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# Phase operator problem and infinitesimal analysis

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**Abstract**

To find an operator representation of the phase variable of a single-mode electromagnetic field, the Schrödinger representation is extended to a larger Hilbert space including states with infinite excitation by nonstandard analysis, so that the self-adjoint phase operator is acting on the extended Hilbert space. It is shown that this phase operator is a Naimark extension of the optimal probability operator-valued measure for the phase parameter already known in the quantum estimation theory, and is naturally considered as the controversial limit of the phase operators on finite dimensional spaces recently proposed by Pegg and Barnett. Eventually, two of recent promising attempts to obtain the correct statistics of the phase variable in quantum mechanics is synthesized in the present framework based on new mathematical ideas concerning the number system including infinite and infinitesimal numbers.

## 1. Introduction

The existence and properties of a self-adjoint operator on a Hilbert space, corresponding to the phase of the electromagnetic field, has provoked many discussions for some time, since Dirac [7] first discussed the problem. According to the uniqueness theorem of the irreducible representations of the canonical commutation relations due to von Neumann, the commutation relation between the number operator and the phase operator, which Dirac presupposed by employing the correspondence between commutators and classical Poisson brackets, cannot be realized. Further, Susskind and Glogower [31] clearly demonstrated that the polar decomposition of the annihilation operator into the unitary operator of the exponential of the phase and the square root of the number operator presupposed by Dirac is also impossible. However, Pegg and Barnett [28] recently made an interesting proposal for the problem. They constructed a phase operator on a finite, but supposedly very large, dimensional space and showed that it obeys a certain new commutation relation with the number operator. According to their proposal, the correct statistical prediction is given after the limit process making the dimension infinity, although they failed to find the operator after the limit process.

On the other hand, another approach to the problem has been established in quantum estimation theory [15,18]. This theory discusses a quite general type of optimization problems of quantum measurements. The statistics of measurements are represented in this theory by the probability operator-valued measures (POM) on Hilbert spaces which extend the conventional description by the self-adjoint operators. In this approach, the optimum POM of the estimation problem of the phase parameter was found by Helstrom [14] and mathematically rigorous development of this approach is given by Holevo [18].

A promising aspect of these two approaches is that the statistics of the phase variable obtained by Pegg and Barnett coincides naturally with the one represented by the optimum POM of the phase parameter. This shows, however, that contrary to their claim the limit of their exponential phase operator is nothing but the well-known Susskind-Glogower exponential phase operator [31], as long as the limit is taken in the original Hilbert space. Since the limit process within the original Hilbert space does not preserve the function calculus and destroys the desired properties of the phase operator such as unitarity of the exponentials, this limit has never the features which Pegg and Barnett [28] described intuitively. According to the Naimark theorem, every POM can be extended to a projection-valued measure on a larger Hilbert space, which gives rise to a self-adjoint operator, by the spectral theory, representing an observable in the standard formulation of quantum mechanics. This suggests that there is the phase operator, somewhere beyond the Schrödinger representation, which is the intuitive limit of the Pegg-Barnett phase operators as well as being an extension of the optimal POM. Thus in order to realize the Pegg and Barnett phase operator on the dimension at infinity, we need an alternative mathematical construction other than limits in the Hilbert space.

For this purpose nonstandard analysis will be used in this paper. The nonstandard analysis was invented by Robinson [29] and has yielded rigorous and fruitful mathematics of infinite and infinitesimal numbers as well as has realized the Leibniz infinitesimal calculus. We shall construct a natural extension of the Schrödinger representation and show that the desired phase operator exists on this extended Hilbert space. The Hilbert space of this extension of the Schrödinger representation is the direct sum of the original Schrödinger representation and the space of the states with infinite excitation which are naturally considered as the classical limits of the ordinary quantum states. Thus our construction supports the following heuristic reasoning on why the phase operator does not exist in the Schrödinger representation: The unitary operator $e^{i\hat{A}}$ of one quantity $A$, from a pair of canonically conjugate quantities $A$ and $B$, changes an eigenvector of the operator $\hat{B}$ to another eigenvector in such a way as the eigenvalue changes in some magnitude, say $\delta B$. Similarly, $e^{i\hat{B}}$ changes an eigenvector of $\hat{A}$ by $\delta A$. Then, it occurs either that both $\delta A$ and $\delta B$ are finite non-infinitesimal numbers or that one of them is infinite and the other is infinitesimal. The first case is the case of so-called the Schrödinger pairs such as the position-momentum pair. The second case arises when one of the pair is a quantized quantity. In this case, the operator of the quantity conjugate to the quantized quantity changes the eigenvalue in magnitude of an infinite number and cannot be represented by an operator on the state space of the Schrödinger representation which contains only eigenstates with a finite eigenvalue. This is true, even if the state space is understood as the Schwarz space extending the Hilbert space, since every self-adjoint operator defined on this space has a complete system of eigenvectors with *finite* eigenvalues.

