Construction of a Kac algebra action on the AFD factor of type $II_1$
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The purpose of this note is to announce the result obtained in [9]. Namely we describe a construction of an "outer" action of a finite-dimensional Kac algebra on the AFD factor of type $II_1$.

§ 1. Kac algebras and their actions

Throughout this note, fix a finite-dimensional Hopf $C^*$-algebra $K=(\mathcal{M}, \Gamma, \kappa, \epsilon)$, i.e.,

(i) $\mathcal{M}$ is a finite-dimensional $C^*$-algebra;

(ii) $\Gamma$ is a coproduct of $\mathcal{M}$, i.e., an injective homomorphism from $\mathcal{M}$ into $\mathcal{M} \otimes \mathcal{M}$ satisfying the coassociativity: $(\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma$;

(iii) $\epsilon$ is a counit of $\mathcal{M}$, i.e., a homomorphism from $\mathcal{M}$ into $\mathbb{C}$ satisfying $(\epsilon \otimes \iota) \circ \Gamma = (\iota \otimes \epsilon) \circ \Gamma = \iota$;

(iv) $\kappa$ is an antipode of $\mathcal{M}$, i.e., a linear mapping from $\mathcal{M}$ into itself satisfying $m_\mathcal{M} \circ (\kappa \otimes \iota) \circ \Gamma(a) = m_\mathcal{M} \circ (\iota \otimes \kappa) \circ \Gamma(a) = \epsilon(a) \cdot 1$, where $m_\mathcal{M}$ is the multiplication of $\mathcal{M}$;

(v) all the morphisms above are $*$-preserving.

Note that (1) $\kappa^2 = \iota$, because of finite-dimensionality of $\mathcal{M}$; (2) if $\varphi$ is a functional on $\mathcal{M}$ defined by

$$\varphi = \oplus_{i=1}^{k} n_i \text{Tr}_{n_i}$$

along with a decomposition of $\mathcal{M}$:

$$\mathcal{M} \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$
where $M_n(C)$ is the full matrix algebra of size $n$ and $\text{Tr}_n$ denotes the ordinary trace on $M_n(C)$, then $\varphi$ is a left-invariant (hence, right-invariant) trace on $\mathcal{M}$: $(\varphi \otimes \iota) \circ \Gamma(a) = (\iota \otimes \varphi) \circ \Gamma(a) = \varphi(a) \cdot 1$. The system $(\mathcal{M}, \Gamma, \kappa, \varphi)$ is a Kac algebra in the sense of Enock-Schwartz, and $\varphi$ is called the Haar weight. We shall mainly work with $K=(\mathcal{M}, \Gamma, \kappa, \varphi)$ instead if $(\mathcal{M}, \Gamma, \kappa, \varepsilon)$, since we often consider $\mathcal{M}$ to be represented on the Hilbert space $L^2(\varphi)$ with respect to this specific $\varphi$. Once a Kac algebra $K$ is given, we immediately obtain three new Kac algebras as follows:

(1) The commutant of $K$, denoted by $K' = (\mathcal{M}', \Gamma', \kappa', \varphi')$. Here $\mathcal{M}'$ is the commutant of $\mathcal{M}$ in $L^2(\varphi)$. The coproduct $\Gamma'$ is defined by $\Gamma'(y) = (J \otimes J)\Gamma(JyJ)(J \otimes J)$ $(y \in \mathcal{M}')$ with $J$ as the modular conjugation of $\varphi$. $\kappa'$ and $\varphi'$ are defined similarly.

(2) The reflection of $K$, denoted by $K^\sigma = (\mathcal{M}, \Gamma^\sigma, \kappa, \varphi)$. The coproduct $\Gamma^\sigma$ is given by $\Gamma^\sigma = \sigma \circ \Gamma$, where $\sigma$ is the flip: $\sigma(x \otimes y) = y \otimes x$.

