$A \ge B \ge 0$ ensures $(B^r A^p B^r)^{1/q} \ge (B^r B^p B^r)^{1/q}$ for $r \ge 0, p \ge 0, q \ge 1$ with $(1 + 2r)q \ge p + 2r$ and its applications (Linear Operators and Inequalities)

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with $(1 + 2r)q \geq p + 2r$ and its applications

In what follows, capital letter means a bounded linear operator on a Hilbert space.

An operator $T$ is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator $T$ is strictly positive (in symbol: $T > 0$) if $T$ is positive and invertible.

As an extension of the Löwner-Heinz theorem [17][20], we established the Furuta inequality [6] which reads as follows. If $A \geq B \geq 0$, then for each $r \geq 0$ (i) $(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$ and (ii) $(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$ hold for $p$ and $q$ such that $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$. We remark that the Furuta inequality yields the Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above: if $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. Alternative proofs of the Furuta inequality are given in [3][8][18] and an elementary proof is shown in [9].

**Theorem A** (Löwner-Heinz 1934). If $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$.

Related to Theorem A, the following result is well known.

**Proposition.** If $A \geq B \geq 0$ does not always ensure $A^p \geq B^p$ for any $p > 1$.

As a generalization of Theorem A and related to Proposition, we established the following result.

**Theorem B** (Furuta 1987). If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$

and

(ii) $(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$

hold for each $p$ and $q$ such that $p \geq 0$, $q \geq 1$ and $(1 + 2r)q \geq p + 2r$.

Inequalities (i) and (ii) in Theorem B hold for the points on $p$, $q$ and $r$ belong to the oblique lines in the following figure.
In this paper, we cite several applications of Theorem B as follows.

**Applications of Theorem B**

(A) **Operator inequalities**

1. Characterizations of operators satisfying $\log A \geq \log B$
2. Generalizations of Ando's theorem
3. Applications to the relative operator entropy
4. Applications to other operator inequalities
5. Applications to the Log-Majorization by Ando and Hiai
6. Application to $p$-hyponormal operators for $0 < p < 1$

(B) **Norm inequalities**

1. Several type generalizations of Heinz-Kato theorem
2. Generalizations of some folk theorem on norm

(C) **Operator equations**

1. Generalizations of Pedersen-Takesaki theorem and related results

Among applications of Theorem B states above, we cite [2][4][5][10] and [11] for (A) operator inequalities and also we cite [12][13][14] and [16] for (B) norm inequalities and finally we cite [7] for (C) operator equations.

Ando-Hiai [1] have established a lot of useful and beautiful results on log-majorization and we are really impressed with these beautiful and useful results. The purpose of this paper is to announce new application [15] of Theorem B to the log-majorization by Ando-Hiai [1]. Precisely speaking, we can interpolate Theorem B and this log-majorization.
§1. AN EXTENSION OF THE FURUTA INEQUALITY

First of all, we state the following extension of the Furuta inequality.

**Theorem 1.1.** If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

\[
F_{p, t}(A, B, r, s) = A^{-r/2} \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-r/2}
\]

is a decreasing function of both $r$ and $s$ for any $s \geq 1$ and $r \geq t$ and the following inequality holds

\[
A^{1-t} = F_{p, t}(A, A, r, s) \geq F_{p, t}(A, B, r, s)
\]

for any $s \geq 1, p \geq 1$ and $r$ such that $r \geq t \geq 0$.

**Corollary 1.2.** If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$,

\[
\{A^{r/2} (A^{-t/2} A^p A^{-t/2})^s A^{r/2}\}^\alpha \geq \{A^{r/2} (A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^\alpha
\]

holds for any $s \geq 0$, $p \geq 0$, $0 \leq \alpha \leq 1$ and $r \geq t$ with $(s-1)(p-1) \geq 0$ and $1-t+r \geq ((p-t)s+r)\alpha$.

**Remark 1.1.** In the case $t = 0$ in Corollary 1.2, we may not assume $A > 0$. Putting $t = 0$ and $s = 1$ in Corollary 1.2, we have (ii) of Theorem B. Hence Corollary 1.2 can be considered as an extension of Theorem B since (i) is equivalent to (ii) in Theorem B.

Corollary 1.2 easily implies the following result when we put $t = 1$.

**Corollary 1.3.** If $A \geq B \geq 0$ with $A > 0$, then

\[
A^r \geq \{A^{r/2} (A^{-1/2} B^p A^{-1/2})^s A^{r/2}\}^{\frac{1}{(s-1)}r+r}
\]

holds for any $s \geq 1, p \geq 1$ and $r \geq 1$.