For bibliography on the phase operator problem, we shall refer to the references of [27,2,4]. For the quantum estimation theory, [35,13,14,15,17,18], and for recent development of quantum measurement theory, [6,5,25,26]. Applications of nonstandard analysis to physics is not new and has been developed in such papers as [3,8,9,12,20,32,21,22,33,34,24,11,10], and in such a monograph as [1]. For mathematical foundations, we shall refer to the following monographs [29,30,19,1].

## 2. The Susskind-Glogower Phase operators

The single-mode electromagnetic field is a well-known physical system which has been modeled by the quantum mechanical harmonic oscillator with unit mass. Let $\mathcal{H}$ be the Hilbert space of the Schrödinger representation of the quantum mechanical harmonic oscillator. Denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. Let $\hat{q}$ and $\hat{p}$ be the position and momentum operators on $\mathcal{H}$. The annihilation operator $\hat{a}$ is defined by

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \tag{2.1}$$

where $\omega$ is the angular frequency, and its adjoint $\hat{a}^\dagger$ is the creation operator. Then the number operator $\hat{N}$ is defined by

$$\hat{N} = \hat{a}^\dagger \hat{a}. \tag{2.2}$$

The number operator $\hat{N}$ has the complete orthonormal basis $\{|n\rangle \mid n = 1, 2, \ldots\}$ of $\mathcal{H}$ for which $\hat{N}|n\rangle = n|n\rangle$. The Hamiltonian $\hat{H}$ of the system is given by $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$.

In his original description of the quantized electromagnetic field, Dirac [7] postulated the existence of a self-adjoint phase operator $\hat{\phi}$ such that the unitary exponential operator $e^{i\hat{\phi}}$ of $\hat{\phi}$ would appear in the polar decomposition of the annihilation operator

$$\hat{a} = e^{i\hat{\phi}} \hat{N}^{-1/2}. \tag{2.3}$$

The difficulty with this approach were clearly pointed out by Susskind and Glogower [31] by showing that the polar decomposition of $\hat{a}$ cannot be realized by any unitary operator. Instead, they introduced the isometries representing the exponentials of the phase variable

$$\hat{e}^{i\phi} = (\hat{N} + 1)^{-1/2} \hat{a}, \tag{2.4}$$

$$\hat{e}^{-i\phi} = \hat{a}^\dagger (\hat{N} + 1)^{-1/2}, \tag{2.5}$$

and the self-adjoint operators representing the sine and cosine of the phase variable

$$\widehat{\cos}\,\phi = \frac{1}{2}(\hat{e}^{i\phi} + \hat{e}^{-i\phi}), \tag{2.6}$$

$$\widehat{\sin}\,\phi = \frac{1}{2i}(\hat{e}^{i\phi} - \hat{e}^{-i\phi}). \tag{2.7}$$

Note that the places of the carets in these Susskind-Glogower operators suggest that these operators are not derived by the function calculus of a certain self-adjoint operator corresponding to $\phi$. These operators are considered to behave well in the limit of large amplitudes but they fail to define well-behaved operators for periodic functions of the phase in the quantum regime. Thus we cannot derive the correct statistics of the phase variable from these operators. However, it turns out that that these operators give the correct mean values of the corresponding quantities $e^{\pm i\phi}$, $\sin\phi$ and $\cos\phi$. A systematic method for obtaining the correct statistics needs a new mathematical concept extending the self-adjoint operators, which is described in the next section.

### 3. Quantum estimation problem of the phase parameter

A positive operator-valued measure $P(d\theta)$ on the Borel field $\mathcal{B}(\mathbf{R})$ of the real line $\mathbf{R}$ with values in $\mathcal{L}(\mathcal{H})$ is called a *probability operator-measure* (POM) if $P(\mathbf{R}) = 1$. Now, we shall start with the following presupposition: *Corresponding to any measurable physical quantity $X$, there is a POM $P_X(d\theta)$ such that the probability distribution of $X$ in state $\psi \in \mathcal{H}$ is given by the probability measure $\langle\psi|P_X(d\theta)|\psi\rangle$, which predicts the statistics of outcomes of ideal measurements of $X$ in $\psi$.* In the conventional framework of quantum mechanics, every observable which has the corresponding self-adjoint operator in $\mathcal{H}$ has the POM as its spectral measure. However, there may be some physical quantities which are considered to have the POM but no self-adjoint operators in $\mathcal{H}$. For measurability of POM's, it is known that for any POM $P(d\theta)$ on $\mathcal{H}$, there is another Hilbert space $\mathcal{K}$, a unit vector $\xi \in \mathcal{K}$, a unitary operator $U$ on $\mathcal{H}\otimes\mathcal{K}$, and a self-adjoint operator in $\mathcal{K}$ with spectral measure $E(d\theta)$ satisfying