(3) The dual of $K$, denoted by $K^\wedge = (\mathcal{M}^\wedge, \Gamma^\wedge, \kappa^\wedge, \varphi^\wedge)$. This is constructed as follows. By considering the adjoint maps of $\Gamma, \kappa, m_\mathcal{M}$ and so on, the dual space $\mathcal{M}^\ast$ can be turned into a Kac algebra. Meanwhile, since $\varphi$ is faithful, $\mathcal{M}^\ast$ can be identified with $\mathcal{M}$ by the correspondence $a \in \mathcal{M} \mapsto \varphi_a \in \mathcal{M}^\ast$, where $\varphi_a(b) = \varphi(ab)$. We write $K^\wedge = (\mathcal{M}^\wedge, \Gamma^\wedge, \kappa^\wedge, \varphi^\wedge)$ for $\mathcal{M}$ with this new Kac algebra structure through this identification, and use notation $f * g$, $f^\dagger$ for the multiplication and the involution of $K^\wedge$. $\mathcal{M}^\wedge$ too is considered to be represented on $L^2(\varphi)$ via the representation $\lambda$: $\lambda(f)g = f * g$.

Combination of these Kac algebras (1) – (3) produces more new Kac algebras such as $K'^\sigma, K^\wedge\sigma$ and so on.

**Definition.** (Nakagami-Takesaki, Enock) An action of $K=(\mathcal{M}, \Gamma, \kappa, \varphi)$ on a von
Neumann algebra $\mathcal{A}$ is an injective unital $\ast$-homomorphism $\beta$ from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{M}$ such that

$$(\beta \otimes \iota) \circ \beta = (\iota \otimes \Gamma) \circ \beta. \quad (*)$$

Here are some simple examples of Kac algebra actions.

(1) $G$ is a (finite) group. Let $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ be an action of $G$ in the ordinary sense. Then the map $\beta : s \in G \mapsto \alpha_s(a) \in \mathcal{A}$ ($a \in \mathcal{A}$) can be viewed as a $\ast$-homomorphism from $\mathcal{A}$ into $\mathcal{A} \otimes \ell^\infty(G)$. Moreover, it enjoys property $(*)$ above. Thus $\beta$ is an action of the commutative Kac algebra $\ell^\infty(G)$ on $\mathcal{A}$. In fact, it is an easy exercise to check that we have a bijective correspondence:

$$\{\alpha : \alpha : G \rightarrow \text{Aut}(\mathcal{A})\} \overset{\text{bijection}}{\longrightarrow} \{\beta : \beta \text{ is an action of the Kac algebra } \ell^\infty(G) \text{ on } \mathcal{A}\}.$$ 

(2) A map $a \in \mathcal{A} \mapsto a \otimes 1 \in \mathcal{A} \otimes \mathcal{M}$ is clearly an action of $K$. This is called the trivial action.

(3) Due to coassociativity of a coproduct, $\Gamma$ itself is an action of $K$ on $\mathcal{M}$. This fact is crucial in the following discussion.

**Definition.** For an action $\beta$ of $K$ on $\mathcal{A}$, the crossed product $\mathcal{A} \times_\beta K$ is by definition generated by $\beta(\mathcal{A})$ and $C_K \otimes \mathcal{M}'$ (assuming that $\mathcal{A}$ is represented on $\mathcal{H}$). On the crossed product, there exists an action $\tilde{\beta}$ of $K'$, called the dual action of $\beta$. $\tilde{\beta}$ maps the generators $\beta(a)$ and $1 \otimes z$ of the crossed product as follows: $\tilde{\beta}(\beta(a)) = \beta(a) \otimes 1$, $\tilde{\beta}(1 \otimes z) = 1 \otimes \Gamma'(z)$. Dual weight construction holds good also in the case of Kac algebra actions. Moreover, Takesaki duality is true.