When we put $s = r$ in Corollary 1.3, we have the following Theorem C obtained by Ando and Hiai [1, Theorem 3.5].
Theorem C [1]. If $A \geq B \geq 0$ with $A > 0$, then

$$A^r \geq \{A^{r/2}(A^{-1/2}B^pA^{-1/2})^{r}A^{r/2}\}^{1/p}$$

holds for any $p \geq 1$ and $r \geq 1$.

Corollary 1.4. If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$

(i) $A^{1+t} \geq (A^{t/2}B^{2p-t}A^{t/2})^{\frac{1+t}{t}} \geq |A^{-t/2}B^p A^{t/2}|^{\frac{1+t}{p}}$

and

(ii) $A^2 \geq (A^{1/2}B^{2p-t}A^{1/2})^{\frac{2}{2p+1-t}} \geq |A^{-t/2}B^p A^{1/2}|^{\frac{2}{2p+1-t}}$

hold for any $2p \geq 1 + t$.

Corollary 1.5. If $A \geq B \geq 0$ with $A > 0$, then

$$A^2 \geq (A^{1/2}B^{2p-1}A^{1/2})^{1/p} \geq |A^{-1/2}B^p A^{1/2}|^{2/p}$$

for any $p \geq 1$.

Corollary 1.6 [4][10][11]. If $A \geq B \geq 0$, then

$$G(p, r) = A^{-r/2}(A^{r/2}B^p A^{r/2})^{(1+r)/(p+r)} A^{-r/2}$$

is a decreasing function of both $p$ and $r$ for $p \geq 1$ and $r \geq 0$.

§2. THE LOG-MAJORIZATION EQUIVALENT TO AN EXTENSION OF THE FURUTA INEQUALITY

Throughout this section, a capital letter means $n \times n$ matrix.

Following after Ando and Hiai [1], let us write $A \prec B$ for positive semidefinite matrices $A, B \geq 0$ and call the log-majorization if

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B), \quad k = 1, 2, \ldots, n - 1,$$

and
\[
\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \text{ i.e. } \det A = \det B,
\]

where \(\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)\) and \(\lambda_1(B) \geq \lambda_2(B) \geq \ldots \geq \lambda_n(B)\) are the eigenvalues of \(A\) and \(B\) respectively arranged in decreasing order. Note that when \(A, B > 0\) (strictly positive) the log-majorization \(A \prec (\log) B\) is equivalent to \(\log A \prec (\log) B\). Also \(A \prec (\log) B\) ensures \(\|A\| \leq \|B\|\) holds for any unitarily invariant norm.

**Definition 1.** When \(0 \leq \alpha \leq 1\), the \(\alpha\)-power mean of \(A, B > 0\) is defined by

\[
A \#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.
\]

Further \(A \#_\alpha B\) for \(A, B \geq 0\) is defined by

\[
A \#_\alpha B = \lim_{\epsilon \downarrow 0}(A + \epsilon I) \#_\alpha (B + \epsilon I).
\]

This \(\alpha\)-power mean is the operator mean corresponding to the operator monotone function \(t^\alpha\). We can see [19] for general theory of operator means.

For the sake of convenience for symbolic expression, we define \(A \mathfrak{h}_s B\) for any \(s \geq 0\) and for \(A > 0\) and \(B \geq 0\) by the following

\[
A \mathfrak{h}_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}.
\]

\(A \mathfrak{h}_\alpha B\) in the case \(0 \leq \alpha \leq 1\) just coincides with the usual \(\alpha\)-power mean denoted by \(A \#_\alpha B\).

We can transform (1.10) of Theorem 1.1 into the following log-majorization inequality by using the method by Ando and Hiai [1].

**Theorem 2.1.** For every \(A > 0\), \(B \geq 0\), \(0 \leq \alpha \leq 1\) and each \(t \in [0, 1]\)

\[
(A \#_\alpha B)^h \succ (\log) A^{1-t+r} \#_\beta (A^{1-t} \mathfrak{h}_s B)
\]

holds for \(s \geq 1\), and \(r \geq t \geq 0\), where \(\beta = \frac{\alpha(1-t+r)}{(1-\alpha t)s + \alpha r}\) and \(h = \frac{(1-t+r)s}{(1-\alpha t)s + \alpha r}\).
Corollary 2.2. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,

\[(A \#_{\alpha} B)^h \succ (A \#_{\alpha} B)^s \quad \text{for } r \geq 1 \text{ and } s \geq 1\]

where $h = [\alpha s^{-1} + (1 - \alpha) r^{-1}]^{-1}$.

