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Extension algebras of $C^*$-algebras via canonical *-endomorphisms

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§1. Introduction. Let $A$ be a unital $C^*$-algebra and $\gamma$ a unit preserving *-endomorphism of $A$. When $\gamma$ satisfies a certain condition which are naturally obtained through the index theory, we call the $\gamma$ canonical. Let $\phi$ be a faithful state on $A$ with $\phi(\gamma(x)) = \phi(x)$ for all $x \in A$. Using the canonical $\gamma$, we define a *-endomorphism $\rho$ of $A$ for which there exists a $\lambda (0 < \lambda < 1)$ so that $\phi(\rho(x)) = \lambda \phi(x)$ for all $x \in A$. From such a *-endomorphism $\rho$, we have a representation $\pi$ of $A$ and a non unitary isometry $W$ which satisfy the following conditions:

(1) $W\pi(a)W^* = \pi(\rho(a))$ for all $a \in A$,

and

(2) $W^*\pi(A)W \subset \pi(A)$.

Let $\langle A, \gamma \rangle$ be the $C^*$-algebra generated by $\pi(A)$ and $W$. We call $\langle A, \gamma \rangle$ the extension algebra of $A$ by the canonical *-endomorphism $\gamma$. We give a condition for $\langle A, \gamma \rangle$ to be simple. We show that the canonical *-endomorphism $\gamma$ of $A$ is always extended to a canonical *-endomorphism $\hat{\gamma}$ of $\langle A, \gamma \rangle$.

The terminology canonical is used by Ocneanu for *-endomorphism on some important *-algebras in the classification theory of subfactors of the hyperfinite II$_1$ factor. For *-endomorphism on the Cuntz algebra $O_n$, same terminology canonical is used by Cuntz. We show that the extension algebra $\langle A, \gamma \rangle$ is always simple for pairs $\{A, \gamma\}$ due to Ocneanu and that the canonical *-
endomorphism of Cuntz is the extension $\hat{\gamma}$ of a special shift $\gamma$

Our extension algebra $\langle A, \gamma \rangle$ is generated by $A$ and an isometry $W$ with the relation (1) and (2). Paschke([Pa]) proved that if $A$ is strongly amenable and simple, then the $C^*$-algebra $C^*(A,W)$ generated by $A$ and $W$ with relations (1) and (2) is always simple. We show that the amenable simple $C^*$-algebra $O_n$ (which is not strongly amenable) has two kinds of isometries $W_1$ and $W_2$ with relations (1) and (2) which give simple $C^*(O_n,W_1)$ and non simple $C^*(O_n,W_2)$.

§2. Crossed product by *-endomorphisms. In this section, we define a crossed product of a $C^*$-algebra $A$ by a *-endomorphism $\rho$ and investigate some properties of the crossed products which we need in the next section. Our method is somewhat similar to the method for von Neumann algebras in [A] and [T]. Related topics are investigated in [D], [Pa] and [S].

Let $A$ be a unital $C^*$-algebra with a faithful state $\phi$ and $\rho$ a *-endomorphisms of $A$ which satisfies for some $\lambda(0 < \lambda < 1)$ that

$$\phi(\rho(x)) = \lambda \phi(x), \quad (x \in A).$$

Let $\xi_0$ be the image of the unit $1$ of $A$ in the Hilbert space $L^2(A, \phi)$. We denote by $p_n$ the projection $\rho^n(1)$. Put

$$H_k = \begin{cases} L^2(A, \phi), & (k \leq 0) \\ p_k L^2(A, \phi), & (k \geq 1) \end{cases}$$

and

$$H = \sum_{k \in \mathbb{Z}} \oplus H_k.$$

Let $\pi$ be the representation of $A$ on $H$ defined by

$$(\pi(x)\eta)_k = \begin{cases} \rho(x)\eta_k, & (k \leq 0) \\ \rho^k(x)\eta_k, & (k \geq 1) \end{cases}.$$
where $\eta = (\eta_k)_{k \in \mathbb{Z}}$ for $\eta_k \in H_k$. Let $W$ be the isometry defined by

$$(W(\eta))_k = \begin{cases} \frac{1}{\sqrt{\lambda}}\rho(x_{k-1})\xi_0, & (k \leq 0) \\ \frac{1}{\lambda}\rho^2(x_{k-1})\xi_0, & (k \geq 1), \end{cases}$$

where $\eta = (\eta_k)_{k \in \mathbb{Z}}$ and $\eta_k = x_k\xi_0$, for some $x_k \in A$. Then $W$ and $\pi$ satisfies

$$W\pi(x) = \pi(\rho(x))W \quad \text{for all} \quad x \in A$$

and $W^kW^{*k} = \pi(p_k)$ for all $k$. We assume that $W^*\pi(A)W \subset \pi(A)$. In the next section, we show that the $*$-endomorphism $\rho$ induced by a canonical $*$-endomorphism $\gamma$ satisfies this assumption.