$$\langle\psi|P(d\theta)|\psi\rangle = \langle\psi\otimes\xi|U^{\dagger}(1\otimes E(d\theta))U|\psi\otimes\xi\rangle, \tag{3.1}$$

for all $\psi \in \mathcal{H}$ [25]. This suggests that for any POM $P(d\theta)$ there is an experiment, with the outcome to be predicted by $P(d\theta)$ statistically, which consists of the following process; 1) preparation of the apparatus, described by $\mathcal{K}$, in state $\xi$, 2) interaction, described by $U$, between the object, described by $\mathcal{H}$, and the apparatus, 3) measurement of the observable in the apparatus corresponding to $E(d\theta)$. We shall call any experiment consisting of the above process and satisfying (3.1) as a *measurement* of POM $P(d\theta)$. Thus, our presupposition is a conservative extension of the standard formulation of quantum mechanics, in the sense that, if every observable can be measured, so can any quantity corresponding to a POM.

The determination of the statistics of the phase variable is thus reduced to determination of POM corresponding to the phase variable. This problem is solved in the quantum estimation theory as follows. Since the phase is canonically conjugate to the action variable in classical mechanics, the number operator is the infinitesimal generator of the phase shift operators $e^{i\theta\hat{N}}$. Thus the POM $P(d\theta)$ corresponding to the phase variable should satisfy the relations

$$\begin{gathered} e^{-i\theta\hat{N}}P(B)e^{i\theta\hat{N}} = P(B_{-\theta}), \\ B, B_{-\theta} \subset [0,2\pi), \quad B_{-\theta} = B - \theta \pmod{2\pi}. \end{gathered} \tag{3.2}$$

Any POM satisfying (3.2) is called a *covariant POM.* It is well known that there is no self-adjoint operators such that their spectral measures satisfy the above relations but there are many solutions among general POM's. In order to select the optimum one, consider the following estimation problem of the phase parameter $\theta$. Let us given a reference state $\psi \in \mathcal{H}$, which is supposed to be modulated by a phase shifter with unknown shift parameter $\theta$ $(0 \leq \theta < 2\pi)$ so that the outgoing state is $\psi_\theta = e^{i\theta\hat{N}}\psi$. The problem is to find an experiment in state $\psi_\theta$ which gives the best estimate of the parameter $\theta$, and is equivalently to find a measurement in the state $\psi_\theta$ the outcome $\bar{\theta}$ of which is the best estimate of the parameter $\theta$. The relevance of this problem to our problem is as follows. Suppose that the reference state $\psi$ were the phase eigenstate $\psi = |\phi = 0\rangle$. Then the outgoing state would also be the phase eigenstate $\psi_\theta = |\phi = \theta\rangle$, for which the the best estimator would give the estimate $\bar{\theta} = \theta$ with probability 1. Thus in this case the best estimator would be the measurement of the phase variable. Thus for a POM $P(d\theta)$ to represent the phase variable, it is necessary that it is the optimum estimator of this estimation problem for reference states which approximate a phase eigenstate well. Although we do not know what and where are phase eigenstates, we can reach the essentially unique solution. For any POM $P(d\theta)$, the joint probability distribution

$$p(d\theta, d\bar{\theta}) = \langle\psi_\theta|P_\phi(d\bar{\theta})|\psi_\theta\rangle\,\frac{d\theta}{2\pi} \tag{3.3}$$

gives naturally the joint probability distribution of the true parameter $\theta$ and the estimate $\bar{\theta}$. Given an appropriate error function $W(\theta - \bar{\theta})$, which gives the penalty for the case $\theta \neq \bar{\theta}$, the optimum estimator should minimize the average error

$$\int_0^{2\pi} W(\theta - \bar{\theta})\,p(d\theta, d\bar{\theta}). \tag{3.4}$$

This type of optimization problem has been studied in quantum estimation theory [15,18]. The common optimum solution, which is also a covariant POM, for a large class of error functions such as $W(x) = 4\sin^2\frac{x}{2}$ or $W(x) = -\delta(x)$, where $\delta(x)$ is the periodic $\delta$-function, is the following POM $P(d\theta)$:

$$\langle n|P(d\theta)|n'\rangle = e^{i(\alpha_n - \alpha_{n'})}e^{i(n-n')\theta}\,\frac{d\theta}{2\pi}, \tag{3.5}$$