§ 2. Construction of a pair of $II_1$ factors
Start with a Kac algebra $K = (\mathcal{M}, \Gamma, \kappa, \varphi)$. Let $A_0 = C$, $A_1 = \mathcal{M}$. Since $\Gamma$ is an action of $K$ on $\mathcal{M}$, we may take its crossed product. We set $A_2 = \mathcal{M} \times_\Gamma K$. On $A_2$, there is the dual action $\tilde{\Gamma}$ of $\Gamma$. So define $A_3 = A_2 \times_\Gamma K'$. By continuing this procedure, we obtain an increasing sequence $\{A_n\}$ of finite-dimensional $C^*$-algebras. Remark that we have in general $K^{-} = K$, $K^{\sigma} = K'$, $K^{\sigma'} = K'^{\sigma}$. From this, it follows that

\[ A_{4n} = A_{4n-1} \times_{\Gamma^{(4n-2)}} K^{\sigma'} \quad (n \geq 1), \]
\[ A_{4n+1} = A_{4n} \times_{\Gamma^{(4n-1)}} K^{-\sigma} \quad (n \geq 0), \]
\[ A_{4n+2} = A_{4n+1} \times_{\Gamma^{(4n)}} K \quad (n \geq 0), \]
\[ A_{4n+3} = A_{4n+2} \times_{\Gamma^{(4n+1)}} K^{-} \quad (n \geq 0), \]

where $\Gamma^{(-1)} = \text{the trivial action of } K^{-}$ on $A_0 = C$, $\Gamma^{(0)} = \Gamma$, and $\Gamma^{(n)} = \text{the dual action of } \Gamma^{(n-1)}$. By Takesaki duality,

\[ A_{2n} \cong \otimes^n M_{\dim \mathcal{M}}(C) \quad (n \geq 1). \]

Next we put $B_0 = \mathcal{M}^{\sigma}$. Then define $B_n$ inductively by

\[ B_{4n} = B_{4n-1} \times_{\delta^{(4n-2)}} K^{\sigma'} \quad (n \geq 1), \]
\[ B_{4n+1} = B_{4n} \times_{\delta^{(4n-1)}} K^{-\sigma} \quad (n \geq 0), \]
\[ B_{4n+2} = B_{4n+1} \times_{\delta^{(4n)}} K \quad (n \geq 0), \]
\[ B_{4n+3} = B_{4n+2} \times_{\delta^{(4n+1)}} K^{-'} \quad (n \geq 0), \]

where $\delta^{(0)} = \delta = \Gamma^{\sigma}$, and $\delta^{(n)} = \text{the dual action of } \delta^{(n-1)}$. Thus we get another increasing sequence $\{B_n\}$ of finite-dimensional $C^*$-algebras. Takesaki duality implies

\[ B_{2n-1} \cong \otimes^n M_{\dim \mathcal{M}}(C) \quad (n \geq 1). \]
Observation 1. For each $n \geq 0$, $A_n$ can be considered as a subalgebra of $B_n$. For example, if $n = 1, 2$, we have

$$A_1 = \mathcal{M}, \quad B_1 = \delta(\mathcal{M}^\wedge) \vee C \otimes \mathcal{M};$$

$$A_2 = \Gamma(\mathcal{M}) \vee C \otimes \mathcal{M}^\prime, \quad B_2 = \delta(\mathcal{M}^\wedge) \otimes C \vee C \otimes \Gamma(\mathcal{M}) \vee C \otimes C \otimes \mathcal{M}^\prime.$$  

Hence $\pi_n(a) = 1 \otimes a$ ($a \in A_n$) in general embeds $A_n$ into $B_n$ so that the diagram

$$\begin{array}{ccc}
B_n & \rightarrow & B_{n+1} \\
\uparrow & & \uparrow \\
A_n & \rightarrow & A_{n+1}
\end{array}$$

commutes. Moreover, we have

Theorem 1. For each $n \geq 0$,

$$\begin{array}{ccc}
B_n & \rightarrow & B_{n+1} \\
\uparrow & & \uparrow \\
A_n & \rightarrow & A_{n+1}
\end{array}$$

forms a commuting square. Here, on each $B_n$, we consider the faithful trace obtained as the dual weight by crossed product construction.