**Proposition 1.** Let $C^*(A, W)$ be the $C^*$-algebra generated by $\pi(A)$ and the above isometry $W$. Then there exists a faithful conditional expectation $E$ of $C^*(A, W)$ onto $\pi(A)$ with $E(aW_k) = 0$, for all $a \in \pi(A), k \geq 1$. The state $\phi$ is extended to a faithful state $\psi$ of $C^*(A, W)$ which satisfies $\psi(aW^k) = 0, (a \in \pi(A), k \geq 1)$.

Let us consider the following condition (*) for a $*$-endomorphism $\gamma$ of $A$ due to Kishimoto[K] in the case of automorphisms.

**Condition (\ast).** For an $a \in A$, a finite set $S$ in $A$, a finite set $F$ in integers $\mathbb{N}$ and $\epsilon > 0$, there exists a positive $x \in A$ with $\|x\| = 1$ such that

$$\|xax\| \geq \|a\| - \epsilon, \quad \|xs\rho^k(x)\| \leq \epsilon (s \in S, k \in F).$$

**Proposition 2.** If $\rho$ satisfies the condition (*), then $C^*(A, W)$ is simple when the only proper ideal $J$ of $A$ for which $WJW^* \subset J$ is the zero ideal.

**Remark 3.** If the $C^*$-algebra $A$ is strongly amenable, then we dont need the condition (*) for simplicity of $C^*(A, W)$ by [Pa]. However, in the case of $A$ which is not strongly amenable, $\langle A, W \rangle$ is not always simple without
any condition like (*). we show in §3 an example of a pair of an amenable $C^*$-algebra $A$ and a canonical *-endomorphism $\gamma$ which implies non simple $C^*(A,W)$.

§3. Canonical *-endomorphisms. In this section we define a canonical *-endomorphism $\gamma$ on $C^*$-algebras, and applying the method of crossed products in §2 to the *-endomorphism $\rho$ induced from $\gamma$, we investigate relations between extension algebras and canonical *-endomorphisms.

**Definition 4.** Let $A$ be a unital $C^*$ algebra with a faithful state $\phi$. Let $\gamma$ be a *-endomorphism of $A$ with $\phi \cdot \gamma = \phi$. If there exist two projections $e \in \gamma(A)' \cap A$ and $f \in \gamma^2(A)' \cap A$ such that:

\[ eAe = \gamma(A)e, \quad \phi(e) = \phi(f) \]

and the projections $\{e_1(= e), e_2(= f), e_3(= \gamma(e))\}$ satisfies Jones relation for $\phi(e) = \phi(f) = \lambda$, that is,

\[ e_i e_j e_i = \lambda e_i, \quad (|i - j| = 1), \quad e_i e_j = e_j e_i, \quad (|i - j| \neq 1) \]

then $\gamma$ is said to be canonical. We call the projection $e$ a basic projection for $\gamma$.

Let $\gamma$ be a canonical *-endomorphism of $A$ and $e$ a basic projection for $\gamma$. We define the *-endomorphism $\rho$ on $A$ by

\[ \rho(a) = e\gamma(a), \quad (a \in A) \]

Then $\rho$ satisfies that $\phi(\rho(x)) = \lambda \phi(x)$ for all $x \in A$, where $\lambda = \phi(e)$.

**Lemma 5.** Let $\rho$ be a *-endomorphism defined by (**) for a canonical *-endomorphism $\gamma$ of $A$. Then the isometry $W$ defined by $\rho$ satisfies that

\[ W^*AW \subset A. \]
By Lemma 5, we can apply the result in §2 to the $\rho$ defined by (***) and we consider the crossed product $C^*(A,W)$ in §2 which we denote by $< A, \gamma >$. We call $< A, \gamma >$ the extension algebra of $A$ via a canonical $^*$-endomorphism $\gamma$.