where $|n\rangle$ $(n = 0, 1, \ldots)$ is the number basis and $\alpha_n = \arg\langle n|\psi\rangle$. Since this problem is not of the estimation of the absolute phase, the optimum solutions depend on the phase factors $\alpha_n$ of the reference state $\psi$. However, this dependence can be interpreted to reflect the arbitrariness of our choice of the phase eigenstate $|\phi = 0\rangle$, and each choice from the optimum POM's $P(d\theta)$ determines a unique $|\phi = 0\rangle$ among *physically equivalent* alternatives. For simplicity, we choose the solution for $\alpha_n = 1$ $(n = 0, 1, \ldots)$ and denote it by $P_\phi(d\theta)$, i.e.,

$$\langle n|P_\phi(d\theta)|n'\rangle = e^{i(n-n')\theta}\,\frac{d\theta}{2\pi}. \tag{3.6}$$

We will call $P_\phi(d\theta)$ as the *phase POM.*

From this result, we can calculate the mean value of any bounded Borel function $f(\phi)$ of the phase variable $\phi$. Indeed, let

$$\widehat{f(\phi)} = \int_0^{2\pi} f(\theta) P_\phi(d\theta). \tag{3.7}$$

Then the mean value of the quantity $f(\phi)$ in a state $\psi$ is given by $\langle\psi|\widehat{f(\phi)}|\psi\rangle$. An interesting result from this is that the Susskind-Glogower phase operators coincide with the operators defined in this way [18, p. 141], i.e.,

$$\hat{e}^{\pm i\phi} = \int_0^{2\pi} e^{\pm i\theta}\, P_{\phi(d\theta)} = \widehat{e^{\pm i\phi}}, \tag{3.8}$$

$$\widehat{\cos}\phi = \int_0^{2\pi} \cos\theta\, P_\phi(d\theta) = \widehat{\cos\phi}, \tag{3.9}$$

$$\widehat{\sin}\phi = \int_0^{2\pi} \sin\theta\, P_\phi(d\theta) = \widehat{\sin\phi}. \tag{3.10}$$

Thus their operators give the correct mean values of $f(\phi) = e^{\pm i\phi}$, $\sin\phi$ and $\cos\phi$ respectively, but none of their powers.

The phase POM $P_\phi(d\theta)$ gives the correct statistics for all functions of phase variable but it gives little information about algebraic relations between other observables.

## 4. The approach due to Pegg and Barnett

Now we shall turn to the recent proposal due to Pegg and Barnett [28]. They start with the $s$-dimensional subspace $\Psi_s$ of $\mathcal{H}$ spanned by $|n\rangle$ with $n = 0, 1, \ldots, s-1$. For $\theta_m = m\Delta\theta$ $(m = 0, 1, \ldots, s-1)$, where $\Delta\theta = 2\pi/s$, their approximate phase state is

$$|\theta_m\rangle = s^{-1/2} \sum_{n=0}^{s-1} e^{in\theta_m}|n\rangle, \tag{4.1}$$

and their approximate phase operator $\hat{\phi}_s$ on $\Psi_s$ is

$$\hat{\phi}_s = \sum_{m=0}^{s-1} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{4.2}$$

Their intrinsic proposal is that the mean value of the quantity $f(\phi)$ in state $\psi$ is the limit of $\langle\psi|f(\hat{\phi}_s)|\psi\rangle$ as $s \to \infty$. Then for state $\psi = \sum_{n=0}^{k} c_n|n\rangle$ $(k < \infty)$, we have

$$\begin{aligned}
\lim_{s\to\infty}\langle\psi|f(\hat{\phi}_s)|\psi\rangle &= \sum_{n,n'=0}^{k} c_n c_{n'}^* \lim_{s\to\infty}\sum_{m=0}^{s-1} f(\theta_m)e^{i(n-n')\theta_m}\frac{\Delta\theta}{2\pi} \\
&= \sum_{n,n'=0}^{k} c_n c_{n'}^* \int_0^{2\pi} f(\theta)e^{i(n-n')\theta}\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} f(\theta)\,\langle\psi|P_\phi(d\theta)|\psi\rangle \\
&= \langle\psi|\widehat{f(\phi)}|\psi\rangle,
\end{aligned}$$

for any continuous function $f(\theta)$ on $[0, 2\pi]$. Thus, their expectations are the same as those given by the phase POM $P_\phi(d\theta)$, and we have

$$\lim_{s\to\infty} f(\hat{\phi}_s) = \int_0^{2\pi} f(\theta)P_\phi(d\theta) = \widehat{f(\phi)}, \tag{4.3}$$

in the weak operator topology. In particular, the limit of their exponential phase operators are the Susskind-Glogower exponential operators, i.e.,

$$\lim_{s\to\infty} e^{\pm i\hat{\phi}_s} = \widehat{e^{\pm i\phi}}. \tag{4.4}$$

We have, therefore, shown that the statistics of the phase variable obtained by Pegg and Barnett coincides with the statistics obtained by the phase POM, and that the limit of the Pegg-Barnett unitary phase operators on finite dimensional spaces is nothing but the Susskind-Glogower exponential operators, as long as the limit is taken in the original Hilbert space.