Proof for $n = 0$. By Takesaki duality, $B_1 \cong \mathcal{L}(L^2(\varphi))$. By keeping track of how this isomorphism $\pi$ was constructed, one has that

$$\pi(B_0) = \mathcal{M}^\wedge, \quad \pi(A_1) = \mathcal{M}.$$  

Thus $\pi$ transforms the diagram in question into

$$\begin{array}{ccc}
\mathcal{M}^\wedge & \rightarrow & \mathcal{L}(L^2(\varphi)) \\
\uparrow & & \uparrow \\
C & \rightarrow & \mathcal{M}.
\end{array}$$

Hence it suffices to show that this diagram is a commuting square. For this purpose, we need to recall the unitary canonically associated to every Kac algebra, called the fundamental unitary (or the Kac-Takesaki operator). It is defined in the following way. Since the Haar weight $\varphi$ is left-invariant, the equation

$$W(f \otimes g) = \Gamma(g)(f \otimes 1) \quad (f, g \in \mathcal{M})$$
defines an isometry on $L^2(\varphi) \otimes L^2(\varphi)$. It is actually a unitary that belongs to $\mathcal{M} \otimes \mathcal{M}^\wedge$. Moreover, $W$ implements the coproduct $\Gamma: \Gamma(a) = W(a \otimes 1)W^*$, and the coassociativity is shown to be equivalent to the so-called the pentagon equation

$$W_{12}W_{23} = W_{23}W_{13}W_{12}.$$  

We see below that $W$ contains more information on the given Kac algebra $K$. First, since $W \in \mathcal{M} \otimes \mathcal{M}^\wedge$, it has the form

$$W = \sum_{i=1}^{d} a_i \otimes \lambda(f_i),$$

where $a_i, f_i \in \mathcal{M}$ ($i = 1, 2, \ldots, n$). We may assume that $\{f_1, f_2, \ldots, f_d\}$ is linearly independent in $\mathcal{M}$.

**Proposition 1.** With the above notation, we have $d = \dim \mathcal{M}$. Thus $\{f_1, f_2, \ldots, f_d\}$ is a basis for $\mathcal{M}$. In fact, for any $f \in \mathcal{M}$,

$$f = \sum_{i=1}^{d} \varphi(f a_i^*) f_i = \sum_{i=1}^{d} \varphi(f^\vee a_i) f_i = \sum_{i=1}^{d} \varphi(f^\vee a_i^*) f_i^*.$$  

Moreover, the set $\{a_1, a_2, \ldots, a_d\}$ also forms a basis for $\mathcal{M}$ and satisfies

$$a = \sum_{i=1}^{d} \varphi(a f_i^\vee) a_i = \sum_{i=1}^{d} \varphi(a f_i^\#) a_i^\# = \sum_{i=1}^{d} \varphi(a^\vee f_i^\#) a_i^\#$$

for any $a \in \mathcal{M}$. Moreover,

$$\Gamma(a) = \sum_{i=1}^{d} a_i \otimes (f_i \ast a) \quad (a \in \mathcal{M});$$

$$\hat{\Gamma}(\lambda(f)) = \sum_{i=1}^{d} \lambda(f_i^\#) \otimes \lambda(a_i^* f)$$

for any $f \in \mathcal{M}$. The algebra $\mathcal{L}(L^2(\varphi))$ coincides with $\text{span}\{\lambda(f_i)a_j : 1 \leq i, j \leq d\}$. The unique conditional expectations $E_{\mathcal{M}}$ and $E_{\mathcal{M}^\wedge}$ from $\mathcal{L}(L^2(\varphi))$ onto $\mathcal{M}$ and $\mathcal{M}^\wedge$ with respect
to the normalized trace on $\mathcal{L}(L^2(\varphi))$ is respectively given by

$$E_{\mathcal{M}}\left(\sum_{i=1}^{d}\lambda(f_i)b_i\right) = \sum_{i=1}^{d}\varepsilon(f_i)b_i \quad (b_i \in \mathcal{M});$$

$$E_{\mathcal{M}^\wedge}\left(\sum_{i=1}^{d}\lambda(k_i)a_i\right) = \sum_{i=1}^{d}\varphi(a_i)\lambda(k_i) \quad (k_i \in \mathcal{M}).$$