**Proposition 6.** Let $A$ be a unital $C^*$-algebra with a faithful state $\phi$ and $\gamma$ a canonical $^*$-endomorphism of $A$. Let $e$ and $f$ be projections in Definition 4. Then there exists a canonical $^*$-endomorphism $\hat{\gamma}$ of $< A, \gamma >$ which satisfies that

$$\hat{\gamma}(x) = \gamma(x), \quad (x \in \pi(A)) \quad \hat{\gamma}(W) = vW,$$

where $v = \lambda^{-1}\gamma(e)f$. Many typical canonical $^*$-endomorphisms $\gamma$ are given as $\gamma = \sigma^2$ for some $^*$-endomorphism $\sigma$. Such a $\sigma$ is also extended to a $^*$-endomorphism $\hat{\sigma}$ of $< A, \gamma >$ which satisfies $\sigma(W) = \lambda^{-1/2}fe$.

§4. **Examples.** In this section we shall restrict ourselves to the case of concrete $C^*$-algebras and show relations between canonical $^*$-endomorphisms.

**Example 1.** Let $A_0$ be the $n$ by $n$ matrix algebra $M_n(\mathbb{C})$ over the complex numbers $\mathbb{C}$. Put $A_i = A_0$ for all integer $i$. Let $A$ be the infinite $C^*$-tensor product $\bigotimes_i^\infty A_i$. Let $\gamma$ be the 1-shift translation to the right on $A$. For a matrix units $e_{i,j}$ of $M_n(\mathbb{C})$. We identify $e_{i,j}$ and $e_{i,j} \otimes 1 \otimes \cdots$. Put $e = e_{1,1}$ and

$$u = \sum_{i=1}^n e_{i,i-1}, \quad f = \sum_{i,j} u^{j-i} \otimes e_{i,j} / n.$$

Then $\gamma, \tau, e$ and $f$ satisfies the conditions in Definition 4 for $\gamma$ to be canonical.

Cuntz [Cu] defined the simple $C^*$-algebra $O_n$ which generated by isometries $(S_j)_{1 \leq i \leq n}$ with $S_iS_j = \delta_{i,j}1$ and $\sum_i S_iS_i^* = 1$. He obtained interesting results
using basically his "canonical" inner *-endomorphism $\Phi$ on $O_n$ defined by

$$\Phi(x) = \sum_j S_j x S_j^* \quad \text{for all} \quad a \in A.$$ 

**Proposition 7.** Let $A$ and $\gamma$ be the same as in Example 1. Then the extension algebra $< A, \gamma >$ is the Cuntz algebra $O_n$ and the extension $\hat{\gamma}$ of $\gamma$ to $< A, \gamma >$ is Cuntz's canonical inner *-endomorphism $\Phi$.

**Remark 8.** By Proposition 5 and 6, Cuntz's endomorphism $\Phi$ is also canonical. Hence we have the extension algebra $< O_n, \Phi >$. By the definition, $< O_n, \Phi >$ is generated by $B = \pi(O_n)$ and an isometry $W(= W_\Phi$, which comes from $\Phi$) with

$$(4) \quad WBW^* \subset B, \quad W^*BW \subset B$$

Paschke proved that if $B$ is strongly amenable and $W$ is a non unitary isometry with the condition (4) then the $C^*$-algebra generated by $B$ and $W$ is always simple. By [Cu], $O_n$ is simple and amenable but not strongly amenable. Following Proposition shows that his result does not hold without strong amenability. We also remark that $O_n$ is generated by $O_n$ and $W$ which satisfy the condition (4), and $O_n$ is simple. Hence $O_n$ has two isometries with relation (4) one of which gives a simple $C^*$-algebra and the other gives a non simple algebra.

**Proposition 9.** Let $\Phi$ be Cuntz's *-endomorphism on $O_n$. Then $< O_n, \Phi >$ is isomorphic to the tensor product $O_n \otimes C^*(u)$. Here $u$ is a unitary with $u^j \neq 1$ for all integer $j$.

**Example 2.** Let $A, \tau$ and $\gamma$ be the same as in Example 1. Put

$$\eta(\gamma^m(e_{p,q})) = \sum_j e_{j,j} \gamma^{m+1}(e_{p+j,q+j})$$
for all $m \geq 0$. Then $\eta$ is extended to the $\tau$ preserving $*$-endomorphism of $A$. Put
\[ e = e_{1,1}, \quad f = \sum_{i,j} \frac{e_{i,j}}{n}. \]
Then $\eta, e$ and $f$ satisfy the conditions for $\eta$ to be canonical.