## 5. Hyperfinite extension of the Schrödinger representation

For the basic framework of nonstandard analysis we shall refer to Hurd-Loeb [19] and Stroyan-Luxemburg [30] as standard textbooks. In what follows we assume that our nonstandard universe is an $\aleph_1$-saturated bounded elementary extension of a superstructure which contains $\mathcal{H}$. To avoid confusions, we shall use symbol $^\dagger$ for the adjoint operation of operators and the involution on a $^*$-algebra.

Let $(E, p)$ be an internal normed linear space with norm $p$ over the hypercomplex number field $^*\mathbf{C}$. We define the *principal galaxy* $E_G$ of $(E, p)$ and the *principal monad* $E_M$ of $(E, p)$ as follows:

$$E_G = \{x \in E \mid p(x) \text{ is finite}\}, \tag{5.1}$$

$$E_M = \{x \in E \mid p(x) \text{ is infinitesimal}\}. \tag{5.2}$$

Then both $E_G$ and $E_M$ are linear spaces over the complex number field $\mathbf{C}$. Let $\hat{E} = E_G/E_M$ and $\hat{p}([x]) = {}^\circ(p(x))$ for $x \in E_G$, where $[x] = x + E_M$, and $^\circ$ stands for the standard part map on the finite hypercomplex numbers. Then $(\hat{E}, \hat{p})$ becomes a normed linear space over $\mathbf{C}$, called the *nonstandard hull* of a normed linear space $(E, p)$. Under the assumption of $\aleph_1$-saturation, the hull completeness theorem holds and concludes that the nonstandard hull $(\hat{E}, \hat{p})$ is a Banach space.

Let ${}^*\mathcal{H}$ be the nonstandard extension of $\mathcal{H}$ and $\widehat{{}^*\mathcal{H}}$ its nonstandard hull. Then $\widehat{{}^*\mathcal{H}}$ is a Hilbert space with inner product $\langle[\xi]|[\eta]\rangle = {}^\circ({}^*\langle\xi|\eta\rangle)$. Let $\nu$ be a nonstandard natural number and $\mathcal{D}$ the internal $\nu$-dimensional subspace of ${}^*\mathcal{H}$ spanned by the hyperfinite set $\{|n\rangle \mid n = 0, 1, \ldots, \nu - 1\}$. Then the nonstandard hull $\hat{\mathcal{D}}$ contains $\mathcal{H}$ as a closed subspace by the canonical isometric embedding $V_H$ which maps $\xi \in \mathcal{H}$ to $[{}^*\xi] \in \hat{\mathcal{D}}$. Since the family of all $[|n\rangle]$ with $0 \le n \le \nu - 1$ is an orthonormal family in $\hat{\mathcal{D}}$, the external dimension of $\hat{\mathcal{D}}$ is at least the cardinality of the continuum. The nonstandard hulls $\widehat{{}^*\mathcal{H}}$ and $\hat{\mathcal{D}}$ are, thus, Hilbert spaces which satisfy the relations

$$\mathcal{H} \subset \hat{\mathcal{D}} \subset \widehat{{}^*\mathcal{H}}. \tag{5.3}$$

Let $\mathcal{A}$ be the algebra of internal linear operators on $\mathcal{D}$. Then $\mathcal{A}$ is a hyperfinite dimensional internal $*$-algebra over ${}^*\mathbf{C}$. For $x \in \mathcal{A}$, let $\|x\|$ be the internal uniform norm of $x$. Let $\mathcal{A}_G$ be the principal galaxy of $(\mathcal{A}, \|\cdot\|)$ and $\mathcal{A}_M$ the principal monad. Obviously, $\mathcal{A}_G$ is a $*$-algebra and $\mathcal{A}_M$ is a self-adjoint two sided ideal of $\mathcal{A}_G$. Thus, from the hull completeness theorem, the nonstandard hull $\hat{\mathcal{A}}$ of $(\mathcal{A}, \|\cdot\|)$ becomes a Banach $*$-algebra with norm $\|[x]\| = {}^\circ\|x\|$. Then the norm $\|\cdot\|$ satisfies the C*-condition, i.e., $\|[x]^*[x]\| = \|[x]\|^2$ for all $[x] \in \hat{\mathcal{A}}$, and hence $(\hat{\mathcal{A}}, \|\cdot\|)$ is a C*-algebra. The internal algebra $\mathcal{A}$ is internally $*$-isomorphic with the internal matrix algebra $\mathrm{M}(\nu, {}^*\mathbf{C})$ of all $\nu \times \nu$ matrices over ${}^*\mathbf{C}$ by a matrix representation of linear operators. In [16], it is shown that the C*-algebra $\hat{\mathcal{A}}$ has a closed maximal ideal $\mathcal{I}$ defined by

