N$ if the fixed point algebra has a unique tracial state.

REFERENCES