In this case,
\[ \rho(x) = e \gamma(UxU^*), \quad \text{for } U = \otimes_{i=1}^{\infty} u_i \text{ all } x \in A, \]
where $u_i$ is the unitary $u$ in the example 1. This implies the following:

In [Cu 2], Cuntz defined the $*$-endomorphism $\lambda_R$ on $O(H)$ of a finite dimensional Hilbert space $H$ defined by
\[ \lambda_R(S) = RS, (S \in H) \]
for $R = FV$. Here $F$ is the flip symmetry of $H \otimes H$ and $V$ is a multiplicative unitary on $H \otimes H$. Example 2 is generalized to a general finite dimensional Hilbert space $H$ by taking a suitable orthonormal basis. The following shows that Cuntz's $\lambda$ is also canonical.

**proposition 10.** Let $A, \tau$ and $\eta$ be the same as in Example 2. Then $<A, \eta>$ is $O_n$ and the extension $\hat{\eta}$ of $\eta$ to $<A, \eta>$ is the $*$-endomorphism $\lambda_R$ due to Cuntz.

**Example 3.** Let $N \subset M$ be an inclusion of type II$_1$ factors with Jones index $[M : N] < \infty$. Put $\lambda = [M : N]^{-1}$. Iterating the basic construction for $N \subset M$, we have the tower of II$_1$ factors:

\[ N \subset M_0 = M \subset M_1 \subset \cdots \subset M_j = <M_{j-1}, e_j> \subset \cdots. \]

Here, $e_j$ is the Jones projection for $M_{j-2} \subset M_{j-1}$. Let $\tau_0$ be the unique trace of $M$. Then $\tau_0$ is extended to the trace $\tau_j$ of $M_j$ via $\tau(xe_j) = \lambda \tau(x)$ for all
$x \in M_{j-1}$. Let

$$A_j = M' \cap M_j$$

for all integer $j$.

For an $x \in \bigcup_j A_j$, put $\tau(x) = \tau_j(x)$ when $x \in A_j$ for some $j$. Let $A$ be the $C^*$-algebra obtained from the GNS construction of $\bigcup_j A_j$ by $\tau$. Then $\tau$ induces a tracial state (which we denote by the same notation $\tau$) on $A$. The antiautomorphism $\gamma_j$ of $A_{2j}$ defined by

$$\gamma_j(x) = J_j x^* J_j, \quad x \in A_{2j},$$

where $J_j$ is the canonical conjugation on $L^2(M_j, \tau_j)$. Then we have ([Ch-H], [O])

$$\gamma_{j+1} \cdot \gamma_j(x) = \gamma_j \cdot \gamma_{j-1}(x), \text{ for all } x \in A_{2j-2} \text{ and } j \geq 1.$$

Since $\gamma_j$ is $\tau_j$ preserving, there is a $\tau$ preserving $*$-endomorphism $\Gamma$ on $A$ defined by $\Gamma(x) = \gamma_{j+1} \gamma_j(x)$ for all $x \in A_{2j}$ ([O]) This $\Gamma$ is called Ocneanu's 2-shift. The Jones projection $e = e_1$ and $f = e_2$ satisfy the conditions for $\Gamma$ to be a canonical $*$-endomorphism for the pair $\{A, \tau\}$.

**Example 6.** Let $G$ be a finite bipartite with a Ocneanu's biunitary connection ([O2]). Then we have the $C^*$-algebra $A$ with a unique trace $\tau$ obtained from path algebras on $G$ and a $*$-endomorphism $\sigma$ on $A$ induced by the connection. Put $\gamma = \sigma^2$. Then $\gamma$ is canonical. the first Jones projection $e_1$ and and the second Jones projection $e_2$ in the path algebra satisfies the conditions in definition 4.

From these canonical $*$-endomorphisms $\gamma$ on $C^*$-algebras $A$, we obtain simple $C^*$- algebras $<A, \gamma>$ because we have always a projection $q$ depending the basic projection for these $\gamma$ as an element $x$ in the condition $(\ast)$. M. Izumi told me these $C^*$-algebras are not always Cuntz algebras because they have different $K_0(<A, \gamma>)$ from $K_0(O_n)$. Any way we have simple amenable but
non strongly amenable $C^*-$algebras $<A,\gamma>$ and canonical $^*$-endomorphisms on $<A,\gamma>$. Anyway, some results are published in a forthcoming paper. Also related algebras are obtained by Izumi and Katayama in this report.

The terminology "canonical" for $^*$-endomorphisms on infinite factors are used in many papers by Longo (for instance, [L1, L2, L3]). We have similar results as in $C^*-$algebras for factors, which are described in [ch].

References


[Ch 2] M. Choda : Type III$_\lambda$ factors for a $^*$-endomorphism of II$_1$ factors with index $\lambda^{-1}$. in preparation.