$$\mathcal{I} = \{[x] \in \hat{\mathcal{A}} \mid (\frac{1}{\nu}\mathrm{Tr}[x^*x])^{1/2} \text{ is infinitesimal}\}, \tag{5.4}$$

where Tr stands for the trace of the corresponding matrix, such that $\hat{\mathcal{A}}/\mathcal{I}$ is a type $\mathrm{II}_1$ factor which is not separably representable nor approximately finite.

Any internal operator $x \in \mathcal{A}_G$ leaves $\mathcal{D}_G$ and $\mathcal{D}_M$ invariant and gives rise to a bounded operator $\lambda(x)$ on $\hat{\mathcal{D}}$ such that $\lambda(x)[\xi] = [x\xi]$ for all $\xi \in \mathcal{D}_G$. Then the correspondence $[x] \mapsto \lambda(x)$ for $x \in \mathcal{A}_G$ defines a faithful $*$-representation of the C*-algebra $\hat{\mathcal{A}}$ on the Hilbert space $\hat{\mathcal{D}}$.

Let $P_D$ be the internal projection from ${}^*\mathcal{H}$ onto $\mathcal{D}$. Any bounded operator $T$ on $\mathcal{H}$ has the nonstandard extension ${}^*T$ which is an internal bounded linear operator on ${}^*\mathcal{H}$. Denote the restriction of $P_D\,{}^*T$ to $\mathcal{D}$ by ${}^*T_D$. Then it is easy to see that ${}^*T_D \in \mathcal{A}_G$. We denote the operator $\lambda({}^*T_D)$ on $\hat{\mathcal{D}}$ by $T_D$, which is called the *standard hyperfinite extension* of $T$ to $\hat{\mathcal{D}}$. Then $T_D = T$ on $\mathcal{H}$ and $\|T_D\| = \|T\|$. Properties of such extensions from $\mathcal{H}$ to $\hat{\mathcal{D}}$ are studied by Moore [23] extensively. Now the following statement is easily established; cf. [23, Lemma 1.3],

**Theorem 5.1.** *The mapping $\mathcal{E}_D : T \mapsto T_D$ is a completely positive isometric injection from $\mathcal{L}(\mathcal{H})$ to $\hat{\mathcal{A}}$ and the mapping $\mathcal{E}_H : \lambda(T) \mapsto V_H^\dagger \lambda(T) V_H$ is a completely positive surjection of $\hat{\mathcal{A}}$ onto $\mathcal{L}(\mathcal{H})$. The composition $\mathcal{E}_H\mathcal{E}_D$ is the identity map on $\mathcal{L}(\mathcal{H})$, and the composition $\mathcal{E}_D\mathcal{E}_H$ is the norm one projection from $\hat{\mathcal{A}}$ onto $\mathcal{E}_D(\mathcal{L}(\mathcal{H}))$.*

## 6. The phase operator in the hyperfinite extension

In this section, we will show that there is a bounded self-adjoint operator $\hat{\phi}$ in $\hat{\mathcal{A}}$ satisfying the following two conditions:

(P1) The spectral measure $E_\phi$ of $\hat{\phi}$ satisfies the relation

$$P_\phi(d\theta) = V_H^\dagger E_\phi(d\theta) V_H, \tag{6.1}$$

where $P_\phi$ is the phase POM.

(P2) The Susskind-Glogower phase operators are given by the relations:

$$\begin{aligned} \widehat{e^{i\phi}} &= V_H^\dagger\, e^{i\hat{\phi}}\, V_H, & (6.2)\\ \widehat{e^{-i\phi}} &= V_H^\dagger\, e^{-i\hat{\phi}}\, V_H, & (6.3)\\ \widehat{\cos}\,\phi &= V_H^\dagger\, \cos\hat{\phi}\, V_H, & (6.4)\\ \widehat{\sin}\,\phi &= V_H^\dagger\, \sin\hat{\phi}\, V_H. & (6.5) \end{aligned}$$

Let $\Delta\theta = 2\pi/\nu$, and $\theta_m = m\Delta\theta$ for each $m$ $(m = 0, 1, \ldots, \nu - 1)$. The internal phase eigenstate $|\theta_m\rangle$ in $\mathcal{D}$ is defined by

$$|\theta_m\rangle = \nu^{-1/2} \sum_{n=0}^{\nu-1} e^{in\theta_m} |n\rangle. \tag{6.6}$$