In particular,

$$E_{\mathcal{M}}(\lambda(f)) = \varepsilon(f)\cdot 1,$$

$$E_{\mathcal{M}^\wedge}(a) = \varphi(a)\cdot 1.$$  

Thus the diagram

$$\begin{array}{ccc}
\mathcal{M}^\wedge & \rightarrow & \mathcal{L}(L^2(\varphi)) \\
\uparrow & & \uparrow \\
\mathcal{C} & \rightarrow & \mathcal{M}.
\end{array}$$

is a commuting square.

Therefore, Proposition 1 proves the preceding Theorem for the case $n = 0$.

Let $A_\infty$ and $B_\infty$ be the approximately finite-dimensional (AF) $C^*$- algebras obtained from the sequences $\{A_n\}$ and $\{B_n\}$, respectively. The algebra $A_\infty$ is regarded as a $C^*$-subalgebra of $B_\infty$ in an obvious way. $B_\infty$ is the $d^\infty$-UHF algebra and thus has the unique faithful factorial tracial state $\tau$. We denote by $Q$ the von Neumann algebra $\pi_\tau(B_\infty)$'" generated by the GNS representation $\pi_\tau$ of $\tau$ on $B_\infty$, which is the AFD factor of type $II_1$. Set $\mathcal{P} = \pi_\tau(A_\infty)^\prime\prime \subseteq Q$. The algebra $\mathcal{P}$ is again the AFD factor of type $II_1$. Therefore, we have constructed a factor-subfactor pair of the AFD factors $\mathcal{P}$ and $Q$.

§ 3. Construction of an action $\beta$ on $\mathcal{P}$

To motivate an idea, we digress and consider a problem of constructing an action $\alpha$ of a group $G$ on a von Neumann algebra $\mathcal{A}$ when $G$ is given. One way to do this is

(i) to find a Hilbert space $\mathcal{H}$ on which $G$ admits a unitary representation $u$ so that $u(s)Au(s)^* = \mathcal{A}$ for any $s \in G$;
(ii) then define $\alpha_s = Adu(s)$.

In terms of the correspondence
\[ \{\alpha : G \rightarrow \text{Aut}(A)\} \overset{\text{bijection}}{\rightarrow} \{\beta : \beta\text{ is an action of the Kac algebra }\ell^\infty(G)\text{ on }A\}, \]
this procedure is the same as

(i) to find a Hilbert space $\mathcal{H}$ for which there exists a unitary $R \in \mathcal{L}(\mathcal{H}) \otimes \ell^\infty(G)$ satisfying $(\iota \otimes \Gamma_G)(R) = R_{12}R_{13}$ ($\Gamma_G$ is the coproduct of $\ell^\infty(G)$) and $R(A \otimes C)R^* \subseteq A \otimes \ell^\infty(G)$;

(ii) then define $\beta(a) = R(a \otimes 1)R^*$.

For a general $K=(\mathcal{M}, \Gamma, \kappa, \varphi)$, the idea is the same. Namely we

(i) find a unitary $R \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}$ satisfying $(\iota \otimes \Gamma)(R) = R_{12}R_{13}$ and $R(A \otimes C)R^* \subseteq A \otimes \mathcal{M}$;

(ii) then define $\beta(a) = R(a \otimes 1)R^*$.

So we will look for such a unitary $R$ below to construct an action $\beta$ on the factor $\mathcal{P}$.