The above log-majorization is equivalent to any one of the following (2.4), (2.5) and (2.6):

\[(2.4) \quad (A \#_{\alpha} B)^{1/r} \succ (A^q \#_{1/q} B^p)^{1/k} \quad \text{for } 0 < r \leq q \text{ and } 0 < r \leq p,\]

where $k = [\alpha p^{-1} + (1 - \alpha) q^{-1}]^{-1}$.

\[(2.5) \quad (A \#_{\alpha} B)^{1/s} \succ (A^p \#_{1/p} B^p)^{1/p} \quad \text{for } 0 < r \leq p \text{ and } 0 < q \leq p,\]

where $s = \alpha q + (1 - \alpha) r$ and $l = [\alpha r^{-1} + (1 - \alpha) q^{-1}]^{-1}$.

\[(2.6) \quad (A \#_{\alpha} B)^{1/u} \succ (A^q \#_{\beta} B^p)^{1/p} \quad \text{for } 0 < r \leq q \leq p,\]

where $u = \frac{\alpha q^2 + (1 - \alpha) pr}{q}$ and $\beta = \frac{\alpha q^2}{\alpha q^2 + (1 - \alpha) pr}$.

Remark 2.1. We remark that $h = [\alpha s^{-1} + (1 - \alpha) r^{-1}]^{-1}$ in Corollary 2.2 is a generalized harmonic mean of $r$ and $s$ and when $\alpha = 1/2$, $h$ is the usual harmonic mean of $r$ and $s$. Also $l$ in (2.5) is a generalized harmonic one of $r$ and $q$, while $s$ in (2.5) is a generalized arithmetic mean of $q$ and $r$.

Corollary 2.2 yields the following result [1, Theorem 2.1].

Theorem D [1]. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$,

\[(A \#_{\alpha} B)^r \succ (A \#_{\alpha} B)^r \quad \text{for } r \geq 1\]

or equivalently

\[(A^q \#_{\alpha} B^p)^{1/q} \succ (A^p \#_{\alpha} B^p)^{1/p} \quad \text{for } 0 < q \leq p.\]

Also we can transform Corollary 1.2 into the following log-majorization.
**Theorem 2.3.** If \( A > 0 \) and \( B \geq 0 \), then for each \( t \in [0,1] \) and \( 0 \leq \alpha \leq 1 \)

\[
(A^{1/2}BA^{1/2})^{\alpha p s} \succ \log (A^{\frac{1}{2}\alpha((p-t)s+r)}(A^{-\frac{r-t}{2}}(A^t h_s B^p)A^{-\frac{r-t}{2}})^{\alpha}A^{\frac{1}{2}\alpha((p-t)s+r)})
\]

holds for any nonnegative numbers \( s, p \) and \( r \) such that \( r \geq t \) and \((s - 1)(p - 1) \geq 0 \) with \( 1 - t + r \geq ((p - t)s + r)\alpha \) where \( q = \alpha(p - t)s + ar - r + t \).

**Theorem 2.4.** If \( A > 0 \) and \( B \geq 0 \), then for each \( t \in [0,1] \) and \( 0 \leq \alpha \leq 1 \)

\[
A^{1/2}(A^{p}\#_{\alpha} B^{p})^{q/p} A^{1/2}
\]

holds for every \( p \geq q > 0 \), \( r \geq t \) and \( s \geq 1 \).

When \( t = 0 \) Theorem 2.4 becomes the following result.

**Corollary 2.5.** If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq \alpha \leq 1 \)

\[
A^{1/2}(A^{p}\#_{\alpha} B^{p})^{q/p} A^{1/2}
\]

holds for every \( p \geq q > 0 \), \( r \geq 0 \) and \( s \geq 1 \).

When \( s = 1 \) and \( r = p \) Corollary 2.5 yields the following Theorem E [1, Theorem 3.3].

**Theorem E [1].** If \( A > 0 \) and \( B \geq 0 \), then

\[
A^{1/2}(A^{p}\#_{\alpha} B^{p})^{q/p} A^{1/2}
\]

for every \( 0 \leq \alpha \leq 1 \) and \( 0 < q \leq p \).

Taking \( s = 2 \) and \( r = p \) in Corollary 2.5 we have
Corollary 2.6. If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq \alpha \leq 1 \)

\[
A^{1/2}(A^{p} \#_{\alpha} B^{p})^{q/p} A^{1/2} \\
\succ A^{\frac{1}{2}(1+\frac{\alpha}{p})} \left\{ (A^{-p/2} B^{p} A^{-p/2})^\alpha A^{p} (A^{-p/2} B^{p} A^{-p/2})^\alpha \right\}^{\frac{q}{p}} A^{\frac{1}{2}(1+\frac{\alpha}{p})}
\]

holds for any \( 0 < q \leq p \).