Then we have

$$\langle\theta_m|\theta_{m'}\rangle = \delta_{m,m'}. \tag{6.7}$$

The internal phase operator $\hat{\phi}_I$ on $\mathcal{D}$ is defined by

$$\hat{\phi}_I = \sum_{m=0}^{\nu-1} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{6.8}$$

Then the internal phase operator $\hat{\phi}_I$ has the spectrum $\{2m\pi/\nu \mid m = 0, 1, \ldots, \nu - 1\}$ and hence is in $\mathcal{A}_G$. Thus we have the self-adjoint operator $\lambda(\hat{\phi}_I)$ on $\hat{\mathcal{D}}$, denoted by $\hat{\phi}$. Denote by $\Lambda(\hat{\phi})$ the spectrum of $\hat{\phi}$ and $\Pi_0(\hat{\phi})$ the point spectrum (eigenvalues) of $\hat{\phi}$.

**Theorem 6.1.** *We have $\Lambda(\hat{\phi}) = \Pi_0(\hat{\phi}) = [0, 2\pi]$. For each $\theta$ ($\leq \theta \leq 2\pi$), the vector $[|\theta_m\rangle]$ with $\theta_m \approx \theta$ is an eigenvector of $\hat{\phi}$ for the eigenvalue $\theta$.*

*Proof.* Let $\theta \in [0, 2\pi]$ and $\theta_m \approx \theta$. Then obviously,

$$\hat{\phi}[|\theta_m\rangle] = [\hat{\phi}_I|\theta_m\rangle] = [\theta_m|\theta_m\rangle] = {}^\circ\theta_m[|\theta_m\rangle] = \theta[|\theta_m\rangle],$$

and hence $[|\theta_m\rangle]$ is an eigenvector of $\hat{\phi}$ corresponding to eigenvalue $\theta$. Thus we have $[0, 2\pi] \subset \Pi_0(\hat{\phi})$. Since $0 \leq \theta_m \leq 2\pi$, we have $0 \leq \langle\psi|\hat{\phi}_I|\psi\rangle \leq 2\pi$ for any unit vector $\psi \in \mathcal{D}$, and hence

$$0 \leq \langle[\psi]|\hat{\phi}|[\psi]\rangle = {}^\circ\langle\psi|\hat{\phi}_I|\psi\rangle \leq 2\pi.$$

It follows that $0 \leq \hat{\phi} \leq 2\pi 1$, so that $\Lambda(\hat{\phi}) \subset [0, 2\pi]$. Therefore, $\Lambda(\hat{\phi}) = [0, 2\pi] = \Pi_0(\hat{\phi})$. □

For each $\theta$ ($0 \leq \theta \leq 2\pi$) and $n \in \mathbf{N}$, define $F(\theta, n)$ to be the internal projection

$$F(\theta, n) = \sum_{\theta_m \leq \theta + n^{-1}} |\theta_m\rangle\langle\theta_m|.$$

Then $F(\theta, n) = 0$ if $\theta + n^{-1} < 0$, and for each $\theta$ the sequence $\{\hat{F}(\theta, n)\}$ is a monotone decreasing sequence of projections on $\hat{\mathcal{D}}$. Define $E_\phi(\theta)$ to be the strong limit of $\hat{F}(\theta, n)$. Then $\{E_\phi(\theta) \mid \theta \in [0, 2\pi]\}$ is the spectral resolution for $\hat{\phi}$ [23, Theorem 4.1].

**Theorem 6.2.** *The bounded self-adjoint operator $\hat{\phi}$ in $\hat{\mathcal{A}}$ with its spectral resolution $E_\phi(\theta)$ satisfies conditions* (P1) *and* (P2).

*Proof.* Let $n, n' \in \mathbf{N}$. We have

$$\begin{aligned}
\langle n|E_\phi(\theta)|n'\rangle &= \lim_{k\to\infty} \langle n|\hat{F}(\theta, k)|n'\rangle \\
&= \lim_{k\to\infty} {}^\circ\langle n|F(\theta, k)|n'\rangle \\
&= \lim_{k\to\infty} {}^\circ \sum_{\theta_m \leq \theta + k^{-1}} e^{i(n-n')\theta_m} \frac{\Delta\theta}{2\pi} \\
&= \lim_{k\to\infty} \int_0^{\theta + k^{-1}} e^{i(n-n')\bar{\theta}} \frac{d\bar{\theta}}{2\pi} \\
&= \int_0^{\theta} e^{i(n-n')\bar{\theta}} \frac{d\bar{\theta}}{2\pi}.
\end{aligned}$$