First, let us look at the embedding, say $\gamma$, of $B_0$ into $Q$:
\[ \gamma : B_0 = \mathcal{M}^{' \vee} \hookrightarrow B_\infty \subseteq Q. \]

Secondly, with $W$ as the fundamental unitary of $K$, consider $S = \sigma W \sigma$ which lies in $\mathcal{M}^{' \otimes} \mathcal{M}$. Put $R = (\gamma \otimes \iota_\mathcal{M})(S) \in Q \otimes \mathcal{M}$.

**Theorem 2.** The unitary $R$ satisfies $(\iota \otimes \Gamma^\sigma)(R) = R_{12}R_{13}$ and $R(\mathcal{P} \otimes C)R^* \subseteq \mathcal{P} \otimes \mathcal{M}$.

Thus the equation
\[ \beta(X) = R(X \otimes 1)R^* \quad (X \in \mathcal{P}) \]
defines an action of the reflection $K^\sigma$ on $\mathcal{P}$. Moreover, the inclusion $\mathcal{P} \subseteq Q$ is spatially isomorphic to $\mathcal{P} \subseteq \mathcal{P} \times_\beta K^\sigma$. 
To ensure that $\beta$ is not a trivial action, we show that it is outer, i.e., the relative commutant $\beta(\mathcal{P})' \cap \mathcal{P} \times_\beta K^\sigma$ is trivial. This is done by proving the following theorem.

**Theorem 3.** With the notation as before, we have

$$E_{B_n}(B_{n+1} \cap A_{n+1}') \subseteq C,$$

where $E_{B_n}$ is the unique conditional expectation from $Q$ onto $B_n$ with respect to the normalized trace on $Q$.

The essential part of the proof of this theorem is to prove the assertion when $n = 0$. If $n = 0$, then, as we noted,

$$\begin{align*}
\mathcal{M} & \to \mathcal{L}(L^2(\varphi)) \\
\uparrow & \equiv \uparrow \\
C & \to \mathcal{M}. \\
\end{align*}$$

From this, we see that the assertion of the theorem is equivalent to $E_{\mathcal{M}'}(\mathcal{M}') \subseteq C$. Thus it suffices to prove that the diagram

$$\begin{align*}
\mathcal{M}^\ast & \to \mathcal{L}(L^2(\varphi)) \\
\uparrow & \uparrow \\
C & \to \mathcal{M}'.
\end{align*}$$

is also a commuting square. But this can be verified exactly the same way as before.

§ 4. **The Jones index of $\mathcal{P} \subseteq \mathcal{Q}$**

To compute the Jones index $[\mathcal{Q} : \mathcal{P}]$, it is enough by Theorem 2 to calculate $[\mathcal{P} \times_\beta K^\sigma : \mathcal{P}]$. For this purpose, we describe the Jones projection $e_\mathcal{P}$ of this inclusion. First, it can be shown that $\tilde{J} \beta(\mathcal{P}) \tilde{J} = \mathcal{P}' \otimes C$, where $\tilde{J}$ is the modular conjugation of the normalized trace on the crossed product. Hence the extension of $\mathcal{P} \subseteq \mathcal{P} \times_\beta K^\sigma$ is $\mathcal{P} \otimes \mathcal{L}(L^2(\varphi))$. So $e_\mathcal{P}$ belongs to $\mathcal{P} \otimes \mathcal{L}(L^2(\varphi))$. It can be proven that it has the form

$$e_\mathcal{P} = 1 \otimes p,$$
where $p$ is a minimal projection in $\mathcal{L}(L^2(\varphi))$. In fact, $p$ is the projection corresponding to the one-dimensional representation of $\mathcal{M}$, i.e., the counit $\varepsilon$. Thus

$$\text{Trace}(ep) = (\dim \mathcal{M})^{-1}.$$ 

Therefore, $[\mathcal{P} \times_\beta K^\sigma : \mathcal{P}] = \dim \mathcal{M}$.

References