Corollary 2.7. If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq r \leq 1 \)

\[
A^{r/2} B^{r} A^{r/2} \succ A^{r/2}(A^{-1/2} B^{1/2} A^{-1/2})^r A^{r}
\]

holds for every \( 0 < \alpha \leq 1 \).

Corollary 2.8. If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq r \leq 1 \)

\[
(A^{1/2} B A^{1/2})^r \succ A^{ru/2}(A^{-u/2} B^{r/u} A^{-u/2})^r A^{ru/2}
\]

holds for every \( 0 < \alpha \leq 1 \) and \( u \geq 0 \).

Corollary 2.7 and Corollary 2.8 imply the following known result [1, Corollary 3.4].

Corollary F [1]. If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq r \leq 1 \)

\[
(A^{1/2} B A^{1/2})^r \succ A^{r/2} B^r A^{r/2} \succ A^r (A^{-1/2} B A^{-1/2})^r A^r
\]

§3. LOGARITHMIC TRACE INEQUALITIES AS AN APPLICATION OF LOG-MAJORIZATION IN §2

Throughout this section, a capital letter means \( n \times n \) matrix.

Theorem 3.1. If \( A > 0 \) and \( B \geq 0 \), then for every \( 0 \leq \alpha \leq 1 \) and \( t \in [0, 1] \)

\[
s\text{Tr} \text{Alog}(A^p \#_{\alpha} B^p) - \text{Tr} \text{Alog}\{A^{-t/2}(A^{p} \#_{\alpha} B^{p})A^{t/2}\} A^{-r/2} \\
\geq (r - st) \text{Tr} \text{Alog} A
\]

holds for any \( s \geq 1 \), \( r \geq t \) and \( p \geq 0 \).

When \( t = 0 \) Theorem 3.1 yields the following result.
Corollary 3.2. If $A > 0$ and $B \geq 0$, then for every $0 \leq \alpha \leq 1$

\[
 s\text{TrAlog}(A^p \#_\alpha B^p) - \text{TrAlog}\{A^{-r/2}[A^p \#_\alpha B^p]^sA^{-r/2}\} \\
 \geq r\text{TrAlog}A
\]

holds for any $s \geq 1$, $r \geq 0$ and $p \geq 0$.

Taking $s = 1$ and $r = p > 0$ in Corollary 3.2 we have the following result [1, Theorem 5.3].

Theorem G [1]. If $A \geq 0$ and $B > 0$, then for every $0 \leq \alpha \leq 1$ and $p > 0$

\[
 \frac{1}{p}\text{TrAlog}(A^p \#_\alpha B^p) + \frac{\alpha}{p}\text{TrAlog}(A^{p/2}B^{-p}A^{p/2}) \\
 \geq \text{TrAlog}A.
\]

Corollary 3.3. If $A > 0$ and $B > 0$, then for every $0 \leq \alpha \leq 1$

\[
 \text{TrAlog}(A^p \#_\alpha B^p) + \text{TrAlog}\{A^{q/2}[A^{-p} \#_\alpha B^{-p}]A^{q/2}\} \\
 \geq q\text{TrAlog}A
\]

holds for any $p \geq 0$ and $q \geq 0$.

We remark that Corollary 3.3 yields Theorem G stated above taking $q = p$.

Also taking $s = 2$, $t = 0$ and $r = p \geq 0$ in Theorem 3.1 we have:

Corollary 3.4. If $A > 0$ and $B > 0$, then for every $0 \leq \alpha \leq 1$

\[
 \text{TrAlog}(A^p \#_\alpha B^p)^2 + \text{TrAlog}\{(A^{p/2}B^{-p}A^{p/2})^\alpha A^{-p}(A^{p/2}B^{-p}A^{p/2})^\alpha\} \\
 \geq p\text{TrAlog}A
\]

holds for any $p \geq 0$. 
At the end of this early announcement, we summarize the following implication relations among results in this paper.

\[(1.10) \text{ in } \textbf{Theorem 1.1} \iff \textbf{Theorem 2.1} \]

\[
\begin{array}{c}
t = 0 \\
\text{Corollary 1.6} \\
\textbf{Theorem B [6]} (\text{Furuta inequality})
\end{array}
\begin{array}{c}
t = 1 \\
\text{Corollary 1.3} \\
\textbf{Theorem C [1]}
\end{array}
\begin{array}{c}
t = 1 \\
\text{Corollary 2.2} \\
\textbf{Theorem D [1]} (\text{log-majorization})
\end{array}
\begin{array}{c}
r = s \\
\textbf{Theorem B [6]} (\text{Furuta inequality})
\end{array}
\begin{array}{c}
r = s \\
\textbf{Theorem C [1]}
\end{array}
\begin{array}{c}
r = s
\end{array}
\]

The details, proofs and related results in this paper will appear in [15].

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