From (3.6), this concludes condition (P1). To prove (P2), by the obvious relations, it suffices to show (6.3). We have

$$\begin{aligned}\langle n|e^{-i\hat{\phi}}|n'\rangle &= \int_0^{2\pi} e^{-i\theta}\,\langle n|dE_\phi(\theta)|n'\rangle \\ &= \int_0^{2\pi} e^{i(n-n'-1)\theta}\,\frac{d\theta}{2\pi} \\ &= \delta_{n,n'+1},\end{aligned}$$

whence (6.3) is obtained from the relation

$$\langle n|(\hat{N}+1)^{-\frac{1}{2}}\hat{a}|n'\rangle = \langle n|n'+1\rangle.$$

□

## 7. Macroscopic states in the hyperfinite extension

We have extended the Schrödinger representation on $\mathcal{H}$ to its hyperfinite extension $\hat{\mathcal{D}}$. The following theorem shows that the augmented states in $\hat{\mathcal{D}} \ominus \mathcal{H}$ can be interpreted naturally as states representing the macroscopic limits of the quantum mechanical states in $\mathcal{H}$.

**Theorem 7.1.** *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that $\langle n|T|n'\rangle$ ($n$, $n' \in \mathbf{N}$) is a Cauchy sequence in $n$ and $n'$. Then for any $k$, $k' \in \{0,1,\ldots,\nu-1\} \setminus \mathbf{N}$ the standard hyperfinite extension $T_D \in \hat{\mathcal{A}}$ of $T$ satisfies the relation*

$$\langle k|T_D|k'\rangle = \lim_{n,n'\to\infty} \langle n|T|n'\rangle. \tag{7.1}$$

*Proof.* Let $a_{n,n'} = \langle n|T|n'\rangle$ for $n$, $n' \in \mathbf{N}$, and $L = \lim_{n,n'\to\infty} a_{n,n'}$. Let ${}^*a_{m,m'}$ ( $m$, $m' \in {}^*\mathbf{N}$) be the nonstandard extension of the sequence $a_{n,n'}$ ($n$, $n' \in \mathbf{N}$). Then, since $a_{n,n'}$ is a Cauchy sequence, $L \approx a_{m,m'}$ for all $m$, $m' \in {}^*\mathbf{N} \setminus \mathbf{N}$. Let $k$, $k' \in \{0,1,\ldots,\nu-1\} \setminus {}^*\mathbf{N}$. By transfer principle, ${}^*a_{k,k'} = \langle k|{}^*T|k'\rangle$, and hence $L \approx \langle k|{}^*T|k'\rangle$. It follows that $L = {}^\circ\langle k|{}^*T|k'\rangle = \langle k|T_D|k'\rangle$. □

**Remark.** Suppose that the nonstandard universe is constructed by a bounded ultrapower of a superstructure based on $\mathbf{R}$, using the index set $I = \mathbf{N}$ and a free ultrafilter $\mathcal{U}$. Then any hyperfinite number $k \in {}^*\mathbf{N} \setminus \mathbf{N}$ is represented by a sequence $s(i)$ ($i \in \mathbf{N}$) of natural numbers in such a way that two sequences $s(i)$ and $s'(i)$ represents the same hyperfinite number $k$ if and only if $\{i \in \mathbf{N}|\ s(i) = s'(i)\} \in \mathcal{U}$. Let $m$ be a hyperfinite number corresponding to a sequence $s(i)$. Let $a(n)$ be a bounded sequence of complex numbers. Then the standard part of ${}^*a_m$ coincides with the ultralimit of the subsequence $a(s(i))$ of $a(n)$, i.e.,

$$°({}^*a_m) = \lim_{i \to \mathcal{U}} a(s(i)). \tag{7.2}$$

Thus, for any $T \in \mathcal{L}(\mathcal{H})$ and $k,\ k' \in {}^*\mathbf{N} \setminus \mathbf{N}$, we have,

$$\langle k|T_D|k'\rangle = \lim_{i,j \to \mathcal{U}} \langle s(i)|T|s'(j)\rangle, \tag{7.3}$$

provided $k$ $k'$ are represented by sequences $s(i)$, $s(j)$.

By the above theorem, it is obvious that the space $\hat{\mathcal{D}}$ contains microscopic states, macroscopic states and superpositions of those states. In the conventional physics, microscopic properties and macroscopic properties are discussed separately in quantum mechanics and in classical mechanics (or quantum mechanics with $\hbar \to \infty$). However, these approaches cannot mention the quantum mechanical coherence between microscopic states and macroscopic states. Our new representation has just realized one of such a coherent description of quantum and classical mechanics — only in such a representation the phase operator behaves as a self-adjoint operator. A potential application of this representation other than phase sensitive effects is the measurement problem, where the unitary time evolution in an amplifier evolves from a microscopic state to a state with infinite excitation which can be observed macroscopically just like Schrödinger's cat. An application of our framework to this area will be published elsewhere.

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